ON *p*-BIHARMONIC SUBMANIFOLDS IN NONPOSITIVELY CURVED MANIFOLDS

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Abstract

Let $u : (M, g) \to (N, h)$ be a map between Riemannian manifolds (M, g) and (N, h). The p-bienergy of u is $\tau_p(u) = \int_M |\tau(u)|^p \, dv_g$, where $\tau(u)$ is the tension field of u and $p > 1$. Critical points of τ_p are called p-biharmonic maps and isometric p-biharmonic maps are called p-biharmonic submanifolds. When $p = 2$, p-biharmonic submanifolds are biharmonic submanifolds and in recent years many nonexistence results are found for biharmonic submanifolds in nonpositively curved manifolds. In this paper we will study the nonexistence result for general p-biharmonic submanifolds.

1. Introduction

In the past several decades harmonic map plays a central role in geometry and analysis. Let $u : (M^m, g) \to (N^n, h)$ be a map between Riemannian manifolds (M, g) and (N, h). The energy of u is defined by

$$E(u) = \int_M \frac{|du|^2}{2} \, dv_g,$$

where dv_g is the volume element on (M, g).

The Euler-Lagrange equation of E is

$$\tau(u) = \sum_{i=1}^{m} \{\tilde{\nabla}_{e_i} \, du(e_i) - du(\nabla_{e_i} e_i)\} = 0,$$

where $\tilde{\nabla}$ is the Levi-Civita connection on the pullback bundle $u^{-1}TN$ and $\{e_i\}$ is a local orthonormal frame field on M. In 1983, Eells and Lemaire [11] proposed to consider the bienergy functional

$$E_2(u) = \int_M \frac{|\tau(u)|^2}{2} \, dv_g,$$

where $\tau(u)$ is the tension field of u. Recall that u is harmonic if $\tau(u) = 0$. The Euler-Lagrange equation for E_2 is

$$\tau_2(u) := \tilde{\triangle}\tau(u) - \sum_{i=1}^{m} R^N(\tau(u), du(e_i)) \, du(e_i) = 0.$$

To further generalize the notion of harmonic maps, Hornung and Moser (see also [14]) considered the p-bienergy $(p > 1)$ functional as follows:

$$E_p(u) = \int_M |\tau(u)|^p \, dv_g.$$

The p-bitension field $\tau_p(u)$ is

$$(1.1) \qquad \tau_p(u) := \tilde{\triangle}(|\tau(u)|^{p-2}\tau(u)) - \sum_{i=1}^{m} (R^N(|\tau(u)|^{p-2}\tau(u), du(e_i)) \, du(e_i)).$$

The Euler-Lagrange equation for E_p is $\tau_p(u) = 0$ and a map u satisfying $\tau_p(u) = 0$ is called p-biharmonic maps. If furthermore $u : (M^m, g) \to (N^n, h)$ is an isometry immersion, then we call u p-biharmonic submanifold and 2-biharmonic submanifolds are called biharmonic submanifolds.

For biharmonic submanifolds, we have the well known Chen's conjecture [6]:

Chen's conjecture: Every biharmonic submanifold in $\mathbf{E}^n$ is minimal.

Chen's conjecture inspires the research on the nonexistence of biharmonic submanifolds in nonpositively curved manifolds ([1]–[10], [12] [17] [18], [20]–[28] etc.). Motivated by Chen's conjecture, Han [13] proposed the following conjecture:

Conjecture: Every complete p-biharmonic submanifolds in non-positively curved Riemannian manifold is minimal.

In [13], using the method developed in [21], Han proved several results on the nonexistence of p-biharmonic submanifolds.

Remark 1.1. In Han's paper, he defined p-biharmonic submanifolds to be isometric p-biharmonic maps where he required $p \geq 2$, but that is not necessary. His nonexistence results also hold for p-biharmonic submanifolds $(p > 1)$, without any change in the proof.

In this paper, using the method in [21] again, we get:

THEOREM 1.2. *Suppose (M, g) is a complete p-biharmonic submanifold in (N, h) with non-positive sectional curcvature (Ricci curvature if M is a hypersurface). If M is of at most polynomial volume growth, it is minimal.*

Here we say (M, g) is of at most polynomial volume growth, if for a fixed point $x_0 \in M$ and geodesic ball $B_r(x_0)$, $Vol(B_r(x_0)) \leq C(1 + r)^s$ for some positive integer s and C independent of r.

