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KODAI MATH. J.
39 (2016), 552–566

HYPERSURFACES IN EUCLIDEAN SPACES WITH FINITE TOTAL CURVATURE

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Abstract

We discuss complete noncompact hypersurfaces in the Euclidean space $\mathbf{R}^{n+1}$ with finite total curvature. We obtain vanishing result and finiteness theorem for the space of L^2 harmonic 2-forms. These results are generalized versions of results for L^2 harmonic 1-forms.

1. Introduction

Shen and Zhu [10] showed that a complete stable immersed minimal hypersurface M in the Euclidean space $\mathbf{R}^{n+1}$ with finite total curvature is hyperplane. Cheng, Cheung and Zhou [4] proved that a complete weakly stable immersed minimal hypersurface M in $\mathbf{R}^{n+1}$ with finite total curvature is hyperplane. Fu and Xu [6] discussed a complete submanifold in $\mathbf{R}^{n+p}$ and obtained the dimension of the space of the L^2 harmonic 1-forms on M is finite if M has finite total curvature (i.e., $\|\Phi\|_{L^n} < +\infty$) and finite total mean curvature (i.e., $\|H\|_{L^n} < +\infty$). Carron [2] obtained the dimension of the space of all L^2 harmonic p-forms is finite if M has finite total curvature and finite total mean curvature. Cavalcante, Mirandola and Vitório [3] proved that if a complete noncompact submanifold M^n $(n \geq 3)$ isometric immersed in $\mathbf{R}^{n+p}$ has finite total curvature, then the dimension of the space of the L^2 harmonic 1-forms on M is finite. Furthermore, they also proved that there exists a positive constant $\delta(n)$, depending only on n, such that if $\|\Phi\|_{L^n} < \delta(n)$, then there admits no non-trivial L^2 harmonic 1-form on M. It was showed in [1] that the space of L^2 harmonic p-forms is related with reduced L^2 cohomology $H_2^p(M)$. The author [12] studied the existence of the symplectic structure and L^2 harmonic 2-forms on complete manifolds by use of the Bochner formula.

In this paper, we discuss a complete noncompact hypersurface M^n in the Euclidean space $\mathbf{R}^{n+1}$ with finite total curvature. We obtain vanishing theorem

2010 *Mathematics Subject Classification*. 53C20; 53C40.

Key words and phrases. Hypersurfaces in the Euclidean space, Total curvature, L^2 harmonic 2-forms.

Received November 20, 2014; revised February 16, 2016.

and finiteness theorem for hypersurfaces in the Euclidean space with finite total curvature as follows:

THEOREM 1.1. *Suppose that* M^n $(n \geq 3)$ *is an n-dimensional complete non-compact hypersurface isometrically immersed in* $\mathbf{R}^{n+1}$. *There exists a positive constant* $\delta(n)$ *depending only on n such that if the total curvature* $\|\Phi\|_{L^n(M)}$ *is less than* $\delta(n)$, *then there admits no non-trivial* L^2 *harmonic* 2*-form on* M *and the second space of reduced* L^2 *cohomology of* M *is trivial.*

THEOREM 1.2. *Let* M^n $(n \geq 3)$ *be an n-dimensional complete noncompact hypersurface isometrically immersed in* $\mathbf{R}^{n+1}$. *If the total curvature is finite, then the dimension of the space of all* L^2 *harmonic* 2*-forms and the dimension of the second space of reduced* L^2 *cohomology of* M *are both finite.*

2. Preliminaries

We recall several definitions. Let M^n be an n-dimensional Riemannian manifold. The Hodge operator $* : \bigwedge^p(M) \to \bigwedge^{n-p}(M)$ is defined as follows:

$$*e^{i_1} \wedge \cdots \wedge e^{i_p} = \operatorname{sgn} \sigma(i_1, i_2, \ldots, i_n) e^{i_{p+1}} \wedge \cdots \wedge e^{i_n},$$

where $\sigma(i_1, i_2, \ldots, i_n)$ denotes a permutation of the set $(i_1, i_2, \ldots, i_n)$ and $\operatorname{sgn} \sigma$ is the sign of σ. The operator $d^* : \bigwedge^p(M) \to \bigwedge^{p-1}(M)$ is given by

$$d^*\omega = (-1)^{(np+p+1)} * d * \omega.$$

The Laplacian operator is defined by

$$\triangle\omega = -dd^*\omega - d^* \, d\omega.$$

A p-form ω is called L^2-harmonic if $\triangle\omega = 0$ and

$$\int_M \omega \wedge *\omega < +\infty.$$

We denote $H^p(L^2(M))$ by the space of all L^2 harmonic p-forms on M.

Suppose that $x : M^n \to \mathbf{R}^{n+1}$ is an isometric immersion of an n-dimensional hypersurface M in an $(n+1)$-dimensional Euclidean space. Let A denote the second fundamental form and H the mean curvature of the immersion x. Let

$$\Phi(X, Y) = A(X, Y) - H\langle X, Y\rangle,$$

for all vector fields X and Y, where $\langle \, , \rangle$ is the induced metric of M. We say the immersion x has finite total curvature if

$$\|\Phi\|_{L^n(M)} < +\infty.$$

We state several results which will be used later.

