H. OHTAKE
KODAI MATH. J.
**39** (2016), 521–529

# ON THE ISOMETRIES FROM THE UNIT DISK TO INFINITE DIMENSIONAL TEICHMÜLLER SPACES

Hiromi Ohtake

**Abstract**

We generalize Earle-Li's polydisk theorem and embedding theorem, and study isometries from the unit disk to infinite dimensional Teichmüller spaces. We also give a simple proof that for any non-Strebel point $\tau$, there exist infinitely many real analytic geodesic disks through $\tau$ and the basepoint in infinitely dimensional Teichmüller spaces.

## 1. Introduction

Troughout this paper, $\Delta$ will be the open unit disk with the hyperbolic metric $(1-|z|^2)^{-1}|dz|$, and $R$ will be a Riemann surface whose universal covering is the unit disk.

For a non-empty set $S$, we denote by $B(S)$ the Banach algebra of all bounded functions on $S$, and by $D(S)$ the open unit ball of $B(S)$. For a subset $S_1$ of $S$, we identify $B(S_1)$ with the subalgebra of all members $f$ in $B(S)$ such that $f|_{S\setminus S_1}=0$. We note that $B(\emptyset)=D(\emptyset)=\{0\}$. It is easily verified that, for two elements $f$ and $g$ in $D(S)$, their Kobayashi distance is

$$d_{D(S)}(f,g)=\sup\{d_\Delta(f(p),g(p)) : p\in S\},$$

where $d_\Delta$ is the hyperbolic distance on $\Delta$, that is,

$$d_\Delta(z,w):=\tanh^{-1}\left|\frac{z-w}{1-\bar{w}z}\right| \quad \text{for } z, w \text{ in } \Delta.$$

Let $L_\infty(R)$ be the Banach space of all Beltrami differentials on $R$, and $Q(R)$ be the Banach space of all integrable holomorphic quadratic differentials on $R$. For each $\mu$ in $L_\infty(R)$, the mapping

$$Q(R)\ni\phi\mapsto\langle\mu,\phi\rangle:=\int_R\mu\phi$$

2010 *Mathematics Subject Classification*. Primary, 30F60; Secondary, 30C62.
*Key words and phrases*. Teichmüller metric, extremal quasiconformal mapping, isometry.
Received December 15, 2015.

is a bounded linear functional on $Q(R)$. The correspondence $L_\infty(R) \ni \mu \mapsto \langle \mu, \cdot \rangle \in Q(R)^*$ induces an isometric linear isomorphism $P$ from the tangent space $L_\infty(R)/Q(R)^\perp$ of $T(R)$ at the basepoint onto $Q(R)^*$.

The following two theorems are extensions of Earle-Li [3, Theorems 4.1, 6.1 and Lemma 7.1].

THEOREM 1. *Let $R$ be a Riemann surface whose Teichmüller space $T(R)$ is infinite dimensional, and let $\Psi$ be a holomorphic mapping from $D(S)$ to $T(R)$ which maps the origin* 0 *of $D(S)$ to the basepoint* [0] *of $T(R)$. Let $S_1$ be a non-empty subset of $S$, and $S_2 = S \backslash S_1$. Suppose that the restriction of the derivative $\Psi'(0)$ to $B(S_1)$ is an isometry to the tangent space of $T(R)$ at* [0], *and that*

$$|P\Psi'(0)f_1(\phi)| + |P\Psi'(0)f_2(\phi)| \le \|f_1 + f_2\| \, \|\phi\|_1$$

*for all $f_1$ in $B(S_1)$, $f_2$ in $B(S_2)$ and $\phi$ in $Q(R)$. Then we have*

$$d_{T(R)}(\Psi(f_1 + f_2), \Psi(g_1 + g_2)) = d_{D(S_1)}(f_1, g_1),$$

*for all $f_1$, $g_1$ in $D(S_1)$ and all $f_2$, $g_2$ in $D(S_2)$ such that*

$$d_{D(S_1)}(f_1, g_1) \ge d_{D(S_2)}(f_2, g_2).$$

*In particular, if $\psi$ is a distance non-increasing mapping from $D(S_1)$ to $D(S_2)$, then the mapping $D(S_1) \ni f \mapsto \Psi(f + \psi(f)) \in T(R)$ is isometric.*

