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VANISHING OF KILLING VECTOR FIELDS ON COMPACT FINSLER MANIFOLDS

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Abstract

In this paper, we define a new Ricci curvature on Finsler manifold named the mean Ricci curvature, which is useful in the study of different symmetric fields on manifolds. By presenting a Bochner type formula of Killing vector fields on general Finsler manifolds, we prove the vanishing theorem of the Killing vector fields on any compact Finsler manifold with a negative mean Ricci curvature. This result involves the vanishing theorem of Killing vector fields in the Riemannian case.

1. Introduction

The Killing vector field, which is a basic concept in Differential Geometry and Physics, is obtained from the isometric transformation on a manifold. Compared to the conformal field or the projective field, it is the simplest symmetric field on manifolds. The Bochner technique shows that any Killing vector field on a compact Riemannian manifold with negative Ricci curvature must be trivial. In particular, this implies that such manifold dose not have a one parameter family of isometries. The detail can be given as the following theorem.

Theorem 1.1 ([5]). *Suppose (M, g) is a compact Riemannian manifold whose Riemannian Ricci tensor is nonnegative, i.e., $Ric \leq 0$. Then every Killing field X is parallel, and $Ric(X, X) = 0$. Furthermore, if the Ricci curvature is negative, i.e., $Ric < 0$, then there is no nontrivial Killing field.*

Indeed one can use the maximum principle or the integral method to prove Theorem 1.1. The isometric transformation is a special kind of conformal or projective transformation. Correspondingly, the Killing vector field is a special

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kind of the conformal or projective vector field. Such geometric vector fields, including the Killing, homothety, conformal, affine and projective vector fields, have attracted many researchers' attentions [2, 9]. In Finsler geometry, using conformal vector fields on a Riemannian manifold with constant curvature, Zhongmin Shen and Qiaoling Xia obtained the expression of the conformal vector fields on a Randers space with weak isotropic flag curvature [8]. Later, Huangjia Tian proved that there is no nontrivial projective vector field on any compact Finsler manifold with negative flag curvature [10]. Since the Killing vector field is also a special kind of projective vector field, one can consider Tian's work as the analog of Theorem 1.1. However, his proof is different from the proof of the Riemannian case. Moreover, the condition in his result is about the flag curvature, which is the sectional curvature when the metric is refined to a Riemannian one. This means that Tian's analog in the Finsler case can not be considered simply as a generalization of Theorem 1.1.

In this paper, we prove a vanishing theorem of Killing vector fields on Finsler manifolds, based on two important concepts, namely, the degenerate elliptic operator and mean Ricci curvature. The presented theorem includes Theorem 1.1. The method we adopted here is the same as that used in the Riemannian case. Details are presented in Theorem 4.2 in Section 4 and the following theorem.

THEOREM 1.2. *Suppose* (M, F) *is a compact Finsler manifold with non-positive mean Ricci curvature* $\widetilde{Ric} \leq 0$. *Then every Killing field V is parallel, and* $\widetilde{Ric}(V, V) = 0$. *Furthermore, if the mean Ricci curvature is negative*, i.e., $\widetilde{Ric} < 0$ *then there is no nontrivial Killing field.*

2. Finsler manifold and mean Ricci curvature

In this section, we present some basic concepts and relations in Finsler geometry, including some important non-Riemannian tensors. At last, we give the definition of mean Ricci curvature.

Let (M, F) be a Finsler manifold. Actually, F is defined on TM, *i.e.*, $F = F(x, y)$, and is smooth on $T_0M := TM \backslash \{0\}$. We call F a Riemannian metric if $F = \sqrt{g_{ij}(x) y^i y^j}$, where all the fundamental tensor components $g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$ are independent of the tangent coordinates y. So there is an important tensor to indicate this fact, called the Cartan tensor, which is defined by $C = C_{ijk}\, dx^i \otimes dx^j \otimes dx^k$. The components are given by

$$C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k} = \frac{1}{4} \frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k}. \tag{1}$$

It follows from the definition that a Finsler metric is a Riemannian one if and only if the Cartan tensor vanishing, *i.e.*, $C = 0$.

Spray G is a special vector field defined on the punched tangent bundle T_0M. Locally, it can be given as

(2) $$G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where G^i are called the spray coefficients, and are 2-homogenous in y, namely, $G^i(x, \lambda y) = \lambda^2 G^i(x, y)$, for any $\lambda > 0$. In the assistance of the fundamental tensor, the spray of a Finsler metric can be expressed as

$$G^i(y) = \frac{1}{4}g^{il}(y)\left\{2\frac{\partial g_{jl}}{\partial x^k}(y) - \frac{\partial g_{jk}}{\partial x^l}(y)\right\}y^j y^k.$$

The spray coefficients arise from the geodesic equation. Actually, the spray gives a canonical horizontal-vertical split of $T(T_0M) = \mathscr{H} \oplus \mathscr{V}$, where $\mathscr{V}$ identical to the TM is called the vertical bundle, and $\mathscr{H}$ is the direct sum complement of $\mathscr{V}$ in $T(T_0M)$. For any vector on M, we can lift it to a horizontal bundle or the vertical bundle by horizontal lifting or vertical lifting, respectively. For any vector $V \in TM$, $V^{\mathscr{V}} \in \mathscr{V}$ denotes the vertical lifting of V, and $V^{\mathscr{H}} \in \mathscr{H}$ denotes the horizontal lifting of V. For instance, the vertical lifting of the unit direction vector $\frac{y^i}{F}\partial_i \in TM$, is a global vertical vector called the *distinguished vector*, whose expression is

(3) $$l^{\mathscr{V}} = \frac{y^i}{F}\frac{\partial}{\partial y^i}.$$

We use $l^{\mathscr{H}}$ and $l^{\mathscr{V}}$ to denote the horizontal and vertical lifting of $\frac{y^i}{F}\partial_i$, respectively. Moreover, the correspondence Θ of $\mathscr{V}$ and $\mathscr{H}$ is an isomorphism. More details can be found in [1].