For p-biharmonic hypersurfaces in a Euclidean space, we have

THEOREM 1.3. *Every weakly convex p-biharmonic hypersurface in a Euclidean space is minimal.*

We say a hypersurface M in N is weakly convex, if its principle curvatures are nonnegative. When $p = 2$, this result was proved in [20]. Furthermore we have

THEOREM 1.4. *Every weakly convex p-biharmonic hypersurface in a nonpositive curvature space form is minimal.*

Remark 1.5. After our paper is completed, the second named author was informed by Yingbo Han that theorem 1.4 was also proved in [15].

We also have

THEOREM 1.6. *Every p-biharmonic submanifold properly immersed in a Euclidean space is minimal.*

We say a map is proper if the preimage of any compact subset is compact. When $p = 2$, this theorem was proved in [1].

The rest of this paper is organized as follows: In section 2 we will briefly recall the theory of p-biharmonic submanifolds and submanifold theory. Our main theorems are proved in section 3.

2.   Preliminaries

In this section we give more details on the definitions of harmonic maps, biharmonic maps, p-biharmonic maps and p-biharmonic submanifolds.

Let $u : (M^m, g) \rightarrow (N^n, h)$ be a map from an m-dimensional Riemannian manifold (M, g) to an n-dimensional Riemannian manifold (N, h). The energy of u is defined by

$$E(u) = \int_M \frac{|du|^2}{2} \, dv_g.$$

The Euler-Lagrange equation of E is

$$\tau(u) := \sum_{i=1}^{m} \{\tilde{\nabla}_{e_i} \, du(e_i) - du(\nabla_{e_i} e_i)\} = 0,$$

where we denote ∇ the Levi-Civita connection on (M, g), and $\tilde{\nabla}$ the induced Levi-Civita connection of the pullback bundle $u^{-1}TN$. A map $u : (M^m, g) \rightarrow (N^n, h)$

is called a harmonic map if $\tau(u) = 0$. To generalize the notion of harmonic maps, Eells and Lemaire [11] proposed to consider the bienergy functional

$$E_2(u) = \int_M \frac{|\tau(u)|^2}{2}\, dv_g.$$

The Euler-Lagrange equation for E_2 is (see [19])

$$\tau_2(u) := \tilde{\triangle}\tau(u) - \sum_{i=1}^m R^N(\tau(u), du(e_i))\, du(e_i) = 0.$$

To further generalize the notion of harmoic maps, Han and Feng [14] (see also [16]) introduced the F-bienergy functional

$$E_F(u) = \int_M F\left(\frac{|\tau(u)|^2}{2}\right) dv_g,$$

where $F : [0, +\infty)$ and $F'(x) > 0$ if $x > 0$.

The critical points of the F-bienergy functional with $F(x) = (2x)^{p/2}(p > 1)$ are called p-biharmonic maps and isometric p-biharmonic maps are called p-biharmonic submanifolds.

The p-bitension field $\tau_p(u)$ is

(2.1) $$\tau_p(u) := \tilde{\triangle}(|\tau(u)|^{p-2}\tau(u)) - \sum_{i=1}^m (R^N(|\tau(u)|^{p-2}\tau(u), du(e_i))\, du(e_i)).$$

A p-biharmonic map satisfies $\tau_p(u) = 0$.

Now we briefly recall the submanifold theory. Let $u : (M, g) \to (N, h)$ be an isometric immersion from an m-dimensional Riemannian manifold into an $m + t$ $(t \geq 1)$-dimensional Riemannian manifold. The second fundamental form $B : TM \times TM \to T^\perp(M)$ is defined by:

(2.2) $$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y), \quad X, Y \in \Gamma(TM),$$

where $\overline{\nabla}$ is the Levi-Civita connection on N and ∇ is the Levi-Civita connection on M. The Weingarten formula is given by