Lemma 2.1 [8]. *If* (M^n, g) *is a Riemannian manifold and* $\omega = a_I \omega_I \in \bigwedge^p(M)$, *then*

$$\triangle|\omega|^2 = 2\langle \triangle\omega, \omega\rangle + 2|\nabla\omega|^2 + 2\langle E(\omega), \omega\rangle,$$

where $E(\omega) = R_{k_\beta i_\beta j_\alpha i_\alpha} a_{i_1 \cdots k_\beta \cdots i_p} e^{i_p} \wedge \cdots \wedge e^{j_\alpha} \wedge \cdots \wedge e^{i_1}$.

Proposition 2.2 [1]. *Let* (M, g) *is a complete Riemannian manifold, then the space of* L^2 *harmonic p-forms* $H^p(L^2(M))$ *is isomorphic to the p-th space of reduced* L^2 *cohomology* $H_2^p(M)$.

Proposition 2.3 [7]. *Let* M^n $(n \geq 3)$ *be a complete noncompact hypersurface isometrically immersed in* $\mathbf{R}^{n+1}$. *Then*

$$\left(\int_M |f|^{2n/(n-2)}\right)^{(n-2)/n} \leq C_0\left(\int_M |\nabla f|^2 + n^2 \int_M H^2 f^2\right)$$

for each $f \in C_0^1(M)$, *where* C_0 *depends only on n and H is the mean curvature of M in* $\mathbf{R}^{n+1}$.

3. An inequality for L^2 harmonic 2-forms

We initially prove an inequality for L^2 harmonic 2-forms on hypersurfaces in $\mathbf{R}^{n+1}$. Suppose $\omega \in H^2(L^2(M))$ and $h = |\omega|$.

Proposition 3.1. *If* M^n $(n \geq 3)$ *is an n-dimensional complete noncompact hypersurface isometrically immersed in* $\mathbf{R}^{n+1}$, *then*

$$h\triangle h \geq \begin{cases} |\nabla h|^2 - |\Phi|^2 h^2 + \dfrac{3}{2} H^2 h^2 & \text{for } n = 3, \\ \dfrac{1}{n-2}|\nabla h|^2 - \dfrac{n-2}{2}|\Phi|^2 h^2 + nH^2 h^2 & \text{for } n \geq 4. \end{cases}$$

Proof. Since $\omega \in H^2(L^2(M))$, we get that

$$\triangle|\omega|^2 = 2|\nabla|\omega||^2 + 2|\omega|\triangle|\omega|. \tag{3.1}$$

Lemma 2.1 implies that

$$\triangle|\omega|^2 = 2|\nabla\omega|^2 + 2\langle E(\omega), \omega\rangle. \tag{3.2}$$

Combining (3.1) with (3.2), we get that

$$|\omega|\triangle|\omega| = |\nabla\omega|^2 - |\nabla|\omega||^2 + \langle E(\omega), \omega\rangle. \tag{3.3}$$

Note that there is the Kato inequality for L^2 harmonic 2-forms [5, 11]:

$$|\nabla\omega|^2 \geq \frac{n-1}{n-2}|\nabla|\omega|\,|^2. \tag{3.4}$$

By (3.3) and (3.4), we get that

$$|\omega|\triangle|\omega| \geq \frac{1}{n-2}|\nabla|\omega|\,|^2 + \langle E(\omega), \omega\rangle. \tag{3.5}$$

Now, we give the estimate of the term $\langle E(\omega), \omega\rangle$. Let $\omega = a_{i_1 i_2} e^{i_2} \wedge e^{i_1} \in \bigwedge^2(M)$, where $a_{i_1 i_2} = -a_{i_2 i_1}$. By Lemma 2.1, we obtain that

$$\begin{aligned} E(\omega) &= R_{k_1 i_1 j_1 i_1} a_{k_1 i_2} e^{i_2} \wedge e^{j_1} + R_{k_2 i_2 j_2 i_2} a_{i_1 k_2} e^{j_2} \wedge e^{i_1} \\ &\quad + R_{k_2 i_2 j_1 i_1} a_{i_1 k_2} e^{i_2} \wedge e^{j_1} + R_{k_1 i_1 j_2 i_2} a_{k_1 i_2} e^{j_2} \wedge e^{i_1} \\ &= Ric_{k_1 j_1} a_{k_1 i_2} e^{i_2} \wedge e^{j_1} + Ric_{k_2 j_2} a_{i_1 k_2} e^{j_2} \wedge e^{i_1} \\ &\quad + R_{k_2 i_2 j_1 i_1} a_{i_1 k_2} e^{i_2} \wedge e^{j_1} + R_{k_1 i_1 j_2 i_2} a_{k_1 i_2} e^{j_2} \wedge e^{i_1}. \end{aligned}$$

