

## A NUMBER THEORETICAL OBSERVATION OF A RESONANT INTERACTION OF ROSSBY WAVES

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### Abstract

Rossby waves are generally expected to dominate the  $\beta$  plane dynamics in geophysics, and here in this paper we give a number theoretical observation of the resonant interaction with a Diophantine equation. The set of resonant frequencies does not have any frequency on the horizontal axis.

### 1. Introduction

We consider three-wave interactions of the Rossby waves in a number theoretical approach. Such waves are observed in an incompressible two-dimensional flow on a  $\beta$  plane (in geophysics). The  $\beta$ -plane approximation was first introduced by meteorologists (see [1, 2]) as a tangent plane of a sphere to approximately describe fluid motion on a rotating sphere, assuming that the Colioris parameter is a linear function of the latitude. A formal derivation of the  $\beta$ -plane approximation is given in [4]. It has been known that in the incompressible two-dimensional flow on a  $\beta$  plane, as time goes on, a zonal pattern emerges, consisting of alternating eastward and westward zonal flows, similar to the zonal band structure observed on Jupiter. From a physical point of view, one of the most important properties of the flow on a  $\beta$  plane is linear waves called “Rossby waves”. The Rossby waves originate from the following dispersion relation (see [6] for example),

$$(1.1) \quad \omega = -\frac{\beta k_1}{k_1^2 + k_2^2},$$

where  $\omega$  and  $(k_1, k_2)$  are the angular frequency and the wavenumber vector. The Rossby waves have been considered to play important roles in the dynamics of geophysical fluids (see [5] for example). In [6], they proved a mathematical rigorous theorem which supports the importance of the resonant pairs of Rossby waves. However, none of studies tried to consider such resonant waves in number theoretical approach, and in this paper we attempt to consider it in an

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elementary number theory. Let us be more precise. We define the wavenumber set consisting of wavenumbers in non-trivial resonance as follows:

DEFINITION 1.1 (Wavenumber set of non-trivial resonance). Let  $\Lambda$  be a wavenumber set such that

$$\Lambda := \left\{ n \in \mathbf{Z}^2 \text{ with } n_1 \neq 0 : \frac{n_1}{n_1^2 + n_2^2} - \frac{x}{x^2 + y^2} - \frac{n_1 - x}{(n_1 - x)^2 + (n_2 - y)^2} = 0, \right. \\ \left. \text{for some } (x, y) \in \mathbf{Z}^2 \text{ with } x \neq 0 \text{ and } n_1 - x \neq 0 \right\}.$$

The role of the above non-trivial resonance  $\Lambda$  can be found in [6] in PDE sense. Thus we omit to explain how it works to the two-dimensional flow on a  $\beta$  plane (in PDE sense). We would like to figure out the exact elements of  $\Lambda$  without any numerical computation. The following remark ensures that  $\Lambda$  has at least infinite elements.

*Remark 1.2 (Infinite elements).* At least,  $n = (n_1, n_2) = (m^4, m\ell^3)$  ( $m, \ell \in \mathbf{N}$ ,  $m \neq \ell$ ) is in  $\Lambda$ . In this case, we just take  $(x, y) = (\ell^4, -m^3\ell)$ . Thus  $\Lambda$  has at least infinite elements.

$\Lambda$  itself is not only mathematically but also physically interesting. In a turbulent flow, every wavenumber component should have nonzero energy. Suppose that the initial energy distribution in a wavenumber space is isotropic. Two-dimensional turbulence is known to transfer the energy from small to largescale motions (energy inverse cascade). If there is no effect of rotation (no Coriolis effect), then the energy therefore becomes concentrated isotropically around the origin in wavenumber space. However, if the rotation effect (Coriolis effect) is dominant, the energy transfer becomes governed by the resonant interaction of Rossby waves  $\Lambda$ , and the number of resonant triads gives a rough estimate of the strength of the nonlinear energy transfer. Therefore, roughly speaking (in a physical point of view), the wavenumbers not in  $\Lambda$  are then expected to gain less energy compared with wavenumbers in  $\Lambda$ . In a numerical computation (see [6]), we can expect that  $\Lambda$  has anisotropic distribution. Thus our aim is to know  $\Lambda$  rigorously, and prove (in a number theoretical approach) that its distribution is anisotropic (however, it seems so difficult that we need to progress little by little). For the first step, in this paper, we give nonexistence of three wave interaction on  $n_1$ -axis by using a Diophantine equation. The main theorem is as follows:

