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RELATIVE PURITY IN LOG ÉTALE COHOMOLOGY

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Abstract

We study log étale cohomology. The goal is to prove relative purity in log étale cohomology.

1. Main results

In the present paper, we prove relative purity in log étale cohomology, i.e.,

Theorem 1.1. *Let S be an fs log scheme; N an fs monoid such that the natural morphism $f : X \overset{\text{def}}{=} S[N] \to S$ is a log smooth morphism; $j : U \overset{\text{def}}{=} S[N^{\mathrm{gp}}] \hookrightarrow X$ the open immersion; $n \in \mathbf{Z}_{>0}$ which is invertible on S; F_0 a sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on the Kummer log étale site of S; $F \overset{\text{def}}{=} f^*F_0$. Then*

$$\mathrm{R}^q j_* j^* F = \begin{cases} F & (q = 0) \\ 0 & (q > 0). \end{cases}$$

The Theorem was conjectured by L. Illusie in [3]. It is known that [5], Lemma 7.6.5 proved the case of Theorem where $S = \operatorname{Spec} k$ with log. str. by $\mathbf{N}^{r'}$ ($r' \in \mathbf{N}$) with k being a separably closed field; $X = S[\mathbf{N}^r]$ ($r \in \mathbf{N}$); $F = \mathbf{Z}/n\mathbf{Z}$.

We outline the proof. First, we prove the case $N = \mathbf{N}$. By replacing the category of log étale sheaves with the category of étale sheaves on which the Galois group acts, we calculate the higher direct images using the group cohomology.

Next, by induction, we prove the case of $N = \mathbf{N}^r$ ($r \in \mathbf{N}$).

Finally, by log-blow-up, the general case reduces to the case in which $N = \mathbf{N}^r$.

2. The case $N = \mathbf{N}$

Let $x \in X$ and $x(\log)$ be a log geometric point of x on X. It suffices to prove that $(\mathrm{R}^q j_* j^* F)_{x(\log)} = 0$ ($q > 0$) and $(j_* j^* F)_{x(\log)} = F_{x(\log)}$. Since the

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problem is local, we may assume that S is a quasi-compact and quasi-separated fs log scheme.

First, we prove the case $N = \mathbf{N}$. It suffices to show it at a point $x \in X$ such that $M_{X,\bar{x}}/\mathcal{O}_{X,\bar{x}}^{\times}$ is isomorphic to $P \overset{\text{def}}{=} (M_{S,\bar{s}}/\mathcal{O}_{S,\bar{s}}^{\times}) \oplus \mathbf{N}$, where $s \overset{\text{def}}{=} f(x)$.

Strict étale locally on X, fix a chart $X \to \operatorname{Spec} \mathbf{Z}[P]$ around x. Let $m \in \mathbf{Z}_{>0}$ be invertible at x. For each m, $X_m \overset{\text{def}}{=} X \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Spec} \mathbf{Z}[P^{1/m}]$; $\tilde{X} \overset{\text{def}}{=} \operatorname{Spec} \mathcal{O}_{X,\bar{x}} \to X$; $\tilde{X}_m \overset{\text{def}}{=} \tilde{X} \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Spec} \mathbf{Z}[P^{1/m}]$; $\tilde{U} \overset{\text{def}}{=} U \times_X \tilde{X}$; $\tilde{U}_m \overset{\text{def}}{=} \tilde{U} \times_{\tilde{X}} \tilde{X}_m$. Let $\pi_m : \tilde{U}_m \to U$ be the projection morphism.

Lemma 2.1. *It holds that*

$$(\mathrm{R}^q j_* j^* F)_{x(\log)} = \varinjlim_m \mathrm{H}^q(\tilde{U}_m, \pi_m^* j^* F),$$

where $m \in \mathbf{Z}_{>0}$ runs over the set of integers invertible at x.

Proof. It follows immediately from the definitions that

$$(\mathrm{R}^q j_* j^* F)_{x(\log)} = \varinjlim_V \mathrm{H}^q(U \times_X V, (U \times_X V \to U)^* j^* F),$$

where $V \to X$ is a két (i.e., log étale and of Kummer type) neighborhood at $x(\log)$. Then

$$= \varinjlim_Q \varinjlim_V \mathrm{H}^q(U \times_X V, (U \times_X V \to U)^* j^* F),$$