Let $N$ be $1 \le N \le \infty$. When $N = \infty$, let $\Delta^N$ be the open unit ball of the complex Banach space $\ell^\infty$ of all bounded infinite sequences, and when $N < \infty$, let $\Delta^N$ be the $N$-ary Cartesian power of the unit disk $\Delta$. In both cases, the Kobayashi distance between two points $z = (z_j)_{j=1}^N$ and $w = (w_j)_{j=1}^N$ in $\Delta^N$ is

$$d_{\Delta^N}(z, w) = \sup_{1 \le j \le N} d_\Delta(z_j, w_j).$$

THEOREM 2. *Let $R$ be a Riemann surface whose Teichmüller space is infinite dimensional, $N$ and $N'$ be integers such that $N \ge 1$, $N' \ge 0$ and $N + N' \ge 2$, and $(\mu_j)_{j=1}^N$ be a sequence of extremal Beltrami differentials on $R$ with norm one. When $N' = 0$, if*

$$\sum_{j=1}^N |\mu_j| \le 1,$$

*then the mapping*

$$\Delta^N \ni \zeta = (\zeta_j)_{j=1}^N \mapsto \left[\sum_{j=1}^N \zeta_j \mu_j\right] \in T(R)$$

*is isometric.*

*When $N' > 0$, let $(\nu_j)_{j=1}^{N'}$ be a sequence of Beltrami differentials on $R$ such that*

$$\sum_{j=1}^{N} |\mu_j| + \sum_{j=1}^{N'} |\nu_j| \leq 1, \tag{1}$$

*and let $(g_j)_{j=1}^{N'}$ be a sequence of distance non-increasing mappings form $\Delta^N$ to $\Delta$, then the mapping*

$$\Delta^N \ni \zeta = (\zeta_j)_{j=1}^{N} \mapsto \left[ \sum_{j=1}^{N} \zeta_j \mu_j + \sum_{j=1}^{N'} g_j(\zeta) \nu_j \right] \in T(R)$$

*is isometric.*

Let $\tau$ be a non-Strebel point other than the basepoint $[0]$ in $T(R)$, and $\mu$ be its extremal representative of $\tau$. Let $E$ be a compact subset of $R$, and define Beltrami differentials $\mu_1$, $\nu_1$ and a mapping $g$ from $\Delta$ to itself by $\mu_1 := 0$ on $E$, $\mu_1 := \mu/\|\mu\|_\infty$ on $R \backslash E$, $\nu_1 := \mu/\|\mu\|_\infty$ on $E$, $\nu_1 := 0$ on $R \backslash E$, and $g(\zeta) := \zeta\|\mu\|_\infty / \max\{|\zeta|, \|\mu\|_\infty\}$. Then we can see that $\mu_1$ is an extremal Beltrami differental with norm one, and that $g$ is distance non-increasing. Hence, by the above theorem, we can give another proof that the mapping $\Gamma : \Delta \ni \zeta \mapsto [\zeta\mu_1 + g(\zeta)\nu_1] \in T(R)$ is isometric (Li [4]). By using these mappings for various $E$, he has proved that, for any non-Strebel point $\tau \neq [0]$, there exist infinitely many geodesic disks containing $[0]$ and $\tau$. See also Yao [9].

We can show the following.