So, we have that

$$\begin{aligned} \langle E(\omega), \omega\rangle &= Ric_{k_1 j_1} a_{k_1 i_2} a_{j_1 i_2} + Ric_{k_2 j_2} a_{i_1 k_2} a_{i_1 j_2} \\ &\quad + R_{k_2 i_2 j_1 i_1} a_{i_1 k_2} a_{j_1 i_2} + R_{k_1 i_1 j_2 i_2} a_{k_1 i_2} a_{i_1 j_2}. \end{aligned} \tag{3.6}$$

By Gauss equation, we have that

$$R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk}.$$

A direct computation shows that

$$Ric_{k_1 j_1} = nH h_{k_1 j_1} - h_{k_1 i} h_{i j_1}; \tag{3.7}$$

$$Ric_{k_2 j_2} = nH h_{k_2 j_2} - h_{k_2 i} h_{i j_2}; \tag{3.8}$$

$$R_{k_2 i_2 j_1 i_1} = h_{k_2 j_1} h_{i_2 i_1} - h_{k_2 i_1} h_{i_2 j_1} \tag{3.9}$$

and

$$R_{k_1 i_1 j_2 i_2} = h_{k_1 j_2} h_{i_1 i_2} - h_{k_1 i_2} h_{i_1 j_2}. \tag{3.10}$$

Since the operator is linear $\langle E(\omega), \omega\rangle$ and zero-th order differential operator, it is sufficient to compute $\langle E(\omega), \omega\rangle$ at a point p. We can choose an orthonormal frame $\{e_i\}$ such that

$$h_{ij} = \lambda_i \delta_{ij}$$

at p. Note that

$$nH = \lambda_1 + \cdots + \lambda_n.$$

By (3.6)–(3.10), we obtain that

$$\begin{aligned}\langle E(\omega),\omega\rangle &= \sum nH\lambda_{k_1}(a_{k_1i_2})^2 - \sum \lambda_{k_1}^2(a_{k_1i_2})^2 \\ &\quad + \sum nH\lambda_{k_2}(a_{i_1k_2})^2 - \sum \lambda_{k_2}^2(a_{i_1k_2})^2 \\ &\quad - \sum \lambda_{k_2\lambda_{i_2}}(a_{k_2i_2})^2 - \sum \lambda_{j_2\lambda_{i_2}}(a_{j_2i_2})^2 \\ &= 2\sum_{i\neq j}((\lambda_1+\cdots+\lambda_n)\lambda_i - \lambda_i^2 - \lambda_i\lambda_j)(a_{ij})^2.\end{aligned}$$

Note that

$$|A|^2 = |\Phi|^2 + nH^2.$$

For $n = 3$, we get that

$$\begin{aligned}\langle E(\omega),\omega\rangle &= 2\sum_{i\neq j}((\lambda_1+\lambda_2+\lambda_3)\lambda_i - \lambda_i^2 - \lambda_i\lambda_j)(a_{ij})^2 \\ &= \sum_{i\neq j}((\lambda_1+\lambda_2+\lambda_3)(\lambda_i+\lambda_j) - (\lambda_i^2+\lambda_j^2) - 2\lambda_i\lambda_j)(a_{ij})^2 \\ &= \sum_{i\neq j}\left(\frac{1}{2}(3H)^2 - \frac{1}{2}\sum_{k=1,k\neq i,j}^{3}\lambda_k^2 - \frac{1}{2}(\lambda_i+\lambda_j)^2\right)(a_{ij})^2 \\ &\geq \sum_{i\neq j}\left(\frac{1}{2}(3H)^2 - \frac{1}{2}\sum_{k=1,k\neq i,j}^{3}\lambda_k^2 - (\lambda_i^2+\lambda_j^2)\right)(a_{ij})^2 \\ &\geq \sum_{i\neq j}\left(\frac{9}{2}H^2 - |A|^2\right)(a_{ij})^2 = \left(\frac{3}{2}H^2 - |\Phi|^2\right)|\omega|^2.\end{aligned} \tag{3.11}$$

For $n \geq 4$, we have that

$$\begin{aligned}\langle E(\omega),\omega\rangle &= 2\sum_{i\neq j}((\lambda_1+\cdots+\lambda_n)\lambda_i - \lambda_i^2 - \lambda_i\lambda_j)(a_{ij})^2 \\ &= \sum_{i\neq j}((\lambda_1+\cdots+\lambda_n)(\lambda_i+\lambda_j) - (\lambda_i^2+\lambda_j^2) - 2\lambda_i\lambda_j)(a_{ij})^2 \\ &= \sum_{i\neq j}((\lambda_1+\cdots+\widehat{\lambda_i}+\cdots+\widehat{\lambda_j}+\cdots+\lambda_n)(\lambda_i+\lambda_j))(a_{ij})^2 \\ &= \sum_{i\neq j}\left(\frac{1}{2}(nH)^2 - \frac{1}{2}\left(\sum_{k=1,k\neq i,j}^{n}\lambda_k\right)^2 - \frac{1}{2}(\lambda_i+\lambda_j)^2\right)(a_{ij})^2\end{aligned} \tag{3.12}$$

$$\geq \sum_{i \neq j} \left(\frac{1}{2}(nH)^2 - \frac{n-2}{2} \left(\sum_{k=1, k \neq i,j}^{n} \lambda_k^2 \right) - (\lambda_i^2 + \lambda_j^2) \right) (a_{ij})^2$$