There are several important connections on a Finsler manifold. In this paper, we choose the *Chern connection*, which is the unique torsion free and almost compatible affine connection. Components of *Christoffel symbol of Chern connection* are locally defined by

$$\Gamma^l_{ij} = \frac{1}{2}g^{lk}\left(\frac{\delta g_{ik}}{\delta x^j} + \frac{\delta g_{jk}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^k}\right).$$

The *horizontal derivatives* are given by $\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}$ where

(4) $$N^i_j = \frac{\partial G^i}{\partial y^j} = \Gamma^i_{jk} y^k.$$

Noticing the homogeneity of spray coefficients, we have $G^i = \frac{1}{2}\Gamma^i_{jk}y^j y^k$.

The *nonlinear connection coefficient* N^i_j only depends on G^i. From now on, we use components of a tensor to denote the tensor itself. If $T = T^i_j e_i \otimes \omega^j$ is a $(1,1)$-tensor, where e_i and ω^j are frames and dual frames on the pull back bundle, respectively, the *horizontal covariant derivatives about the Chern connection* $\nabla_k T^i_j$ are given by

$$\nabla_k T^i_j = \frac{\delta T^i_j}{\delta x^k} + \Gamma^i_{kl} T^l_j - \Gamma^l_{kj} T^i_l. \tag{5}$$

In particular, when operated on a function $f(x)$ defined on the whole manifold, the horizontal covariant derivatives become

$$\nabla_k f = \frac{\partial f}{\partial x^k}. \tag{6}$$

Since the Chern connection is almost compatible with the Finsler metric, the curvature tensors include two parts, namely, the *hh*-curvature tensor and the *hv*-curvature tensor as

$$\Omega^i_j = \frac{1}{2} R^i_{jkl}\omega^k \wedge \omega^l + P^i_{jkl}\omega^k \wedge \omega^{n+l}, \tag{7}$$

when we choose the dual frame as $\{\omega^1, \ldots, \omega^n, \omega^{n+1}, \ldots, \omega^{2n}\}$. The *hh*-curvature is also called *Chern Riemannian curvature tensor* whose components are locally defined by

$$R^i_{jkl} := \frac{\delta \Gamma^i_{jl}}{\delta x^k} - \frac{\delta \Gamma^i_{jk}}{\delta x^l} + \Gamma^i_{km}\Gamma^m_{jl} - \Gamma^i_{lm}\Gamma^m_{jk}. \tag{8}$$

The *hv*-curvature is also called *Chern non-Riemannian curvature tensor* whose components are obtained in local coordinates by

$$P^i_{jkl} = -\frac{\partial \Gamma^i_{jl}}{\partial y^k}. \tag{9}$$

It follows from the definition that $P^i_{jkl} = P^i_{kjl}$. If we denote $P_{jikl} = g_{im}P^i_{jkl}$, then

$$P_{jikl} + P_{ijkl} = 2(C_{ijs}L^s_{kl} - C_{ijl|k}), \tag{10}$$

where L^s_{kl} are the components of Landsberg tensor. The Landsberg tensor is defined by $L = L^i_{jk}(x,y)\frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k$ with components locally related to Cartan tensor and Chern non-Riemannian curvature as

$$L^i_{jk} = C^i_{jk|l}y^l = -y^l P^i_{ljk}. \tag{11}$$

We denote $L_{ijk} = g_{il}L^l_{jk}$. More details about the Riemannian and non Riemannian curvature tensor and their relations can be referred to [6] and [7].

Because curvatures on a Finsler manifold are related to a tangent coordinate y, *i.e.*, a tangent direction, the *flag curvature* should be dependent on not only the section (called 'flag') spanned by two tangent vectors, but also a special direction (called 'pole'). Its expression is

$$K(\Pi_y) := \frac{-R_{ijkl}\,y^i v^j y^k v^l}{(g_{ik}g_{jl} - g_{il}g_{jk})\,y^i v^j y^k v^l}, \tag{12}$$

where $R_{ijkl} := R^s_{ikl}g_{sj}$, and $\Pi_y = span\{y, v\}$ is a 2-dimensional section.

When we denote the y contraction of Chern Riemannian curvature tensor by $R^i_{kl} = y^s R^i_{skl}$, it follows from (8) and the equation $N^i_j = y^k \Gamma^i_{jk}$ that

$$R^i_{jk} = \frac{\partial N^i_k}{\partial x^j} - \frac{\partial N^i_j}{\partial x^k} + N^s_k \frac{\partial N^i_j}{\partial y^s} - N^s_j \frac{\partial N^i_k}{\partial y^s}. \tag{13}$$

If we define

$$R^i_k := y^j R^i_{jkl} y^l, \quad R_{jk} := g_{ij} R^i_k = -R_{ijlk}\,y^i y^l,$$

and assume that v is a unit vector and is orthogonal to y with respect to g_y, that is, v satisfies that $g_y(v, v)$ and $g_y(y, v)$, we obtain

$$K(\Pi_y)(v) = F^{-2} R_{jk} v^j v^k. \tag{14}$$

So R^i_k or R_{jk} are also called components of the *flag curvature tensor*.