(2.3) $$\overline{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi, \quad X \in \Gamma(TM),$$

where A_ξ is called the Weingarten map w.r.t. $\xi \in T^\perp M$, and $\nabla^\perp$ denotes the normal connection on the normal bundle of M in N. For any $x \in M$, the mean curvature vector field H of M at x is

$$\vec{H} = \frac{1}{m} \sum_{i=1}^m B(e_i, e_i).$$

If u is an isometric immersion, we see that $\{du(e_i)\}$ is a local orthonormal frame of M. In addition, for any $X, Y \in \Gamma(TM)$,

$$(2.4) \qquad \nabla \, du(X, Y) = \tilde{\nabla}_X(du(Y)) - du(\nabla_X^Y) = B(X, Y),$$

where $\tilde{\nabla}$ is the connection on the pull back bundle $u^{-1}TN$, whose fiber at a point $x \in M$ is $T_{u(x)}N = T^{\perp}M \oplus TM$. Therefore if u is an isometric immersion,

$$\tau(u) = tr \, \nabla \, du = tr \, B = m\vec{H},$$

and a p-biharmonic submanifold satisfies the following equatuion:

$$(2.5) \qquad \tau_p(u) = \tilde{\triangle}(|\vec{H}|^{p-2}\vec{H}) - \sum_{i=1}^{m}(R^N(|\vec{H}|^{p-2}\vec{H}, e_i)e_i) = 0,$$

where $\tilde{\triangle} = \sum_{i=1}^{m}(\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i} - \tilde{\nabla}_{\nabla_{e_i}e_i})$, $\tilde{\nabla}$ is the connection on the pullback bundle, and R^N is the Riemanian curvature tensor on N.

From (2.3), we get for any vector field $\xi \in \Gamma(T^{\perp}M)$:

$$\begin{aligned}
\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i}\xi &= \tilde{\nabla}_{e_i}(\nabla_{e_i}^{\perp}\xi - A_{\xi}e_i) \\
&= \nabla_{e_i}^{\perp}\nabla_{e_i}^{\perp}\xi - \tilde{\nabla}_{e_i}A_{\xi}e_i - A_{\nabla_{e_i}^{\perp}\xi}e_i \\
&= \nabla_{e_i}^{\perp}\nabla_{e_i}^{\perp}\xi - \nabla_{e_i}A_{\xi}e_i - B(e_i, A_{\xi}e_i) - A_{\nabla_{e_i}^{\perp}\xi}(e_i),
\end{aligned}$$

and

$$\tilde{\nabla}_{\nabla_{e_i}e_i}\xi = \nabla_{\nabla_{e_i}e_i}^{\perp}\xi - A_{\xi}(\nabla_{e_i}e_i).$$

Combining the above two identities, we get

$$\begin{aligned}
\tilde{\triangle}\xi &= \nabla_{e_i}^{\perp}\nabla_{e_i}^{\perp}\xi - \nabla_{e_i}A_{\xi}e_i - B(e_i, A_{\xi}e_i) - A_{\nabla_{e_i}^{\perp}\xi}(e_i) - \nabla_{\nabla_{e_i}e_i}^{\perp}\xi + A_{\xi}(\nabla_{e_i}e_i) \\
&= \triangle^{\perp}\xi - (\nabla_{e_i}A_{\xi})e_i + A_{\xi}(\nabla_{e_i}e_i) - B(e_i, A_{\xi}e_i) - A_{\nabla_{e_i}^{\perp}\xi}(e_i).
\end{aligned}$$

Therefore by decomposing the p-biharmonic submanifold equation into its normal and tangential parts respectively we get ([13]):

$$(2.6) \quad \Delta^{\perp}(|\vec{H}|^{p-2}\vec{H}) - \sum_{i=1}^{m} B(A_{|\vec{H}|^{p-2}\vec{H}}e_i, e_i) + \sum_{i=1}^{m}(R^N(|\vec{H}|^{p-2}\vec{H}, e_i)e_i)^{\perp} = 0,$$

$$(2.7) \quad Tr_g(\nabla A_{|\vec{H}|^{p-2}\vec{H}}) + Tr_g[A_{\nabla^{\perp}|\vec{H}|^{p-2}\vec{H}}(\cdot)] - \sum_{1}^{m}(R^N(|\vec{H}|^{p-2}\vec{H}, e_i)e_i)^{\top} = 0.$$