$$\geq \sum_{i \neq j} \left(\frac{1}{2}(nH)^2 - \frac{n-2}{2} |A|^2 \right) (a_{ij})^2$$

$$= \left(nH^2 - \frac{n-2}{2} |\Phi|^2 \right) |\omega|^2.$$

By (3.5), (3.11) and (3.12), we obtain the desired result. □

4. Vanishing theorem on hypersurfaces in $\mathbf{R}^{n+1}$

In this section, we give the proof of Theorem 1.1. If η is a compactly supported piecewise smooth function on M, then

$$div(\eta^2 h \nabla h) = \eta^2 h \triangle h + \eta^2 |\nabla h|^2 + 2\eta h \langle \nabla \eta, \nabla h \rangle.$$

Integrating by parts on M, we have that

$$\int_M \eta^2 h \triangle h + \int_M \eta^2 |\nabla h|^2 + 2 \int_M \eta h \langle \nabla \eta, \nabla h \rangle = 0. \tag{4.1}$$

Case I: $n = 3$. By Proposition 3.1 and (4.1), we obtain that

$$-2\int_M \eta h \langle \nabla \eta, \nabla h \rangle - 2\int_M \eta^2 |\nabla h|^2 + \int_M |\Phi|^2 \eta^2 h^2 - \frac{3}{2}\int_M H^2 h^2 \eta^2 \geq 0. \tag{4.2}$$

Note that

$$-2\int_M \eta h \langle \nabla \eta, \nabla h \rangle \leq a_1 \int_M \eta^2 |\nabla h|^2 + \frac{1}{a_1}\int_M h^2 |\nabla \eta|^2, \tag{4.3}$$

for any positive real number a_1. Set $\phi_1(\eta) := \left(\int_{Supp\,\eta} |\Phi|^3\right)^{1/3}$. Then

$$\begin{aligned} \int_M |\Phi|^2 \eta^2 h^2 &\leq \left(\int_{Supp\,\eta} (|\Phi|^2)^{3/2} \right)^{2/3} \cdot \left(\int_M (\eta^2 h^2)^3 \right)^{1/3} \\ &= \phi_1(\eta)^2 \cdot \left(\int_M (\eta h)^6 \right)^{1/3} \\ &\leq C_0 \phi_1(\eta)^2 \cdot \left(\int_M |\nabla(\eta h)|^2 + 9\int_M H^2 (\eta h)^2 \right) \\ &\leq C_0 \phi_1(\eta)^2 \cdot \left(\left(1 + \frac{1}{b_1}\right) \int_M h^2 |\nabla \eta|^2 \right. \\ &\qquad \left. + (1 + b_1) \int_M \eta^2 |\nabla h|^2 + 9\int_M H^2 (\eta h)^2 \right), \end{aligned} \tag{4.4}$$

for any positive real number b_1, where the second inequality holds because of Proposition 2.3. By (4.2)–(4.4), we obtain that

$$\mathscr{A}_1 \int_M \eta^2 |\nabla h|^2 + \mathscr{B}_1 \int_M H^2 \eta^2 h^2 \le \mathscr{C}_1 \int_M h^2 |\nabla \eta|^2, \tag{4.5}$$

where

$$\mathscr{A}_1 := (2 - C_0 \phi_1(\eta)^2) - (a_1 + b_1 C_0 \phi_1(\eta)^2),$$
$$\mathscr{B}_1 := \frac{3}{2} - 9 C_0 \phi_1(\eta)^2$$

and

$$\mathscr{C}_1 := \frac{1}{a_1} + C_0 \phi_1(\eta)^2 \left(1 + \frac{1}{b_1}\right).$$

Since the total curvature $\|\Phi\|_{L^3(M)}$ is less than $\delta(3) = \sqrt{\frac{1}{6C_0}}$, $\mathscr{B}_1$ and $\mathscr{C}_1$ are positive. Choose a_1 and b_1 small enough such that $\mathscr{A}_1$ is positive. Suppose B_r is a geodesic ball of radius r on M centered at a fixed point p_0. Choose $\eta \in C_0^\infty(M)$ such that

$$\begin{cases} 0 \le \eta \le 1, \\ \eta \equiv 1 \quad on \ B\left(\frac{r}{2}\right), \\ \eta \equiv 0 \quad on \ M \backslash B(r), \\ |\nabla \eta| \le \frac{2}{r}. \end{cases}$$

So (4.5) reduces to

$$\mathscr{A}_1 \int_M \eta^2 |\nabla h|^2 + \mathscr{B}_1 \int_M H^2 \eta^2 h^2 \le \frac{4\mathscr{C}_1}{r} \int_M h^2.$$

Since $\int_M h^2$ is finite, taking $r \to +\infty$, we obtain that h is constant and $H^2 h^2 = 0$. If $h \ne 0$, then $H = 0$. Hence, M has infinite volume, contracting the finiteness of $\int_M h^2$. Therefore, $h = 0$.