For any Finsler metric, a *related Riemannian metric* defined in [3] is given by

$$a_{ij}(x) = \int_{S_1} g_{ij}(x, y)\omega_x = \int_{S_1} \frac{\partial^2 F^2}{\partial y^i \partial y^j}\omega_x, \tag{15}$$

where ω_x is the volume form on $S_1 := \{\xi \in \mathbf{R}^n \mid F(\xi) = 1\}$. One can use the Busemann-Hausdorff volume form or Holmes-Thompson volume form according to the concrete problem.

Since $\{y^i\}$ can be considered as the homogeneous coordinates on fiber $S_xM = \{y \in T_xM \mid F(y) = 1\}$, for any point x on a Finsler manifold (M, F), there is a volume form called *Holmes-Thompson volume element*. It is defined by

$$dV_F := \sigma_H(x)\,dx, \tag{16}$$

$$\sigma_H(x) := \frac{1}{c_{n-1}} \int_{S_xM} \sqrt{\det(g_{ij})}\,dv, \tag{17}$$

$$dv := \sqrt{\det(g_{ij})} \sum_i (-1)^{i-1} y^i\,dy^1 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^n, \tag{18}$$

where "$\widehat{\ }$" means that the term is suppressed and c_{n-1} denotes the volume of the $(n-1)$-dimensional Euclidean sphere $\mathbf{S}^{n-1}$. The symbol dv is the volume form on the tangent sphere S_xM.

We now introduce the definition of the *mean Ricci curvature* $\widetilde{Ricci}$ (or $\widetilde{Ric}$ for short). Since the Riemannian Ricci curvature is the integral average of the sectional curvature, $\widetilde{Ric}$ becomes the Riemannian Ricci curvature when the metric reduces to Riemannian.

Definition 2.1. The mean Ricci curvature $\widetilde{Ric}$ is a kind of integral average of the flag curvature tensor on the indicatrix S_xM of each point, namely,

$$\widetilde{Ric}(v) = \frac{1}{c_{n-1}} \int_{S_xM} K(\Pi_y)(v) \frac{\sqrt{\det g_{ij}}}{\sqrt{\det a_{ij}}}\, dv = \frac{1}{c_{n-1}} \int_{S_xM} F^{-2} R_{jk} v^j v^k \frac{\sqrt{\det g_{ij}}}{\sqrt{\det a_{ij}}}\, dv, \tag{19}$$

where a_{ij} are the components of the related Riemannian metric defined in (15) with the Holmes-Thompson volume form.

3. Bochner type formula of the Killing vector field

In this paper, we denote the horizontal covariant derivative about the Chern connection by "|" and the vertical covariant derivative about the Chern connection by ";". We will first introduce the Finsler Ricci identity for vector fields.

Lemma 3.1 (Ricci type formula). *For any horizontal vector field $v = v^i(x, y)\delta_i$ on a Finsler manifold, the exchange of horizontal covariant derivatives about the Chern connection satisfies*

$$v_{j|k|l} - v_{j|l|k} = R^m_{jkl} v_m + R^m_{kl} v_{j;m}, \tag{20}$$

where R^m_{jkl} is the Chern-Riemannian curvature tensor.

Proof. For any $v(x, y)$, the first order horizontal covariant is

$$v_{j|k} = \frac{\delta v_j}{\delta x^k} - \Gamma^l_{jk} v_l,$$

where Γ^k_{ij} is the Christoffel symbol of the Chern connection. The second order horizontal covariant derivative is

$$\begin{aligned} v_{j|k|l} &= \frac{\delta^2 v_j}{\delta x^k \delta x^l} - \Gamma^m_{jk} \frac{\delta v_m}{\delta x^l} - \frac{\delta \Gamma^m_{jk}}{\delta x^l} v_m - \Gamma^m_{jl} v_{m|k} - \Gamma^m_{lk} v_{j|m} \\ &= \frac{\delta^2 v_j}{\delta x^k \delta x^l} - \Gamma^m_{jk} \frac{\delta v_m}{\delta x^l} - \frac{\delta \Gamma^m_{jk}}{\delta x^l} v_m - \Gamma^m_{jl} \left(\frac{\delta v_m}{\delta x^k} - \Gamma^i_{mk} v_i \right) - \Gamma^m_{lj} \left(\frac{\delta v_m}{\delta x^j} - \Gamma^i_{mj} v_i \right). \end{aligned} \tag{21}$$