3. Proof of theorems

3.1. Proof of Theorem 1.2

Proof. From equation (2.6), we see that

$$(3.1) \quad \Delta|\vec{H}|^{2p-2} = 2|\nabla(|\vec{H}|^{p-2}\vec{H})|^2 + 2\langle \triangle(|\vec{H}|^{p-2}\vec{H}), |\vec{H}|^{p-2}\vec{H}\rangle$$

$$= 2|\nabla(|\vec{H}|^{p-2}\vec{H})|^2 + 2\left\langle \sum_{i=1}^{m} B(A_{|\vec{H}|^{p-2}\vec{H}}e_i, e_i), |\vec{H}|^{p-2}\vec{H}\right\rangle$$

$$- 2\langle R^N(e_i, |\vec{H}|^{p-2}\vec{H})e_i, |\vec{H}|^{p-2}\vec{H}\rangle$$

$$\geq 2|\nabla(|\vec{H}|^{p-2}\vec{H})|^2 + 2\left\langle \sum_{i=1}^{m} B(A_{|\vec{H}|^{p-2}\vec{H}}e_i, e_i), |\vec{H}|^{p-2})\vec{H}\right\rangle,$$

where we used (2.6) and the fact that the sectional curvature of N is nonpositive. Therefore by the following inequality ([13])

$$(3.2) \quad \sum_{i=1}^{m} \langle B(A_{|\vec{H}|^{p-2}\vec{H}}e_i, e_i), |\vec{H}|^{p-2}\vec{H}\rangle \geq m|\vec{H}|^{2p},$$

we obtain

$$(3.3) \quad \Delta|\vec{H}|^{2p-2} \geq 2|\vec{H}|^{2p-4}|\nabla\vec{H}|^2 + 2m|\vec{H}|^{2p}.$$

For a fixed point $x_0 \in M$, and every $r > 0$, let us consider the following cut off function $\lambda(x)$ on M:

$$(3.4) \quad \begin{cases} 0 \leq \lambda(x) \leq 1 & x \in M \\ \lambda(x) = 1 & x \in B_r(x_0) \\ \lambda(x) = 0 & x \in M \backslash B_{2r}(x_0) \\ |\nabla\lambda(x)| \leq \dfrac{C}{r} & x \in M \end{cases}$$

where C is independent of r, and $B_r(x_0)$ is a geodesic ball of M.

Let $a \geq 2$, and b be positive constants to be determined later. From (3.3), we have

$$(3.5) \quad -\int_M \nabla(\lambda^b|\vec{H}|^a) \cdot \nabla(|\vec{H}|^{2p-2}) = \int_M \lambda^b|\vec{H}|^a \triangle(|\vec{H}|^{2p-2})$$

$$\geq \int_M 2\lambda^b|\vec{H}|^a(|\vec{H}|^{2p-4}|\nabla\vec{H}|^2 + m|\vec{H}|^{2p})$$

$$\geq \int_M 2\lambda^b|\vec{H}|^{a+2p-4}|\nabla\vec{H}|^2 + 2m\int_M \lambda^b|\vec{H}|^{a+2p},$$

where $\nabla f \cdot \nabla g := g^{ij}\dfrac{\partial f}{\partial x_i}\dfrac{\partial g}{\partial x_j}$ in local coordinates, for $f, g \in C^1(M)$, and for simplicity we omit the symbol $\cdot$ in the subsequent.

On the other hand, we have

$$(3.6) \quad -\int_M \nabla(\lambda^b|\vec{H}|^a)\nabla|\vec{H}|^{2p-2} = -\int_M (b\lambda^{b-1}\nabla\lambda|\vec{H}|^a + \lambda^b a|\vec{H}|^{a-2}\langle\vec{H}, \nabla\vec{H}\rangle)$$
$$\times ((2p-2)|\vec{H}|^{2p-4}\langle\vec{H}, \nabla\vec{H}\rangle)$$
$$= -\int_M (2p-2)b\lambda^{b-1}\nabla\lambda|\vec{H}|^{a+2p-4}\langle\vec{H}, \nabla\vec{H}\rangle$$
$$- (2p-2)a\int_M \lambda^b|\vec{H}|^{a+2p-6}\langle\vec{H}, \nabla\vec{H}\rangle^2$$
$$\leq -\int_M (2p-2)b\lambda^{b-1}\nabla\lambda|\vec{H}|^{a+2p-4}\langle\vec{H}, \nabla\vec{H}\rangle.$$