Theorem 3. *If $\tau$ is a non-Strebel point other than the basepoint, then there exist infinitely many real analytic geodesic disks containing the basepoint and $\tau$.*

Let $(\mu_j)_{j=1}^{N}$ be as in Theorem 2, and $(h_j)_{j=1}^{N}$ be a sequence of distance non-increasing mappings from $\Delta$ to itself. Obviously, if at least one $h_j$ is isometric, then so is $\Psi : \Delta \ni \zeta \mapsto [\sum h_j(\zeta)\mu_j] \in T(R)$. Does the converse hold for $N \geq 2$? The next theorem provides an answer to this question.

Theorem 4. (i) *If $N = 2$, then the converse holds, that is, $\Psi$ is isometric if and only if so is $h_1$ or $h_2$.*

(ii) *If $N \geq 3$, then there exists $(h_j)$ such that no $h_j$ is isometric but so is $\Psi$.*

## 2. Definitions and priliminaries

In this section, we introduce some definitions and known facts briefly which are necessary in the next section. For details on the theory of Teichmüller spaces, see Gardiner-Lakic [6], Earle-Gardiner [2].

We denote by $M(R)$ the open unit ball in $L_\infty(R)$. Elements in $M(R)$ are called Beltrami coefficients.

Every quasiconformal mapping from a Riemann surface $R$ onto $R'$ extends to a homeomorphism from the bordered Riemann surface $R \cup \partial R$ onto $R' \cup \partial R'$. Two quasiconformal mappings $f$ and $g$ with domain $R$ are said to be *equivalent* if there exists a conformal mapping $h$ from $f(R)$ onto $g(R)$ such that the extension of $g^{-1} \circ h \circ f$ to $R \cup \partial R$ is homotopic to the identity by a homotopy which fixes $\partial R$ pointwise.

For any Beltrami coefficient $\mu$ on $R$, there exists a quiasiconformal mapping $f^{\mu}$ with domain $R$ and Beltrami coefficient $\mu$, which is uniquely determined up to conformal mappings. Two Beltrami coefficients $\mu$ and $\nu$ are said to be equivalent if two quasiconformal mappings $f^{\mu}$ and $f^{\nu}$ are equivalent. The equivalence class of $\mu$ is denoted by $[\mu]$.

The Teichmüller space $T(R)$ of $R$ is defined as the set of equivalence classes in $M(R)$. It has a complex manifold structure such that the canonical projection $\Phi : M(R) \to T(R)$ is holomorphic. The Teichmüller distance between two points $[\mu_0]$ and $[\nu_0]$ in $T(R)$ is defined by

$$d_{T(R)}([\mu_0], [\nu_0]) := \inf\left\{\tanh^{-1}\left\|\frac{\mu - \nu}{1 - \bar{\nu}\mu}\right\|_{\infty} : \mu \in [\mu_0], \nu \in [\nu_0]\right\}.$$

It is known that this is equal to the Kobayashi distance of $T(R)$. (Royden [8], Gardiner [5])

A point $\tau$ in $T(R)$ is called a *Strebel point* if there exist $\mu$ in $\tau$ and a compact subset $E$ of $R$ such that $\|\mu|_{R \setminus E}\|_{\infty} < k_0(\tau)$, where $k_0(\tau) := \inf\{\|\nu\|_{\infty} : \nu \in \tau\}$.

A Beltrami coefficient is said to be *extremal* if it has the smallest norm in its equivalence class. A sufficent and necessary condition for a Beltrami coefficient to be extremal is known.

Theorem A (Hamilton, Krushkal, Reich-Strebel). *A Beltrami coefficient $\mu$ on a Riemann surface $R$ is extremal if and only if*

$$\|\mu\|_{\infty} = \sup\left\{\left|\int_R \mu\phi\right| : \phi \in Q(R), \|\phi\|_1 = 1\right\}. \tag{2}$$

A Beltrami differential $\mu$ which satisfies (2) is also said to be extremal. We call a sequence $(\phi_n)_{n=1}^{\infty}$ with norm one in $Q(R)$ a *Hamilton sequence* for an extremal Beltrami differential $\mu$ if

$$\lim_{n \to \infty}\left|\int_R \mu\phi_n\right| = \|\mu\|_{\infty}.$$

A Hamilton sequence $(\phi_n)$ is said to be *degenerate* if $\lim \int_K |\phi_n| = 0$ for any compact subset $K$ of $R$.