We have

$$\frac{\delta v_i}{\delta x^j} = \frac{\partial v_i}{\partial x^j} - N^k_j \frac{\partial v_i}{\partial y^k}, \tag{22}$$

and

$$\frac{\delta^2 v_i}{\delta x^k \delta x^j} = \frac{\partial^2 v_i}{\partial x^j \partial x^k} - \frac{\partial N_j^l}{\partial x^k}\frac{\partial v_i}{\partial y^l} - N_j^l \frac{\partial^2 v_i}{\partial x^k \partial y^l} - N_k^m \frac{\partial^2 v_i}{\partial y^m \partial x^j} + N_k^m \frac{\partial N_j^l}{\partial y^m}\frac{\partial v_i}{\partial y^l} + N_k^m N_j^l \frac{\partial^2 v_i}{\partial y^m \partial y^l}. \tag{23}$$

Plugging equations (22) and (23) back into (21), we get

$$\begin{aligned} v_{j|k|l} - v_{j|l|k} &= \left(\frac{\delta \Gamma^m_{jl}}{\delta x^k} - \frac{\delta \Gamma^m_{jk}}{\delta x^l} + \Gamma^m_{ki}\Gamma^i_{jl} - \Gamma^m_{li}\Gamma^i_{jk}\right) v_m \\ &\quad - \left(N_k^s \frac{\partial N_l^m}{\partial y^s} - N_l^s \frac{\partial N_k^m}{\partial y^s} + \frac{\partial N_k^m}{\partial x^l} - \frac{\partial N_l^m}{\partial x^k}\right) v_{j;m}. \end{aligned} \tag{24}$$

Considering formulae (8) and (13), we get (20). Q.E.D.

Now we will focus on Killing vector fields, which can induce isometric transformations on manifolds. That is, $V = V^i(x)\partial_i$ is a killing vector field if and only if the Lie derivative of the metric about the complete lifting $\tilde{V}$ vanishes, i.e.,

$$\mathscr{L}_{\hat{V}} F = 0.$$

A *complete lifting* $\tilde{V}$ of a vector field V from TM to TTM is always defined by $\hat{V} = V^i \frac{\partial}{\partial x^i} + y^j \frac{\partial V^i}{\partial x^j}\frac{\partial}{\partial y^i}$.

Using the correspondence Θ between horizontal and vertical bundles, we can extend the Chern connection to $l^{\mathscr{V}}$. Locally,

$$\nabla_{\delta/\delta x^i} \frac{\partial}{\partial y^j} = \Gamma^k_{ij}\frac{\partial}{\partial y^k}. \tag{25}$$

Moreover, under this correspondence, $\Theta^{-1}(V^{\mathscr{H}}) = V^{\mathscr{V}}$. Then, we can express the complete lifting in a global way as

$$\hat{V} = V^{\mathscr{H}} + F\nabla_{l^{\mathscr{H}}} V^{\mathscr{V}}. \tag{26}$$

Actually,

$$\begin{aligned} \hat{V} &= V^i \frac{\partial}{\partial x^i} + y^j \frac{\partial V^i}{\partial x^j}\frac{\partial}{\partial y^i} \\ &= V^i \frac{\delta}{\delta x^i} + y^j\left(\frac{\partial V^i}{\partial x^j} + V^k \Gamma^i_{kj}\right)\frac{\partial}{\partial y^i} \\ &= V^i \frac{\delta}{\delta x^i} + y^j \nabla_{\delta/\delta x^j} V^{\mathscr{V}}. \end{aligned} \tag{27}$$

Locally, $\mathscr{L}_{\hat{V}}F = 0$ is equal to

$$V^i \frac{\partial F}{\partial x^i} + y^j \frac{\partial V^i}{\partial x^j}\frac{\partial F}{\partial y^i} = 0, \tag{28}$$

or

$$y^j y^k V_{k|j} = 0. \tag{29}$$

By taking the second derivatives of (29) about y^i, y^j, one can directly conclude that V satisfies the following equation

$$V_{i|j} + V_{j|i} + 2C^p_{ij}V_{p|q}y^q = 0, \tag{30}$$

where $C^p_{ij} := g^{pq}C_{qij}$ and $C_{qij} = \frac{1}{4}[F^2]_{y^i y^j y^q} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^q}$ is the Cartan tensor.

Before giving the Bochner type formula of the Killing vector field on Finsler manifold, we define a degenerate elliptic operator $\triangle^D$. First, Laplacian on a Riemannian manifold is defined by $\triangle = \nabla\cdot\nabla = g^{ij}(\nabla_{\partial/\partial x^i}\nabla_{\partial/\partial x^j} - \nabla_{\nabla_{\partial/\partial x^i}\partial/\partial x^j})$, where $\nabla\cdot\nabla$ means taking trace by the Riemannian metric g^{ij}. Now, we replace the g^{ij} by a degenerate matrix y^iy^j to define the degenerate elliptic operator, that is

Definition 3.2. A degenerate elliptic operator $\triangle^D$ is defined as the second order derivative about the Chern connection contracting with a symmetric semi-positive definite matrix $a^{ij} = y^iy^j$, namely,

$$\triangle^D := \nabla\diamond\nabla = y^iy^j(\nabla_{\delta/\delta x^i}\nabla_{\delta/\delta x^j} - \nabla_{\nabla_{\delta/\delta x^i}\delta/\delta x^j}), \tag{31}$$

where $\diamond$ means taking trace with respect to the matrix y^iy^j, and ∇ means the horizontal covariant derivative with respect to the Chern connection.