where Q is a sharp P-fs monoid such that $Q^{\mathrm{gp}} \supset P^{\mathrm{gp}}$ and such that there exists an integer m being invertible at x and satisfying $ma = 0$ for any $a \in Q^{\mathrm{gp}}/P^{\mathrm{gp}}$, and $V \to X \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Spec} \mathbf{Z}[Q]$ is a strict étale neighborhood at the lift of $x(\log)$. Since there exists an $m \in \mathbf{Z}_{>0}$ such that m is invertible at x and the natural homomorphism $P \to P^{1/m}$ factors through Q, we have

$$= \varinjlim_m \varinjlim_V \mathrm{H}^q(U \times_X V, (U \times_X V \to U)^* j^* F),$$

where $V \to X_m$ is a strict étale neighborhood at the lift of $x(\log)$. Thus,

$$= \varinjlim_m \mathrm{H}^q(U \times_X \operatorname{Spec} \mathcal{O}_{X_m,\bar{x}}, (U \times_X \operatorname{Spec} \mathcal{O}_{X_m,\bar{x}} \to U)^* j^* F).$$

Since $U \times_X \operatorname{Spec} \mathcal{O}_{X_m,\bar{x}} \simeq \tilde{U}_m$,

$$= \varinjlim_m \mathrm{H}^q(\tilde{U}_m, \pi_m^* j^* F). \qquad \square$$

Definition 2.2. For a log scheme Y, Y^{cl} denotes the log scheme (the underlying scheme of Y, the trivial log structure). The natural morphism $\varepsilon : Y \to Y^{\mathrm{cl}}$ is called the forgetting-log morphism.

Let $\tilde{U}_\infty^{\mathrm{cl}} \stackrel{\mathrm{def}}{=} \varprojlim_m \tilde{U}_m^{\mathrm{cl}}$ and $\rho_m : \tilde{U}_\infty^{\mathrm{cl}} \to \tilde{U}_m^{\mathrm{cl}}$. For any $m, n \in \mathbf{Z}_{>0}$, which are invertible at x, we consider the diagram

$$\begin{tikzcd}
 & \tilde{U}_{mn} \arrow[r, "\varepsilon_{mn}"'] \arrow[dl, "\pi_{mn}"'] \arrow[d] & \tilde{U}_{mn}^{\mathrm{cl}} \arrow[d] & \tilde{U}_\infty^{\mathrm{cl}} \arrow[l, "\rho_{mn}"'] \arrow[dl, "\rho_m"] \\
U & \tilde{U}_m \arrow[l, "\pi_m"'] \arrow[r, "\varepsilon_m"'] & \tilde{U}_m^{\mathrm{cl}}. &
\end{tikzcd}$$

We consider

$$\varinjlim_m \rho_m^* \mathrm{R}^p \varepsilon_{m*} \pi_m^* j^* F.$$

LEMMA 2.3. *It holds that*

$$\varinjlim_m \rho_m^* \mathrm{R}^p \varepsilon_{m*} \pi_m^* j^* F = 0 \quad (p > 0).$$

Proof. By proper base change theorem (cf. [5], Theorem 5.1), we reduce to the case in which S is the spectrum of a separably closed field k. Let

$$I_m \stackrel{\mathrm{def}}{=} \varprojlim_{v:\,\text{prime to }\mathrm{ch}(k)} \mathrm{Hom}((P^{1/m})^{\mathrm{gp}}, \mathrm{Ker}(v : k^\times \to k^\times)).$$

By [5], Proposition 4.6, let $G_m \in I_m$-$\mathbf{Z}/n\mathbf{Z}$-$\mathrm{Mod}_{/\tilde{U}_m}$ (cf. [5], Notation 4.5) which corresponds to $\pi_m^* j^* F \in \mathscr{S}_{\tilde{U}_m}^{\mathbf{Z}/n\mathbf{Z}}$ (cf. [5], Notation 2.3), then $\mathrm{H}^p(I_m, G_m)$ corresponds to $\rho_m^* \mathrm{R}^p \varepsilon_{m*} \pi_m^* j^* F$. Since $\varinjlim_m \mathrm{H}^p(I_m, G_m) = 0$ $(p > 0)$, it holds that $\varinjlim_m \rho_m^* \mathrm{R}^p \varepsilon_{m*} \pi_m^* j^* F = 0$ $(p > 0)$. □

LEMMA 2.4. *It holds that*

$$\varinjlim_m \mathrm{H}^q(\tilde{U}_m, \pi_m^* j^* F) = \varinjlim_m \mathrm{H}^q(\tilde{U}_m^{\mathrm{cl}}, \varepsilon_{m*} \pi_m^* j^* F).$$