THEOREM 1 (Nonexistence of the three wave interaction on  $n_1$ -axis). *If  $n_1, x, y \in \mathbf{Z}$  and*

$$(1.2) \quad \frac{1}{n_1} = \frac{x}{x^2 + y^2} + \frac{n_1 - x}{(n_1 - x)^2 + y^2},$$

*then  $n_1x(n_1 - x) = 0$ .*

*Remark 1.3.* In order to consider more general setting, namely, to figure out whether  $(n_1, n_2)$  ( $n_1, n_2 \in \mathbf{Z}$ ,  $n_1 \neq 0$ ) belongs to  $\Lambda$  or not, we need to consider the following equality (just derived from Definition 1.1):

$$y^4 - 2n_2y^3 - 2x(n_1 - x)y^2 + 2n_2x\left(n_1 - x + \frac{n_2^2}{n_1}\right)y - x(n_1 - x)(x^2 - n_1x + n_1^2 + 2n_2^2) - \frac{n_2^4x}{n_1} = 0$$

for  $x, y \in \mathbf{Z}$  with  $x \neq 0$ .

This equality might be related to “elliptic curve” more or less. In this point of view, the ideas of Mordell’s theorem and “infinite descent” might be useful.

## 2. Proof of the Theorem 1

Assume there is  $n_1$  and  $x$  such that  $n_1x(n_1 - x) \neq 0$ , and we only consider the case  $n_1 > x > 0$ . The other cases are the same. From (1.2), we see

$$\begin{aligned} (x^2 + y^2)\{(n_1 - x)^2 + y^2\} &= n_1x\{(n_1 - x)^2 + y^2\} + n_1(n_1 - x)(x^2 + y^2) \\ \Leftrightarrow y^4 + \{x^2 + (n_1 - x)^2 - n_1x - n_1(n_1 - x)\}y^2 \\ &\quad + \{x^2(n_1 - x)^2 - n_1x(n_1 - x)^2 - n_1x^2(n_1 - x)\} = 0 \\ \Leftrightarrow y^4 - 2x(n_1 - x)y^2 + x^2(n_1 - x)^2 - n_1^2x(n_1 - x) &= 0 \\ \Leftrightarrow y^2 = x(n_1 - x) \pm n_1\sqrt{x(n_1 - x)}. \end{aligned}$$

Clearly, we do not treat complex numbers in this consideration, thus  $n_1 - x > 0$ . By  $0 < x(n_1 - x) < n_1^2$  (we have already assumed that  $n_1 > x > 0$ , thus  $n_1 - x \leq n_1$ ), we have  $0 < x(n_1 - x) < n_1\sqrt{x(n_1 - x)}$ . Thus

$$y^2 = x(n_1 - x) + n_1\sqrt{x(n_1 - x)}.$$

Otherwise,  $y$  becomes a complex number. In particular,  $x(n_1 - x) =: p^2$  ( $p \in \mathbf{N}$ ) and  $p^2 + n_1p$  are square numbers (if  $x(n_1 - x)$  is not square number, then  $y^2$  is not in  $\mathbf{Z}$  and it is in contradiction).