Case II: $n \ge 4$. By Proposition 3.1 and (4.1), we get that

$$\begin{aligned} -2 \int_M \eta h \langle \nabla \eta, \nabla h \rangle - \frac{n-1}{n-2} \int_M \eta^2 |\nabla h|^2 \\ + \frac{n-2}{2} \int_M |\Phi|^2 \eta^2 h^2 - n \int_M H^2 h^2 \eta^2 \ge 0. \end{aligned} \tag{4.6}$$

Note that

$$-2\int_M \eta h\langle\nabla\eta,\nabla h\rangle \le a_2\int_M \eta^2|\nabla h|^2 + \frac{1}{a_2}\int_M h^2|\nabla\eta|^2, \tag{4.7}$$

for any positive real number a_2. We set $\phi_2(\eta) := \left(\int_{Supp\ \eta}|\Phi|^n\right)^{1/n}$ and get

$$\begin{aligned}
\int_M |\Phi|^2\eta^2h^2 &\le \left(\int_{Supp\ \eta}(|\Phi|^2)^{n/2}\right)^{2/n}\cdot\left(\int_M(\eta^2h^2)^{n/(n-2)}\right)^{(n-2)/n} \\
&= \phi_2(\eta)^2\cdot\left(\int_M(\eta h)^{2n/(n-2)}\right)^{(n-2)/n} \\
&\le C_0\phi_2(\eta)^2\cdot\left(\int_M|\nabla(\eta h)|^2 + n^2\int_M H^2(\eta h)^2\right) \\
&\le C_0\phi_2(\eta)^2\cdot\Bigg(\int_M\left(1+\frac{1}{b_2}\right)h^2|\nabla\eta|^2 \\
&\qquad + (1+b_2)\eta^2|\nabla h|^2 + n^2\int_M H^2(\eta h)^2\Bigg),
\end{aligned} \tag{4.8}$$

for any positive real number b_2, where the second inequality holds because of Proposition 2.3. By (4.6)–(4.8), we have that

$$\mathscr{A}_2\int_M \eta^2|\nabla h|^2 + \mathscr{B}_2\int_M H^2\eta^2h^2 \le \mathscr{C}_2\int_M h^2|\nabla\eta|^2, \tag{4.9}$$

where

$$\mathscr{A}_2 := \left(\frac{n-1}{n-2} - \frac{n-2}{2}C_0\phi_2(\eta)^2\right) - \left(a_2 + \frac{n-2}{2}b_2C_0\phi_2(\eta)^2\right),$$

$$\mathscr{B}_2 := n - \frac{n^2(n-2)}{2}C_0\phi_2(\eta)^2$$

and

$$\mathscr{C}_2 := \frac{1}{a_2} + \frac{n-2}{2}\left(1+\frac{1}{b_2}\right)C_0\phi_2(\eta)^2.$$

Since the total curvature $\|\Phi\|_{L^n(M)}$ is less than $\delta(n) = \sqrt{\dfrac{2}{n(n-2)C_0}}$, we have $\mathscr{B}_2$ and $\mathscr{C}_2$ are positive. Choose a_2 and b_2 small enough such that $\mathscr{A}_2$ is positive. Let B_r be a geodesic ball of radius r on M centered at a fixed point p_0. Choose $\eta \in C_0^\infty(M)$ such that

$$\begin{cases} 0 \le \eta \le 1, \\ \eta \equiv 1 \quad on \ B\left(\frac{r}{2}\right), \\ \eta \equiv 0 \quad on \ M \backslash B(r), \\ |\nabla \eta| \le \frac{2}{r}. \end{cases}$$

Let $r \to +\infty$ in (4.9). We obtain that $h = 0$, which is similar to Case I.

Therefore, there admits no nontrivial L^2-harmonic 2-form on M. By Corollary 1.6 in [1], we get that the second space of reduced L^2 cohomology of M is trivial.

5. Finiteness theorem on hypersurfaces in $\mathbf{R}^{n+1}$

In this section, we prove Theorem 1.2.

Suppose $n = 3$. By (4.5), we obtain that

$$\mathscr{A}_1 \int_M \eta^2 |\nabla h|^2 + \mathscr{B}_1 \int_M H^2 \eta^2 h^2 \le \mathscr{C}_1 \int_M h^2 |\nabla \eta|^2, \tag{5.1}$$

where

$$\mathscr{A}_1 := (2 - C_0 \phi_1(\eta)^2) - (a_1 + b_1 C_0 \phi_1(\eta)^2),$$
$$\mathscr{B}_1 := \frac{3}{2} - 9 C_0 \phi_1(\eta)^2$$

and

$$\mathscr{C}_1 := \frac{1}{a_1} + C_0 \phi_1(\eta)^2 \left(1 + \frac{1}{b_1}\right).$$

Since the total curvature $\|\Phi\|_{L^3(M)}$ is finite, we can choose a fixed r_0 such that

$$\|\Phi\|_{L^3(M - B_{r_0})} < \delta_1 = \sqrt{\frac{1}{12 C_0}}.$$

Set

$$\tilde{\mathscr{A}}_1 := (2 - C_0 \delta_1^2) - (a_1 + b_1 C_0 \delta_1^2),$$
$$\tilde{\mathscr{B}}_1 := \frac{3}{2} - 9 C_0 \delta_1^2$$

and

$$\tilde{\mathscr{C}}_1 := \frac{1}{a_1} + C_0 \delta_1^2 \left(1 + \frac{1}{b_1}\right).$$