By direct computation, one can easily see that

$$\begin{aligned}\triangle^D &= y^iy^j(\nabla_{\delta_i}\nabla_{\delta_j} - \nabla_{\nabla_{\delta_i}\delta_j})\\ &= \nabla_{y^i\delta_i}\nabla_{y^j\delta_j} - y^i(-N^j_i)\nabla_{\delta_j} - \nabla_{y^iy^j\nabla_{\delta_i}\delta_j}\\ &= \nabla_{y^i\delta_i}\nabla_{y^j\delta_j}.\end{aligned}$$

Thus we can present the following definition on the sphere bundle.

Definition 3.3. On the tangent sphere, the operator also can be given by

$$\triangle^{SD} := \frac{\triangle^D}{F^2} = (\nabla_{l^{\mathscr{H}}})^2, \tag{32}$$

where $l^{\mathscr{H}} = \frac{y^i}{F}\frac{\delta}{\delta x^i}$ is the horizontal correspondence of the distinguished vector $l^{\mathscr{V}}$.

From Definition 3.2, we know this degenerate elliptic operator can be calculated as the usual Laplacian. In geometric analysis, the Laplacian is always written as $\triangle = g^{ij}\nabla_i\nabla_j$, where ∇_i means the covariant derivative. Here we can express the degenerate elliptic operator as $\triangle^D = y^i y^j \nabla_i \nabla_j$ or $\triangle^{SD} = l^i l^j \nabla_i \nabla_j$. The symbol ∇_i means the horizontal covariant derivative with respect to the Chern connection. Based on these facts, we can prove the following Bochner type formula for Killing vector fields on Finsler manifolds.

PROPOSITION 3.4 (Bochner type formula). *The Killing vector field $V = V^i(x)\partial_i$ satisfies the following formula*

$$\triangle^D(|V|^2) = 2|\nabla\tilde{V}|^2 - 2R(V, V), \tag{33}$$

where $\triangle^D$ is given by (31), *$\tilde{V} = V_i y^i$, and R is the flag curvature tensor.*

Proof. Contracting (30) by y^i, it follows that

$$y^i(V_{i|j} + V_{j|i}) = 0. \tag{34}$$

Plugging it back into (30) yields

$$V_{i|j} + V_{j|i} + 2C^p_{ij}(V_q y^q)_{|p} = 0, \tag{35}$$

where we have used the fact that $y^i_{|p} = \nabla_p y^i = 0$. Taking the second order covariant derivative, one can get

$$(g^{ij}V_iV_j)_{|k|l} = 2g^{ij}V_{i|k}V_{j|l} + 2g^{ij}V_iV_{j|k|l}, \tag{36}$$

where we have used the fact that $g_{ij|p} = \nabla_p g_{ij} = 0$.

Since the component of the Killing vector field is only dependent on x, by Lemma 3.1, the vector satisfies

$$V_{j|k|l} - V_{j|l|k} = R^m_{jkl}V_m + 2R^m_{kl}V_pC^p_{jm}. \tag{37}$$

Contracting (36) by y^k, y^l, we can acquire that

$$\begin{aligned}
\triangle^D(g^{ij}V_iV_j) &= 2g^{ij}y^k y^l V_{i|k}V_{j|l} + 2g^{ij}y^k y^l V_i V_{j|k|l} \\
&= 2g^{ij}(V_k y^k)_{|i}(V_l y^l)_{|j} - 2V^j y^l y^k V_{k|j|l} \\
&= 2g^{ij}(V_k y^k)_{|i}(V_l y^l)_{|j} - 2V^j y^l y^k (V_{k|l|j} + R^m_{kjl}V_m) \\
&= 2|\nabla(\tilde{V})|^2 - 2R(V, V),
\end{aligned} \tag{38}$$

where $\tilde{V} = V_i y^i$ and R is the flag curvature tensor. Q.E.D.

The following proposition indicates that the parallel of V is equal to the parallel of $\tilde{V}$ with respect to the Chern connection, when V is a Killing vector field.

PROPOSITION 3.5. *Suppose V is a Killing vector field. Then V is parallel if and only if $\tilde{V}$ is a function on TM whose horizontal derivatives vanish.*

Proof. $\tilde{V}$ is a function with vanishing horizontal derivatives if and only if $\frac{\delta}{\delta x^i}\tilde{V} = 0$. That is,

$$0 = \frac{\delta}{\delta x^i}\tilde{V} = \frac{\delta}{\delta x^i}V_j y^j + V_j \frac{\delta}{\delta x^i}y^j = \frac{\delta}{\delta x^i}V_j y^j - V_j N_i^j = V_{j|i}y^j = \nabla_i \tilde{V}. \tag{39}$$

So if V is parallel, it is obvious that $\tilde{V}$ is a function on TM with vanishing horizontal derivatives. On the other hand, if $\frac{\delta}{\delta x^i}\tilde{V} = \nabla_i \tilde{V} = 0$, then considering the definition of Killing fields, we assert that

$$V_{j|i}y^i = V_{i|j}y^i = 0. \tag{40}$$

Thus it follows from (9) and (10) that

$$\begin{aligned} 0 = \frac{\partial}{\partial y^k}(V^i_{|j}y_i) &= V^i_{|j}\frac{\partial}{\partial y^k}y_i + y_i \frac{\partial \Gamma^i_{jl}}{\partial y^k}V^l \\ &= V_{k|j} - y^i P_{jilk}V^l \\ &= V_{k|j} - y^i(-P_{ijlk} + 2C_{ijs}L^s_{lk} - 2C_{ijk|l})V^l \\ &= V_{k|j} - L_{jlk}V^l. \end{aligned} \tag{41}$$