## 3. Proofs of results

*Proof of Theorem* 1. We can prove this theorem by using an argument almost the same as in [3, Theorem 6.1].

Step 1 is to verify that

$$d_{T(R)}(\Psi(f_1+f_2),\Psi(0)) = d_\Delta(\|f_1+f_2\|,0)$$

for all $f_1$ in $D(S_1)$ and $f_2$ in $D(S_2)$ such that $\|f_1\| \geq \|f_2\|$. Consider the holomorphic mapping $F : \Delta \ni \zeta \mapsto \Psi(\zeta(f_1+f_2)/\|f_1+f_2\|) \in T(R)$. It suffices to show that $\|F'(0)\| = 1$, that is, $\|\Psi'(0)(f_1+f_2)\| = \|f_1+f_2\|$. For $\phi \in Q(R)$ with $\|\phi\|_1 = 1$,

$$\begin{aligned}\|f_1+f_2\| &\geq |P\Psi'(0)(f_1+f_2)(\phi)| \\ &\geq |P\Psi'(0)f_1(\phi)| - |P\Psi'(0)f_2(\phi)| \\ &\geq 2|P\Psi'(0)f_1(\phi)| - \|f_1+f_2\|.\end{aligned}$$

Since $\sup_\phi|P\Psi'(0)f_1(\phi)| = \|\Psi'(0)f_1\| = \|f_1\| = \|f_1+f_2\|$, $\|\Psi'(0)(f_1+f_2)\| = \sup_\phi|P\Psi'(0)(f_1+f_2)(\phi)| = \|f_1+f_2\|$.

Step 2 to verify that, for $p \in S_1$, $f \in D(S)$ such that $f(p) = 0$ and for $\zeta, \zeta' \in \Delta$,

$$d_{T(R)}(\Psi(f+\zeta\chi_p),\Psi(f+\zeta'\chi_p)) = d_\Delta(\zeta,\zeta'),$$

where $\chi_p$ is the characteristic function of the singleton $\{p\}$, and the last step to prove the assertion are the same with [3]. □

To prove Theorem 2, we use the following lemma.

Lemma 1. *Let $\mu$ and $\nu$ be Beltrami differentials on $R$ such that*

$$|\mu| + |\nu| \leq \|\mu\|_\infty. \tag{3}$$

*If $\mu$ is extremal and $(\phi_n)$ is a Hamilton sequence for $\mu$, then*

$$\lim_{n\to\infty}\int_R |\nu|\,|\phi_n| = 0,$$

*$\|\mu+\nu\|_\infty = \|\mu\|_\infty$ and $(\phi_n)$ is a Hamilton sequence for $\mu+\nu$. In particular, $\mu+\nu$ is extremal.*

*Proof.* From (3), we see that $\|\mu+\nu\|_\infty \leq \|\mu\|_\infty$, and

$$\begin{aligned}\int_R |\nu|\,|\phi_n| &\leq \int_R (\|\mu\|_\infty - |\mu|)|\phi_n| \\ &\leq \|\mu\|_\infty - \left|\int_R \mu\phi_n\right| \xrightarrow{n\to\infty} 0,\end{aligned}$$

$$\begin{aligned}\|\mu+\nu\|_\infty &\geq \left|\int_R (\mu+\nu)\phi_n\right| \\ &\geq \left|\int_R \mu\phi_n\right| - \int_R |\nu|\,|\phi_n| \xrightarrow{n\to\infty} \|\mu\|_\infty.\end{aligned}$$

□

*Proof of Theorem* 2. We prove only the case where $N = N' = \infty$. Proofs of the other cases are almost the same.