Thus,

$$\tilde{\mathscr{A}_1}\int_M \eta^2|\nabla h|^2 + \tilde{\mathscr{B}}_1\int_M H^2\eta^2h^2 \le \tilde{\mathscr{C}}_1\int_M h^2|\nabla\eta|^2, \tag{5.2}$$

for any $\eta \in C_0^\infty(M - B_{r_0})$, where $\tilde{\mathscr{A}_1}$, $\tilde{\mathscr{B}}_1$ and $\tilde{\mathscr{C}}_1$ are positive. By Proposition 2.3, we have

$$\begin{aligned}\frac{1}{C_0}\left(\int_M (\eta h)^6\right)^{1/3} &\le \int_M |\nabla(\eta h)|^2 + 9\int_M H^2(\eta h)^2 \\ &\le \left(1+\frac{1}{c_1}\right)\int_M h^2|\nabla\eta|^2 \\ &\quad + (1+c_1)\int_M \eta^2|\nabla h|^2 + 9\int_M H^2(\eta h)^2,\end{aligned} \tag{5.3}$$

for any positive real number c_1. By (5.2) and (5.3), we have

$$\begin{aligned}&\frac{1}{C_0}\left(\int_M (\eta h)^6\right)^{1/3} \\ &\le \left(1+\frac{1}{c_1}\right)\int_M h^2|\nabla\eta|^2 + (1+c_1)\int_M \eta^2|\nabla h|^2 + 9\int_M H^2(\eta h)^2 \\ &\le \left(1+\frac{1}{c_1}+(1+c_1)\frac{\tilde{\mathscr{C}}_1}{\tilde{\mathscr{A}_1}}\right)\int_M h^2|\nabla\eta|^2 + \left(9-(1+c_1)\frac{\tilde{\mathscr{B}}_1}{\tilde{\mathscr{A}_1}}\right)\int_M H^2\eta^2h^2.\end{aligned} \tag{5.4}$$

Choose a sufficient large c_1 such that

$$9-(1+c_1)\frac{\tilde{\mathscr{B}}_1}{\tilde{\mathscr{A}_1}} < 0.$$

Then (5.4) implies that

$$\left(\int_M (\eta h)^6\right)^{1/3} \le \tilde{A}\int_M h^2|\nabla\eta|^2, \tag{5.5}$$

for any $\eta \in C_0^\infty(M - B_{r_0})$, where $\tilde{A}$ is a positive constant.

Suppose $n \ge 4$. By (4.9), we get

$$\mathscr{A}_2\int_M \eta^2|\nabla h|^2 + \mathscr{B}_2\int_M H^2\eta^2h^2 \le \mathscr{C}_2\int_M h^2|\nabla\eta|^2, \tag{5.6}$$

where

$$\mathscr{A}_2 := \left(\frac{n-1}{n-2} - \frac{n-2}{2}C_0\phi_2(\eta)^2\right) - \left(a_2 + \frac{n-2}{2}b_2C_0\phi_2(\eta)^2\right),$$

$$\mathscr{B}_2 := n - \frac{n^2(n-2)}{2}C_0\phi_2(\eta)^2$$

and

$$\mathscr{C}_2 := \frac{1}{a_2} + \frac{n-2}{2}\left(1+\frac{1}{b_2}\right) C_0 \phi_2(\eta)^2.$$

Since the total curvature $\|\Phi\|_{L^n(M)}$ is finite, we can choose a fixed r_0 such that

$$\|\Phi\|_{L^n(M-B_{r_0})} < \delta_2 = \sqrt{\frac{1}{n(n-2)C_0}}.$$

$$\tilde{\mathscr{A}_2} := \left(\frac{n-1}{n-2} - \frac{n-2}{2} C_0 \delta_2^2\right) - \left(a_2 + \frac{n-2}{2} b_2 C_0 \delta_2^2\right),$$

$$\tilde{\mathscr{B}}_2 := n - \frac{n^2(n-2)}{2} C_0 \delta_2^2$$

and

$$\tilde{\mathscr{C}_2} := \frac{1}{a_2} + \frac{n-2}{2}\left(1+\frac{1}{b_2}\right) C_0 \delta_2^2.$$