Taking the derivative of (40) again yields

$$\begin{aligned} 0 = \frac{\partial}{\partial y^k}(V_{j|i}y^i) &= \frac{\partial}{\partial y^k}(V^l_{|i}g_{lj}y^i) \\ &= 2C_{ljk}y^i V^l_{|i} + V_{j|k} - g_{lj}y^i P^l_{imk}V^m \\ &= V_{j|k} + 2C^l_{jk}V_{l|i}y^i + L_{jlk}V^l. \end{aligned} \tag{42}$$

Plugging (41) into (42) and noticing that $y^i V_{l|i} = 0$, we have

$$L_{jlk}V^l = 0,$$

for any Killing vector field V. Therefore,

$$V_{j|k} = 0,$$

which means V is parallel. Q.E.D.

4. Vanishing theorem of Killing vector fields

Before taking a closer look at the Bochner formula, we will describe the local structure of the operator in the left hand of (33).

For a function h on the tangent bundle TM, we have

$$h_{|k} = \frac{\delta h}{\delta x^k} = \frac{\partial h}{\partial x^k} - N^l_k \frac{\partial h}{\partial y^l}, \tag{43}$$

and

$$h_{|k|l} = \frac{\delta h_{|k}}{\delta x^l} - \Gamma^m_{kl} h_{|m} = \frac{\delta}{\delta x^l}\left(\frac{\delta h}{\delta x^k}\right) - \Gamma^m_{kl}\left(\frac{\partial h}{\partial x^m} - N^i_m \frac{\partial h}{\partial y^i}\right) \tag{44}$$

$$= \left(\frac{\partial}{\partial x^l} - N^i_l \frac{\partial}{\partial y^i}\right)\left(\frac{\partial h}{\partial x^k} - N^m_k \frac{\partial h}{\partial y^m}\right) - \Gamma^m_{kl}\left(\frac{\partial h}{\partial x^m} - N^i_m \frac{\partial h}{\partial y^i}\right)$$

$$= \frac{\partial^2 h}{\partial x^k \partial x^l} - \frac{\partial N^m_k}{\partial x^l}\frac{\partial h}{\partial y^m} - N^m_k \frac{\partial^2 h}{\partial x^l \partial y^m} - N^i_l \frac{\partial^2 h}{\partial x^k \partial y^i}$$

$$+ N^i_l N^m_k \frac{\partial^2 h}{\partial y^i \partial y^m} - \Gamma^m_{kl}\left(\frac{\partial h}{\partial x^m} - N^i_m \frac{\partial h}{\partial y^i}\right).$$

Hence by (4),

$$\triangle^D h = y^k y^l \frac{\partial^2 h}{\partial x^k \partial x^l} - 2y^l \frac{\partial G^m}{\partial x^l}\frac{\partial h}{\partial y^m} - 2G^m \frac{\partial^2 h}{\partial x^l \partial y^m} y^l - 2G^i \frac{\partial^2 h}{\partial x^k \partial y^i} y^k \tag{45}$$

$$+ 4G^i G^m \frac{\partial^2 h}{\partial y^i \partial y^m} - 2G^m\left(\frac{\partial h}{\partial x^m} - N^i_m \frac{\partial h}{\partial y^i}\right)$$

$$= SOP(h) + FOP(h),$$

where $SOP(h)$ denotes the second order derivative part of h and $FOP(h)$ denotes the first order derivative part of h. Indeed, It follows from (2) that,

$$SOP(h) = y^k y^l \frac{\partial^2 h}{\partial x^k \partial x^l} - 2G^m \frac{\partial^2 h}{\partial x^l \partial y^m} y^l - 2G^i \frac{\partial^2 h}{\partial x^k \partial y^i} y^k + 4G^i G^m \frac{\partial^2 h}{\partial y^i \partial y^m} \tag{46}$$

$$= \begin{pmatrix} y^k \\ -2G^k \end{pmatrix}^T \begin{pmatrix} \dfrac{\partial^2 h}{\partial x^k \partial x^l} & \dfrac{\partial^2 h}{\partial x^k \partial y^l} \\ \dfrac{\partial^2 h}{\partial y^k \partial x^l} & \dfrac{\partial^2 h}{\partial y^k \partial y^l} \end{pmatrix} \begin{pmatrix} y^l \\ -2G^l \end{pmatrix}$$

$$= Hessian(h)(G, G),$$

where $Hessian(h)$ is the locally Euclidean Hessian of x, y.

At the maximum point of h, $Hessian(h)$ is semi-negative definite, hence

$$\triangle^D h \le 0. \tag{47}$$

Now we can get the following corollary.

COROLLARY 4.1. *Suppose (M, F) is a compact Finsler manifold with negative flag curvature, then there is no nontrivial Killing field.*

Proof. From Proposition 3.4 and (47), one can obtain that

$$0 \ge \triangle^D (g^{ij} V_i V_j) = 2|\nabla(\tilde{V})|^2 - 2R(V, V), \tag{48}$$

holds at the maximum point of $|V|^2$. Since the flag curvature is negative, there must be

$$V_{i|j}y^i = 0, \quad \text{and} \quad R(V,V) = 0, \tag{49}$$

which means $V = 0$ at the maximum point of $|V|^2$. It is equal to $|V|^2 = 0$, hence $V = 0$. Q.E.D.