Let $b = a + 2p$, we have

$$(3.7) \quad 2\int_M \lambda^{a+2p}|\vec{H}|^{a+2p-4}|\nabla\vec{H}|^2 + 2m\int_M \lambda^{a+2p}|\vec{H}|^{a+2p}$$
$$\leq -(2p-2)(a+2p)\int_M \lambda^{a+2p-1}\nabla\lambda|\vec{H}|^{a+2p-4}\langle\vec{H}, \nabla\vec{H}\rangle$$
$$\leq \int_M \lambda^{a+2p}|\vec{H}|^{a+2p-4}|\nabla\vec{H}|^2 + \int_M \lambda^{a+2p}|\vec{H}|^{a+2p} + C(a,p)\int_M |\nabla\lambda|^{a+2p},$$

where in the last inequality we used the following young's inequality twice:

$$(3.8) \qquad\qquad ab \leq \varepsilon^p\frac{a^p}{p} + \frac{1}{\varepsilon^q}\frac{b^q}{q},$$

where $\epsilon > 0$ and p, q are positive constants satisfying $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

From (3.7), we see that if M is of at most polynomial volume growth then

$$(2m-1)\int_{B_r(x_0)} |\vec{H}|^{a+2p}\, dv_g \leq C(a,p)\frac{1}{r^{a+2p}}(1+2r)^s.$$

We finish the proof by letting a be big enough and $r \to +\infty$. □

3.2. Proof of Theorem 1.3 and Theorem 1.4

3.2.1. Proof of Theorem 1.3

Proof. From (2.6) and (2.7) we have:

$$(3.9) \qquad\qquad \Delta^\perp(|\vec{H}|^{p-2}\vec{H}) - \sum_{i=1}^m B(A_{|\vec{H}|^{p-2}\vec{H}}e_i, e_i) = 0,$$

$$(3.10) \qquad\qquad Tr_g(\nabla A_{|\vec{H}|^{p-2}\vec{H}}) + Tr_g[A_{\nabla^\perp|\vec{H}|^{p-2}\vec{H}}(\cdot)] = 0.$$

Note that for any $X, Y \in TM$ and $\xi \in T^\perp M$ we have

$$\langle A_\xi X, Y \rangle = \langle B(X, Y), \xi \rangle.$$

Assume that $\vec{H} = Hv$, where v is the unit normal vector field on M. Note that by the assumption that M is weakly convex, we have $H \geq 0$. Define

$$B = \{p \in M : H(p) > 0\}.$$

We will prove that B is an empty set by a contradiction argument, and so M is minimal and we are done. If B is not empty, we see that B is an open subset of M. We assume that B_1 is a nonempty connect component of B. We will prove that $H = 0$ in B_1, thus a contradiction. We prove it in two steps.

STEP 1. H is a constant in B_1.

Let $q \in B_1$ be a point. Around q we choose a local orthonormal frame $\{e_k, k = 1, \ldots, m\}$ such that $\nabla_{e_i} e_j(q) = 0$ and $\langle h, v \rangle$ is a diagonal matrix at q, where v is the unit normal vector field of M. For any $1 \leq k \leq m$, we have at q

$$(3.11) \qquad \left\langle \sum_{i=1}^m A_{\nabla_{e_i}^\perp |\vec{H}|^{p-2} \vec{H}} e_i, e_k \right\rangle = \sum_{i=1}^m \langle B(e_i, e_k), \nabla_{e_i}^\perp |\vec{H}|^{p-2} \vec{H} \rangle$$

$$= \langle B(e_k, e_k), \nabla_{e_k}^\perp |\vec{H}|^{p-2} \vec{H} \rangle$$

From (3.10) and (3.11), we have at q

$$(3.12) \qquad 0 = \sum_{i=1}^m \langle (\nabla_{e_i} A_{|\vec{H}|^{p-2} \vec{H}}) e_i, e_k \rangle + \left\langle \sum_{i=1}^m A_{\nabla_{e_i}^\perp |\vec{H}|^{p-2} \vec{H}} e_i, e_k \right\rangle$$

$$= \sum_{i=1}^m e_i \langle A_{|\vec{H}|^{p-2} \vec{H}} e_i, e_k \rangle + \left\langle \sum_{i=1}^m A_{\nabla_{e_i}^\perp |\vec{H}|^{p-2} \vec{H}} e_i, e_k \right\rangle$$

$$= \sum_{i=1}^m e_i \langle B(e_i, e_k), |\vec{H}|^{p-2} \vec{H} \rangle + \left\langle \sum_{i=1}^m A_{\nabla_{e_i}^\perp |\vec{H}|^{p-2} \vec{H}} e_i, e_k \right\rangle$$