Obviously, $\tilde{\mathscr{A}_2}$, $\tilde{\mathscr{B}}_2$ and $\tilde{\mathscr{C}_2}$ are positive. Thus,

$$\tilde{\mathscr{A}_2} \int_M \eta^2 |\nabla h|^2 + \tilde{\mathscr{B}}_2 \int_M H^2 \eta^2 h^2 \le \tilde{\mathscr{C}_2} \int_M h^2 |\nabla \eta|^2, \tag{5.7}$$

for any $\eta \in C_0^\infty(M - B_{r_0})$. Combining with Proposition 2.3, we get that

$$\begin{aligned} \frac{1}{C_0}\left(\int_M |\eta h|^{2n/(n-2)}\right)^{(n-2)/n} &\le \int_M |\nabla(\eta h)|^2 + n^2 \int_M H^2 (\eta h)^2 \\ &\le (1+c_2)\int_M \eta^2 |\nabla h|^2 + \left(1+\frac{1}{c_2}\right)\int_M h^2|\nabla\eta|^2 + n^2 \int_M H^2\eta^2 h^2, \end{aligned} \tag{5.8}$$

for any positive real number c_2. By (5.7) and (5.8), we have

$$\begin{aligned} \frac{1}{C_0}\left(\int_M |\eta h|^{2n/(n-2)}\right)^{(n-2)/n} \le{}& \left(1+\frac{1}{c_2} + (1+c_2)\frac{\tilde{\mathscr{D}}_2}{\tilde{\mathscr{A}_2}}\right)\int_M h^2|\nabla\eta|^2 \\ &+ \left(n^2 - (1+c_2)\frac{\tilde{\mathscr{B}}_2}{\tilde{\mathscr{A}_2}}\right)\int_M H^2\eta^2h^2. \end{aligned} \tag{5.9}$$

We choose a sufficient large c_2 such that

$$n^2 - (1+c_2)\frac{\tilde{\mathscr{B}}_2}{\tilde{\mathscr{A}_2}} < 0.$$

Then (5.9) implies that

$$\left(\int_M (\eta h)^{2n/(n-2)}\right)^{(n-2)/n} \le \tilde{A}\int_M h^2|\nabla\eta|^2, \tag{5.10}$$

for any $\eta \in C_0^\infty(M - B_{r_0})$, where $\tilde{A}$ is a positive constant depending only on n.

Therefore, we show that

$$\left(\int_M (\eta h)^{2n/(n-2)}\right)^{(n-2)/n} \le \tilde{A}\int_M h^2|\nabla\eta|^2, \tag{5.11}$$

for any $\eta \in C_0^\infty(M - B_{r_0})$, where $\tilde{A}$ is a positive constant depending only on n $(n \ge 3)$.

Next, the proof follows standard techniques (after inequality (33) in [3]) and uses a Moser iteration argument (lemma 11 in [9]). We include a proof here for the sake of completeness. Choose $r > r_0 + 1$ and $\eta \in C_0^\infty(M - B_{r_0})$ such that

$$\begin{cases} \eta = 0 \quad on\ B_{r_0} \cup (M - B_{2r}), \\ \eta = 1 \quad on\ B_r - B_{r_0+1}, \\ |\nabla\eta| < \tilde{c} \quad on\ B_{r_0+1} - B_{r_0}, \\ |\nabla\eta| \le \tilde{c}r^{-1} \quad on\ B_{2r} - B_r, \end{cases}$$

for some positive constant $\tilde{c}$. Then (5.11) becomes that

$$\left(\int_{B_r - B_{r_0+1}} h^{2n/(n-2)}\right)^{(n-2)/n} \le \tilde{A}\int_{B_{r_0+1}-B_{r_0}} h^2 + \frac{\tilde{A}}{r^2}\int_{B_{2r}-B_r} h^2.$$

Letting $r \to \infty$ and noting that $h \in L^2(M)$, we obtain that

$$\left(\int_{M - B_{r_0+1}} h^{2n/(n-2)}\right)^{(n-2)/n} \le \tilde{A}\int_{B_{r_0+1}-B_{r_0}} h^2. \tag{5.12}$$

By Hölder inequality

$$\int_{B_{r_0+2}-B_{r_0+1}} h^2 \le \left(\int_{B_{r_0+2}-B_{r_0+1}} h^{2n/(n-2)}\right)^{(n-2)/n} \cdot \left(\int_{B_{r_0+2}-B_{r_0+1}} 1^{n/2}\right)^{2/n},$$

we have that

$$\int_{B_{r_0+2}} h^2 \le (1 + \tilde{A}\ Vol(B_{r_0+2})^{2/n})\int_{B_{r_0+1}} h^2. \tag{5.13}$$

Set

$$\Psi = \begin{cases} \left| |\Phi|^2 - \dfrac{3}{2}H^2 \right|, \quad for\ n = 3, \\ \left| \dfrac{n-2}{2}|\Phi|^2 - nH^2 \right|, \quad for\ n \ge 4. \end{cases}$$

Fix $x \in M$ and take $\tau \in C_0^1(B_1(x))$. By Proposition 3.1, we have

$$h \triangle h \geq \alpha |\nabla h|^2 - \Psi h^2,$$

where

$$\alpha = \begin{cases} \frac{1}{2}, & \textit{for } n = 3, \\ \frac{1}{n-2}, & \textit{for } n \geq 4. \end{cases}$$

Then, for $p > 2$, there exsits

$$\int_M \tau^2 h^{p-1} \triangle h \geq \alpha \int_M \tau^2 h^{p-2} |\nabla h|^2 - \int_M \tau^2 \Psi h^p.$$