Furthermore, by the weak maximum principle of degenerate elliptic operator in [4], we can prove the following theorem, which contains the above corollary.

THEOREM 4.2. *Suppose (M,F) is a compact Finsler manifold with non-positive flag curvature R. Then every Killing field V is parallel with respect to the Chern connection, and $R(V,V) = 0$. Furthermore, if the flag curvature is negative, then there is no nontrivial Killing field.*

Proof. We will only prove the first part with non-positive flag curvature. It follows from Proposition 3.4 and (32) that for a Killing vector field V,

$$\triangle^{SD}(|V|^2) = 2\left|\nabla\left(\frac{\tilde{V}}{F}\right)\right|^2 - 2\frac{R(V,V)}{F^2} \geq 0, \tag{50}$$

on the whole sphere bundle. By (46), we know that the degenerate elliptic operator has at least a non-degenerate direction G. Since $|V|^2 \geq 0$, by Theorem 2.1 of [4], on any domain Ω with boundary $\partial\Omega$, $\sup_\Omega |V|^2 \leq \sup_{\partial\Omega} |V|^2$. However, the sphere bundle SM is compact since M is compact. Then $|V|^2$ is a constant. Hence

$$0 = 2\left|\nabla\left(\frac{\tilde{V}}{F}\right)\right|^2 - 2\frac{R(V,V)}{F^2} \geq 0, \tag{51}$$

which means $\nabla\left(\frac{\tilde{V}}{F}\right) = 0$ and $R(V,V) = 0$. It asserts from Proposition 3.5 that the equation $\nabla\left(\frac{\tilde{V}}{F}\right) = 0$ is equal to $V_{i|j} = 0$, that is, V is parallel with respect to the Chern connection. Q.E.D.

Now let's look into the degenerate elliptic operator $\triangle^{SD}$. We have the following proposition. Firstly, we denote the *integral inner product about the Holmes-Thompson volume form* on the sphere bundle SM by $(\cdot,\cdot)$.

PROPOSITION 4.3. *Let (M,F) be a compact Finsler manifold. The degenerate elliptic operator $\triangle^{SD}$ is self-adjoint with respect to the integral inner product about the Holmos-Thompson volume form on the sphere bundle. In other words, for any two functions u, w on SM,*

$$(\triangle^{SD}u, w) = (u, \triangle^{SD}w). \tag{52}$$

Proof. For the convenience of readers, we will present the details in local coordinates here. Considering (32), we only need to prove that

$$\nabla_{l^{\mathscr{H}}}\left[(\det g_{ij})\sum_i(-1)^i y^i\, dy^1\wedge\cdots\wedge\widehat{dy^i}\wedge\cdots\wedge dy^n\wedge dx\right]=0. \tag{53}$$

It follows from the definition of Chern connection and the correspondence Θ between horizontal and vertical bundles that

$$\nabla_{\delta/\delta x^i}\frac{\delta}{\delta x^j}=\Gamma^k_{ij}\frac{\delta}{\delta x^k},\quad \nabla_{\delta/\delta x^i}\frac{\partial}{\partial y^j}=\Gamma^k_{ij}\frac{\partial}{\partial y^k}. \tag{54}$$

By the duality of dx^i and $\frac{\delta}{\delta x^i}$, δy^i and $\frac{\partial}{\partial y^i}$, one can obtain that

$$\begin{aligned}\nabla_{\delta/\delta x^i}\, dx^k &= -\Gamma^k_{ij}\, dx^j,\\ \nabla_{\delta/\delta x^i}\, dy^k &= -\Gamma^k_{ij}\, dy^j-\left(\frac{\delta}{\delta x^i}N^k_l-N^k_j\Gamma^j_{il}+\Gamma^k_{ij}N^j_l\right)dx^l.\end{aligned} \tag{55}$$

Then (53) follows from the direct calculation that

$$\begin{aligned}
LHS &= g^{rs}l^{\mathscr{H}}(g_{rs})(\det g_{ij})\sum_i(-1)^i y^i\, dy^1\wedge\cdots\wedge\widehat{dy^i}\wedge\cdots\wedge dy^n\wedge dx\\
&\quad+(\det g_{ij})\sum_{i,k}(-1)^i(-N^i_k l^k)\, dy^1\wedge\cdots\wedge\widehat{dy^i}\wedge\cdots\wedge dy^n\wedge dx\\
&\quad+(\det g_{ij})\sum_{i,j,k,l,m}(-1)^i y^i\, dy^1\\
&\quad\wedge\cdots\wedge\left\{l^m\left[-\Gamma^j_{mk}\, dy^k-\left(\frac{\delta N^j_l}{\delta x^m}-N^j_k\Gamma^k_{ml}+\Gamma^j_{mk}N^k_l\right)dx^l\right]\right\}\\
&\quad\wedge\cdots\wedge\widehat{dy^i}\wedge\cdots\wedge dy^n\wedge dx+(\det g_{ij})\sum_{i,j,k,l}(-1)^i y^i\, dy^1\\
&\quad\wedge\cdots\wedge\widehat{dy^i}\wedge\cdots\wedge dy^n\wedge dx^1\wedge\cdots\wedge(-\Gamma^j_{kl}l^k\, dx^l)\wedge\cdots\wedge dx^n\\
&=(\det g_{ij})\sum_{i,j,k,l,r,s}(-1)^i\left[y^i g^{rs}\left(g_{rl}\Gamma^l_{ks}\frac{y^k}{F}+g_{ls}\Gamma^l_{rk}\frac{y^k}{F}\right)-\Gamma^i_{kl}\frac{y^k y^l}{F}\right.\\
&\qquad\left.-y^i\Gamma^j_{kj}\frac{y^k}{F}-y^i\Gamma^j_{kj}\frac{y^k}{F}\right]dy^1\wedge\cdots\wedge\widehat{dy^i}\wedge\cdots\wedge dy^n\wedge dx\\
&\quad-(\det g_{ij})\sum_{i,j,k}(-1)^i\left((-1)^{i-j-1}y^i\Gamma^j_{ki}\frac{y^k}{F}\right)dy^1\wedge\cdots\wedge\widehat{dy^j}\wedge\cdots\wedge dy^n\wedge dx\\
&=0.
\end{aligned}$$