Here, we can assume  $x$  and  $n_1$  are relatively prime (namely,  $x$  and  $n_1 - x$  are relatively prime). In fact, if the greatest common divisor is  $d > 1$ , we set  $x' = x/d \in \mathbf{N}$  and  $n'_1 = n_1/d \in \mathbf{N}$  and then

$$(y/d)^2 = x'(n'_1 - x') + n'\sqrt{x'(n'_1 - x')}.$$

Since the left hand side of the above equality is a rational number, then  $x'(n'_1 - x')$  is a square number. This gives us that the right hand side is a natural number. Thus  $y' := y/d \in \mathbf{N}$ . Therefore we can regard  $n'_1$ ,  $x'$  and  $y'$  the same as  $n_1$ ,  $x$  and  $y$ .

Since  $x$  and  $(n_1 - x)$  are relatively prime and  $x(n_1 - x)$  is a square number,  $x$  and  $n_1 - x$  are also square numbers. In fact, if either  $x$  or  $n_1 - x$  is not square number, then (at least) two  $p_j$  in the following expression

$$x(n_1 - x) = p^2 = p_1^2 p_2^2 \cdots p_N^2$$

( $p_1, \dots, p_N$  are prime numbers, and some  $p_i$  and  $p_j$  ( $i \neq j$ ) may be the same) must belong to both  $x$  and  $(n_1 - x)$ . In this case,  $x$  and  $n_1 - x$  are not relatively prime. Therefore

$$(2.1) \quad x = q^2, \quad (n_1 - x) = r^2, \quad n_1 = q^2 + r^2, \quad p = qr,$$

where  $q, r \in \mathbf{N}$  are relatively prime.

We see that  $q, r$  and  $q^2 + qr + r^2$  are all relatively prime. For example, if  $q$  and  $q^2 + qr + r^2$  are not relatively prime, there is a prime number  $p_1$  such that  $q = s_1 p_1$  and  $q^2 + qr + r^2 = s_2 p_1$  ( $s_1, s_2 \in \mathbf{N}$ ). Since  $q^2 + qr$  is multiple of  $q$  (namely, multiple of  $p_1$ ) then  $r^2$  is also multiple of  $p_1$ . However, if  $r^2$  is multiple of  $p_1$ , then  $r$  itself must be multiple of  $p_1$ . This means that  $q$  and  $r$  are not relatively prime. It is in contradiction to (2.1).

Recall that  $p^2 + n_1 p = q^2 r^2 + (q^2 + r^2)qr = qr(q^2 + qr + r^2)$  is a square number. Since  $q, r$  and  $q^2 + qr + r^2$  are all relatively prime, each  $q, r$  and  $q^2 + qr + r^2$  are square numbers, we can rewrite

$$q = s^2, \quad r = t^2, \quad s^4 + s^2 t^2 + t^4 = u^2, \quad s, t, u \in \mathbf{N}.$$

However it is in contradiction to the following lemma. This concludes the proof of Theorem 1.

LEMMA 2.1 (see [3] for example). *The following Diophantine equation*

$$X^4 + X^2 Y^2 + Y^4 = Z^2, \quad X, Y, Z \in \mathbf{Z}$$

*only has trivial integer solutions:  $X = 0$  or  $Y = 0$ .*

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#### REFERENCES

- [1] J. CHARNEY, The dynamics of long waves in a baroclinic westerly current, *J. Meteorol.* **4** (1947), 135–162.
- [2] J. G. CHARNEY, On the scale of atmospheric motions, *Geof. Publ.* **17** (1948), 17–30.
- [3] L. E. DICKSON, *History of the theory of numbers, Diophantine analysis 2*, Dover Publications, Inc. Mineola, New York, 2005.

- [ 4 ] J. PEDLOSKY, *Grophysical fluid dynamics*, 2nd ed., Springer-Verlag, New York, 1987.
- [ 5 ] L. M. SMITH AND F. WALEFFE, Transfer of energy to two-dimensional large-scales in forced, rotating three-dimensional turbulence, *Phys. Fluids*, **11** (1999), 1608–1622.
- [ 6 ] M. YAMADA AND T. YONEDA, Resonant interaction of Rossby waves in two-dimensional flow on a  $\beta$  plane, *Physica D* **245** (2013), 1–7.

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