$$= \sum_{i=1}^m \langle |\vec{H}|^{p-2} \vec{H}, \nabla_{e_i}^\perp B(e_i, e_k) \rangle + 2 \left\langle \sum_{i=1}^m A_{\nabla_{e_i}^\perp |\vec{H}|^{p-2} \vec{H}} e_i, e_k \right\rangle$$

$$= \sum_{i=1}^m \langle |\vec{H}|^{p-2} \vec{H}, \nabla_{e_k}^\perp B(e_i, e_i) \rangle + 2 \left\langle \sum_{i=1}^m A_{\nabla_{e_i}^\perp |\vec{H}|^{p-2} \vec{H}} e_i, e_k \right\rangle$$

$$= m |\vec{H}|^{p-1} \nabla_{e_k} H + 2 \langle B(e_k, e_k), \nabla_{e_k}^\perp |\vec{H}|^{p-2} \vec{H} \rangle$$

$$= m |\vec{H}|^{p-1} \nabla_{e_k} H + 2 \lambda_k \langle v, \nabla_{e_k}^\perp |\vec{H}|^{p-2} \vec{H} \rangle$$

$$= m|\vec{H}|^{p-1}\nabla_{e_k}H + 2\lambda_k\langle v, (p-2)|\vec{H}|^{p-2}\nabla_{e_k}\vec{H} + |\vec{H}|^{p-2}\nabla_{e_k}\vec{H}\rangle$$

$$= m|\vec{H}|^{p-1}\nabla_{e_k}H + 2\lambda_k(p-1)|\vec{H}|^{p-2}\langle v, \nabla_{e_k}\vec{H}\rangle$$

$$= (m|\vec{H}|^{p-1} + 2\lambda_k(p-1)|\vec{H}|^{p-2})\nabla_{e_k}H,$$

where in the fifth equality we used the Codazzi equation.

Because $\lambda_k \geq 0$, $p > 1$, we have $(m|\vec{H}|^{p-1} + 2\lambda_k(p-1)|\vec{H}|^{p-2}) > 0$ and so

$$\nabla_{e_k}H(q) = 0, \tag{3.13}$$

for any $k = 1, \ldots, m$, which implies that

$$\nabla H(q) = 0. \tag{3.14}$$

Because q is an arbitrary point in B_1, we see that in B_1

$$\nabla H = 0. \tag{3.15}$$

Therefore we get that H is a constant in B_1 and $\triangle|\vec{H}|^{2p-2} = 0$.

STEP 2. H is zero in B_1.

Let $q \in B_1$, by step 1, we see that $\triangle|\vec{H}|^{2p-2} = 0$. On the other hand, by equation (3.9), we have

$$\triangle|\vec{H}|^{2p-2} = 2|\nabla(|\vec{H}|^{p-2})\vec{H}|^2 + 2\langle|\vec{H}|^{p-2}\vec{H}, \triangle(|\vec{H}|^{p-2}\vec{H})\rangle \tag{3.16}$$

$$\geq 2\langle B(A_{|\vec{H}|^{p-2}\vec{H}}, e_i), |\vec{H}|^{p-2}\vec{H}\rangle$$

$$= 2\langle A_{|\vec{H}|^{p-2}\vec{H}}e_i, A_{|\vec{H}|^{p-2}\vec{H}}e_i\rangle$$

$$= 2|\vec{H}|^{2p-2}\langle A_v e_i, A_v e_i\rangle$$

$$\geq 2m|\vec{H}|^{2p}.$$

Therefore $H = 0$ in B_1, a contradiction. This completes the proof of Theorem 1.3. □

3.2.2. Proof of Theorem 1.4

Proof. The proof of Theorem 1.4 is similar with the proof of Theorem 1.3. If M is a p-biharmonic submanifold in the space form $N(c)$ ($c \leq 0$), then it satisfies the following two equations:

$$\Delta^{\perp}(|\vec{H}|^{p-2}\vec{H}) - \sum_{i=1}^{m} B(A_{|\vec{H}|^{p-2}\vec{H}}e_i, e_i) + c|\vec{H}|^{p-2}\vec{H} = 0, \tag{3.17}$$

$$Tr_g(\nabla A_{|\vec{H}|^{p-2}\vec{H}}) + Tr_g[A_{\nabla^{\perp}|\vec{H}|^{p-2}\vec{H}}(\cdot)] = 0. \tag{3.18}$$

From equation (3.18) we will get that if H is not identically zero in a nonempty component, then it must be a constant and by equation (3.17) we can deduce that it is zero, a contradiction. □

3.3. Proof of Theorem 1.6

Proof. We will prove the theorem by a contradiction argument. Suppose that $\vec{H}(x_0) \neq 0$ for some $x_0 \in M$.