Proof. We consider the spectral sequence

$$E_2^{s,t} = \varinjlim_m \mathrm{H}^s(\tilde{U}_m^{\mathrm{cl}}, \mathrm{R}^t \varepsilon_{m*} \pi_m^* j^* F) \Rightarrow \varinjlim_m \mathrm{H}^{s+t}(\tilde{U}_m, \pi_m^* j^* F).$$

E_2-terms are

$$\varinjlim_m \mathrm{H}^s(\tilde{U}_m^{\mathrm{cl}}, \mathrm{R}^t \varepsilon_{m*} \pi_m^* j^* F) = \mathrm{H}^s\left(\tilde{U}_\infty^{\mathrm{cl}}, \varinjlim_m \rho_m^* \mathrm{R}^t \varepsilon_{m*} \pi_m^* j^* F\right).$$

By Lemma 2.3,

$$\varinjlim_m \rho_m^* \mathrm{R}^t \varepsilon_{m*} \pi_m^* j^* F = 0 \quad (t > 0).$$

Thus, $E_2^{s,t} = 0$ $(t > 0)$ and

$$\varinjlim_m \mathrm{H}^s(\tilde{U}_m^{\mathrm{cl}}, \varepsilon_{m*}\pi_m^* j^* F) = \varinjlim_m \mathrm{H}^s(\tilde{U}_m, \pi_m^* j^* F). \qquad \square$$

LEMMA 2.5. *It holds that*

$$\varinjlim_m \mathrm{H}^q(\tilde{U}_m^{\mathrm{cl}}, \varepsilon_{m*}\pi_m^* j^* F) = 0 \quad (q > 0),$$

$$\varinjlim_m \mathrm{H}^0(\tilde{U}_m^{\mathrm{cl}}, \varepsilon_{m*}\pi_m^* j^* F) = F_{x(\log)}.$$

Proof. We consider the spectral sequence

$$E_2^{p,q} = \varinjlim_m \mathrm{H}^p(\tilde{X}_m^{\mathrm{cl}}, \mathrm{R}^q j_* \varepsilon_{m*}\pi_m^* j^* F) \Rightarrow \varinjlim_m \mathrm{H}^{p+q}(\tilde{U}_m^{\mathrm{cl}}, \varepsilon_{m*}\pi_m^* j^* F).$$

Since $\tilde{X}_\infty^{\mathrm{cl}} \stackrel{\mathrm{def}}{=} \varprojlim_m \tilde{X}_m^{\mathrm{cl}}$ is strictly henselian, it holds that $E_2^{p,q} = 0$ $(p > 0)$. In particular,

$$\varinjlim_m \mathrm{H}^q(\tilde{U}_m^{\mathrm{cl}}, \varepsilon_{m*}\pi_m^* j^* F) = \varinjlim_m \Gamma(\tilde{X}_m^{\mathrm{cl}}, \mathrm{R}^q j_* \varepsilon_{m*}\pi_m^* j^* F).$$

Let $\tilde{Z}_m^{\mathrm{cl}} \stackrel{\mathrm{def}}{=} \tilde{X}_m^{\mathrm{cl}} \backslash \tilde{U}_m^{\mathrm{cl}}$ and give $\tilde{Z}_m^{\mathrm{cl}}$ the reduced induced subscheme structure; $i : \tilde{Z}_m^{\mathrm{cl}} \hookrightarrow \tilde{X}_m^{\mathrm{cl}}$ the closed immersion; $S_m = S \times_{\operatorname{Spec} \mathbf{Z}[M_{S,\bar{s}}/\mathcal{O}_{S,\bar{s}}^\times]} \operatorname{Spec} \mathbf{Z}[(M_{S,\bar{s}}/\mathcal{O}_{S,\bar{s}}^\times)^{1/m}]$. By proper base change theorem (cf. [5], Theorem 5.1),

$$\begin{array}{ccc} \tilde{U}_m & \longrightarrow & \tilde{U}_m^{\mathrm{cl}} \\ \big\downarrow & & \big\downarrow \\ S & \xrightarrow[\text{proper}]{} & S^{\mathrm{cl}}, \end{array}$$

it holds that

$$\varinjlim_m \Gamma(\tilde{X}_m^{\mathrm{cl}}, \mathrm{R}^q j_* j^* (\tilde{X}_m^{\mathrm{cl}} \to S_m^{\mathrm{cl}} \to S^{\mathrm{cl}})^* \varepsilon_{S*} F_0) \xrightarrow{\sim} \varinjlim_m \Gamma(\tilde{X}_m^{\mathrm{cl}}, \mathrm{R}^q j_* \varepsilon_{m*}\pi_m^* j^* F).$$