Since SM is compact, we can directly compute that,

$$(\triangle^{SD}u, w) = \frac{1}{c_{n-1}} \int_M dx \int_{S_xM} (\triangle^{SD}u) w (\det g_{ij}) \times \sum_i (-1)^i y^i \, dy^1 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^n$$
$$= -\frac{1}{c_{n-1}} \int_M dx \int_{S_xM} (\nabla_{l\mathscr{H}} u)(\nabla_{l\mathscr{H}} w) \sqrt{\det g_{ij}} \, dv$$
$$= \frac{1}{c_{n-1}} \int_M dx \int_{S_xM} u(\triangle^{SD}w) \sqrt{\det g_{ij}} \, dv$$
$$= (u, \triangle^{SD}w). \tag{56}$$

This equation can also be acquired from the fact that $\nabla F = \nabla y^i = y^k \nabla_k g_{ij} = 0$ for Chern connection ∇. Q.E.D.

Using the mean Ricci curvature, we can further get Theorem 1.2, which includes Theorem 1.1.

Proof of Theorem 1.2. For any Killing vector field V, it follows from Proposition 3.4 that,

$$\triangle^{SD}(|V|^2) = 2\left|\nabla\left(V^i \frac{y^i}{F}\right)\right|^2 - 2\frac{R(V, V)}{F^2} \geq 0, \tag{57}$$

on the sphere bundle. Taking the integral of both sides on SM, one can get

$$\int_{SM} \triangle^{SD}(|V|^2) \sqrt{\det g_{ij}} \, dvdx = 2\int_{SM} \left|\nabla\left(V^i \frac{y^i}{F}\right)\right|^2 \sqrt{\det g_{ij}} \, dvdx - 2\int_{SM} \frac{R(V, V)}{F^2} \sqrt{\det g_{ij}} \, dvdx. \tag{58}$$

By (19) in Definition 2.1, Proposition 4.3 and the condition, we can get

$$0 = \int_{SM} \triangle^{SD}(|V|^2) \sqrt{\det g_{ij}} \, dvdx = 2\int_{SM} \left|\nabla\left(V^i \frac{y^i}{F}\right)\right|^2 \sqrt{\det g_{ij}} \, dvdx - 2\int_M \widetilde{Ric}(V, V) \sqrt{\det a_{ij}} \, dx \geq 0. \tag{59}$$

If the mean Ricci curvature is non-positive, then

$$\nabla \tilde{V} = 0, \quad \text{and} \quad \widetilde{Ric}(V, V) = 0. \tag{60}$$

The first one asserts from Proposition 3.5 that V is parallel with respect to the Chern connection. If the mean Ricci curvature is negative, then

$$\widetilde{Ric}(V, V) = 0, \tag{61}$$

which means $V = 0$. Q.E.D.

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References

[1] M. Abate and G. Patrizio, Finsler metric-A global approach, Springer-Verlag, Berlin Heidelberg, 1994.

[2] R. Lovas, Affine and projective vector fields on spary manifolds, Period. Math. Hung. **48** (2004), 937–949.

[3] V. Matveev, H. Rademacher, M. Troyanov and A. Zeghib, Finsler conformal Lichnerowicz-Obata conjecture, Ann. Inst. Fourier **59** (2009), 937–949.

[4] D. Monticelli, Maximum principles and the method of moving planes for a class of degenerate elliptic linear operators, J. Eur. Math. Soc. **12** (2010), 611–654.

[5] P. Petersen, Riemannian geometry, 2nd ed., Springer-Verlag, New York, 2006.

[6] Y. B. Shen and Z. M. Shen, Introduction to modern Finsler geometry, Higer Education Press, Beijing, 2013.

[7] Z. M. Shen, Lectures on Finsler geometry, Singapore, World Scientific, 2001.

[8] Z. M. Shen and Q. L. Xia, On conformal vector fields on Randers manifolds, Sci. China Math. **55** (2012), 1869–1882.

[9] Y. Tashiro, Complete Riemannian manifolds and some vector fields, Trans. Amer. Math. Soc. **117** (1965), 251–275.

[10] H. J. Tian, Projective vector fields on Finsler manifolds, Appl. Math. J. Chinese Univ. Ser. B. **29** (2014), 217–229.

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