Set $u(x) = |\vec{H}(x)|^{2p-2}$ for $x \in M$. For each $\rho > 0$ let

$$F(x) = F_\rho(x) = (\rho^2 - |X(x)|^2)^{2p-2}u(x),$$

for $x \in M \cap X^{-1}(\overline{B}_\rho)$, where $X : M \to \mathbf{R}^n$ is the isometric immersion and B_ρ is the standard ball in $\mathbf{R}^n$ with radius ρ.

Assume that $x_0 \in X^{-1}(B_{\rho_0})$. For each $\rho \geq \rho_0$, $F = F_\rho$ is a nonnegative function which is not identically zero on $M \cap X^{-1}(\overline{B}_\rho)$. Assume that $q \in M \cap X^{-1}(B_\rho)$ is the maximum point of F. Then $\nabla F(q) = 0$ and hence we get at q

$$(3.19) \qquad \frac{\nabla u}{u} = \frac{(2p-2)\nabla|X(x)|^2}{\rho^2 - |X(x)|^2}.$$

By $\Delta F(q) \leq 0$ we get at q

$$(3.20) \qquad \frac{\Delta u}{u} \leq \frac{(2p-2)(2p-1)|\nabla|X(q)|^2|^2}{(\rho^2 - |X(q)|^2)^2} + \frac{(2p-2)\Delta|X(q)|^2}{\rho^2 - |X(q)|^2}.$$

By $\Delta X = m\vec{H}$ we see that

$$|\nabla|X(x)|^2|_g^2 \leq 4m|X(x)|^2,$$

and

$$\Delta|X(x)|^2 = 2\sum_{i=1}^m |\nabla_{e_i}X(x)|^2 + 2\langle X(x), \Delta X(x)\rangle \leq 2m + 2m|X(x)|\,|\vec{H}(x)|,$$

where $m = dim\, M$. Recall that $\Delta|\vec{H}|^{2p-2} \geq 2m|\vec{H}|^{2p}$, i.e. $\Delta u \geq u^{2p/(2p-2)}$, thus we obtain

$$u(q)^{1/(p-1)} \leq \frac{2(2p-2)(2p-1)|X(q)|^2}{(\rho^2 - |X(q)|^2)^2} + \frac{(2p-2)(1 + |X(q)|u^{1/(2p-2)})}{\rho^2 - |X(q)|^2}.$$

From the last inequality one gets

$$(3.21) \qquad u(q) \leq C(p)\left[\frac{|X(q)|^{2p-2}}{(\rho^2 - |X(q)|^2)^{2p-2}} + \frac{1}{(\rho^2 - |X(q)|^2)^{p-1}} \right.$$

$$\left. + \sqrt{u(q)}|X(q)|^{p-1}\frac{1}{(\rho^2 - |X(q)|^2)^{p-1}} \right],$$

where $C(p)$ is a constant depends only on p. Therefore

$$F(q) \leq C(p)[|X(q)|^{2p-2} + (\rho^2 - |X(q)|^2)^{p-1} + \sqrt{F(q)}|X(q)|^{p-1}],$$

which implies that

$$F(q) \leq C(p)\rho^{2p-2}.$$

Since q is the maximum of F, for any $x \in M \cap B_\rho$ we have

$$F(x) \leq F(q) \leq C(p)\rho^{2p-2}.$$

Therefore

$$|\vec{H}(x)|^{2p-2} \leq \frac{C(p)\rho^{2p-2}}{(\rho^2 - |X(x)|^2)^{2p-2}},$$

for any $x \in M \cap B_\rho$ and $\rho \geq \rho_0$. Letting $\rho \rightarrow +\infty$ in the last inequality and $x = x_0$ we get $|\vec{H}(x_0)| = 0$, a contradiction. Therefore M is minimal. $\square$

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