That is,

$$(5.14) \qquad -2 \int_{B_1(x)} \tau h^{p-1} \langle \nabla \tau, \nabla h \rangle \geq (\alpha + (p-1)) \int_{B_1(x)} \tau^2 h^{p-2} |\nabla h|^2 - \int_{B_1(x)} \tau^2 \Psi h^p.$$

Note that

$$\begin{aligned} -2\tau h^{p-1} \langle \nabla \tau, \nabla h \rangle &= -2 \langle h^{p/2} \nabla \tau, \tau h^{p/2-1} \nabla h \rangle \\ &\leq \frac{1}{\alpha} h^p |\nabla \tau|^2 + \alpha \tau^2 h^{p-2} |\nabla h|^2. \end{aligned}$$

By (5.14), we have that

$$(5.15) \qquad (p-1) \int_{B_1(x)} \tau^2 h^{p-2} |\nabla h|^2 \leq \int_{B_1(x)} \Psi \tau^2 h^p + \frac{1}{\alpha} \int_{B_1(x)} |\nabla \tau|^2 h^p.$$

Combining Cauchy-Schwarz inequality with (5.15), we get that

$$(5.16) \qquad \int_{B_1(x)} |\nabla (\tau h^{p/2})|^2 \leq \int_{B_1(x)} \mathscr{A} \Psi \tau^2 h^p + \mathscr{B} |\nabla \tau|^2 h^p,$$

where $\mathscr{A} = \frac{1}{p-1} \left(\frac{p^2}{4} + \frac{p}{2} \right)$ and $\mathscr{B} = \left(1 + \frac{p}{2} \right) + \frac{1}{\alpha(p-1)} \left(\frac{p^2}{4} + \frac{p}{2} \right)$. Choose $f = \tau h^{p/2}$ in Proposition 2.3. By (5.16), we obtain that

$$(5.17) \qquad \left(\int_{B_1(x)} (\tau h^{p/2})^{2n/(n-2)} \right)^{(n-2)/2} \leq p \mathscr{C} \int_{B_1(x)} (\tau^2 + |\nabla \tau|^2) h^p,$$

where $\mathscr{C}$ depends on n and $\sup_{B_1(x)} \Psi$. Set $p_k = \frac{2n^k}{(n-2)^k}$ and $\rho_k = \frac{1}{2} + \frac{1}{2^{k+1}}$ for $k = 0, 1, 2, \ldots$. Take a function $\tau_k \in C_0^\infty(B_{\rho_k(x)})$ satisfying:

$$\begin{cases} 0 \le \tau_k \le 1, \\ \tau_k = 1 \quad on \ B_{\rho_{k+1}}(x), \\ |\nabla \tau_k| \le 2^{k+3}. \end{cases}$$

Choosing $p = p_k$ and $\tau = \tau_k$ in (5.17), we obtain that

$$\left(\int_{B_{\rho_{k+1}}(x)} h^{p_{k+1}} \right)^{1/(p_{k+1})} \le (\mathscr{C} p_k 4^{k+4})^{1/p_k} \left(\int_{B_{\rho_k}(x)} h^{p_k} \right)^{1/p_k}. \tag{5.18}$$

By recurrence, we have

$$\|h\|_{L^{p_{k+1}}(B_{1/2}(x))} \le \prod_{i=0}^{k} p_i^{1/p_i} 4^{i/p_i} (\mathscr{C} 4^4)^{1/p_i} \|h\|_{L^2(B_1(x))} \le \mathscr{D} \|h\|_{L^2(B_1(x))}, \tag{5.19}$$

where $\mathscr{D}$ is a positive constant depending only on n, $Vol(B_{r_0+2})$ and $\sup_{B_{r_0+2}} \Psi$. Letting $k \to \infty$, we get

$$\|h\|_{L^\infty(B_{1/2}(x))} \le \mathscr{D} \|h\|_{L^2(B_1(x))}. \tag{5.20}$$

Now, choose $y \in \bar{B}_{r_0+1}$ such that $\sup_{B_{r_0+1}} h^2 = h(y)^2$. Note that $B_1(y) \subset B_{r_0+2}$. (5.20) implies that

$$\sup_{B_{r_0+1}} h^2 \le \mathscr{D} \|h\|^2_{L^2(B_1(y))} \le \mathscr{D} \|h\|^2_{L^2(B_{r_0+2})}. \tag{5.21}$$

By (5.13), we have

$$\sup_{B_{r_0+1}} h^2 \le \mathscr{F} \|h\|^2_{L^2(B_{r_0+1})}, \tag{5.22}$$

where $\mathscr{F}$ depends only on n, $Vol(B_{r_0+2})$ and $\sup_{B_{r_0+2}} \Psi$. In order to show the finiteness of the dimension of $H^2(L^2(M))$, it suffices to prove that the dimension of any finite dimensional subspaces of $H^2(L^2(M))$ is bounded above by a fixed constant. Combining (5.22) with Lemma 11 in [9], we show that $\dim H^2(L^2(M)) < +\infty$. By Proposition 2.2, we have that the dimension of the second space of reduced L^2 cohomology of M is finite.

Acknowledgements. This work was partially supported by NSFC Grants 11471145, 11371309 and Qing Lan Project.

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