By relative purity (cf. [2], XVI 3.7),

$$\varinjlim_m \Gamma(\tilde{X}_m^{\mathrm{cl}}, \mathrm{R}^q j_* j^* (\tilde{X}_m^{\mathrm{cl}} \to S^{\mathrm{cl}})^* \varepsilon_{S*} F_0) = 0 \quad (q > 1),$$

$$\varinjlim_m \Gamma(\tilde{X}_m^{\mathrm{cl}}, j_* j^* (\tilde{X}_m^{\mathrm{cl}} \to S^{\mathrm{cl}})^* \varepsilon_{S*} F_0) = F_{x(\log)}.$$

If $q = 1$,

$$\varinjlim_m \Gamma(\tilde{X}_m^{\mathrm{cl}}, \mathrm{R}^1 j_* j^* (\tilde{X}_m^{\mathrm{cl}} \to S^{\mathrm{cl}})^* \varepsilon_{S*} F_0) = \varinjlim_m \mathrm{H}^2(\tilde{Z}_m^{\mathrm{cl}}, (\tilde{Z}_m^{\mathrm{cl}} \to S^{\mathrm{cl}})^* \varepsilon_{S*} F_0).$$

Since the transition map

$$\mathrm{H}^2(\tilde{Z}_m^{\mathrm{cl}}, (\tilde{Z}_m^{\mathrm{cl}} \to S^{\mathrm{cl}})^* \varepsilon_{S*} F_0) \to \mathrm{H}^2(\tilde{Z}_{mn}^{\mathrm{cl}}, (\tilde{Z}_{mn}^{\mathrm{cl}} \to S^{\mathrm{cl}})^* \varepsilon_{S*} F_0)$$

is 0 map, it holds that

$$\varinjlim_m \mathrm{H}^2(\tilde{Z}_m^{\mathrm{cl}}, (\tilde{Z}_m^{\mathrm{cl}} \to S^{\mathrm{cl}})^* \varepsilon_{S*} F_0) = 0. \qquad \square$$

This completes the proof of Theorem 1.1 in the case where $N = \mathbf{N}$.

3. General case

Next, in the case of $N = \mathbf{N}^r$ ($r \in \mathbf{Z}_{\geq 1}$), we consider the decomposition

$$U = S[\mathbf{Z}^r] = S[\mathbf{Z}][\mathbf{Z}^{r-1}] \xrightarrow{j_1} S[\mathbf{Z}][\mathbf{N}^{r-1}] = S[\mathbf{N}^{r-1}][\mathbf{Z}] \xrightarrow{j_2} S[\mathbf{N}^{r-1}][\mathbf{N}] = S[\mathbf{N}^r] = X.$$

By Theorem 1.1 in the case where $N = \mathbf{N}$ and by induction,

$$\mathrm{R}j_* j^* F \simeq \mathrm{R}j_{2*}\mathrm{R}j_{1*} j_1^* j_2^* F \simeq \mathrm{R}j_{2*} j_2^* F \overset{\mathrm{qis}}{\simeq} F.$$

Finally, in the general case, we consider a log-blow-up

$$\pi : X' \to S[N],$$

where X' is covered by $S[\mathbf{N}^s \oplus \mathbf{Z}^t]$ with various s and t. Thus, we obtain a commutative diagram

$$\begin{tikzcd}
S[N^{\mathrm{gp}}] \times_{S[N]} X' \arrow[r, "j'"'] \arrow[d, "\pi'"'] & X' \arrow[d, "\pi"'] \\
S[N^{\mathrm{gp}}] \arrow[r, "j"'] & S[N].
\end{tikzcd}$$

By the previous case,

$$\pi^* F \overset{\mathrm{qis}}{\simeq} \mathrm{R}j'_* j'^* \pi^* F.$$

Then it holds that

$$\mathrm{R}\pi_* \pi^* F \overset{\mathrm{qis}}{\simeq} \mathrm{R}\pi_* \mathrm{R}j'_* j'^* \pi^* F \overset{\mathrm{qis}}{\simeq} \mathrm{R}j_* \mathrm{R}\pi'_* \pi'^* j^* F.$$

By [1], (2.7) (cf. [4], §6),

$$\mathrm{R}\pi'_* \pi'^* = \mathrm{id}, \quad \mathrm{R}\pi_* \pi^* = \mathrm{id}.$$

Therefore,

$$F \overset{\mathrm{qis}}{\simeq} \mathrm{R}j_* j^* F.$$

We complete the proof of Theorem 1.1.

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