Let $A$ be the bounded linear mapping

$$\ell^\infty \times \ell^\infty \ni ((\xi_j), (\eta_j)) \mapsto \sum_{j=1}^{\infty} \xi_j \mu_j + \sum_{j=1}^{\infty} \eta_j \nu_j \in L_\infty(R).$$

and $\Psi = \Phi \circ A : \Delta^\infty \times \Delta^\infty \to T(R)$, where $\Phi$ is the canonical projection from $M(R)$ onto $T(R)$. Let $\xi = (\xi_j)$ and $\eta = (\eta_j)$ be arbitrary points in $\ell^\infty$. To prove Theorem 2, by Theorem 1, it suffices to show that

$$\|PA(\xi, 0)\|_{Q(R)^*} = \|\xi\|_\infty, \tag{4}$$

$$|PA(\xi, 0)(\phi)| + |PA(0, \eta)(\phi)| \le \|(\xi, \eta)\|_\infty \|\phi\|_1 \quad \text{for all } \phi \in Q(R). \tag{5}$$

In an arbitrary neibourhood of $\xi$, there exists a point $\xi' = (\xi'_j)$ such that $|\xi'_m| = \|\xi'\|_\infty$ for some index $m$. To show (4), we may assume that $\xi$ itself is such a point. Write $\mu := \xi_m \mu_m$ and $\nu := A(\xi, 0) - \mu = \sum_{j \ne m} \xi_j \mu_j$. Then $\mu$ is an extremal Beltrami defferential with norm $|\xi_m|$, and

$$|\mu| + |\nu| \le \sum |\xi_j|\, |\mu_j| \le \|\xi\|_\infty \sum |\mu_j| \le \|\xi\|_\infty = |\xi_m| = \|\mu\|_\infty.$$

Therefore, by Lemma 1, we see that $A(\xi, 0)$ is extremal, and $\|PA(\xi, 0)\|_{Q(R)^*} = \|\mu\|_\infty = \|\xi\|_\infty$. Inequality (5) easily follows from Triangle inequality. □

*Proof of Theorem* 3. Let $\mu$ be an extremal representative of $\tau$. By strong Strebel frame mapping criterion [3, Theorem 5.4], $\mu$ has a degenarate Hamilton sequence. Thus, by [3, Theorem 6.2], there exists an infinite sequence $(\mu_j)$ of extremal Beltrami differentials with norm one such that $\mu/\|\mu\|_\infty = \sum \mu_j$ and $\mu_j$ have disjoint supports.

For each real number $t$ in the interval $(0, 1/2]$, let $.\varepsilon_1 \varepsilon_2 \varepsilon_3 \cdots$ be its binary digit (infinite) expansion. Note that $\varepsilon_1 = 0$. Define the sequence $(h_j)$ of self-isometries of $\Delta$ by

$$h_j(\zeta, t) := \begin{cases} \zeta & \text{if } \varepsilon_j = 0, \\ \bar{\zeta} & \text{if } \varepsilon_j = 1. \end{cases}$$

Then, for each $t$, the mapping $H_t : \Delta \ni \zeta \mapsto [\sum h_j(\zeta, t)\mu_j] \in T(R)$ is a real analytic isometry, and $H_t(0) = [0]$, $H_t(\|\mu\|_\infty) = \tau$. Suppose that $H_t(\zeta) = H_{t'}(\zeta')$ for distinct $t$ and $t'$. Then $h_j(\zeta, t) = h_j(\zeta', t')$ for all $j$ by Theorem 2. Thus $\zeta = \zeta' = \bar{\zeta}$. Therefore $H_t(\Delta) \cap H_{t'}(\Delta) = H_{1/2}(\Delta \cap \mathbf{R})$. □

To prove Theorem 4, we use the following lemma.

Lemma 2. *Let $h$ be a distance non-increasing mapping from $\Delta$ to itself, and let $r$ be $0 < r < 1$. Put*

$$E := \{z \in \partial\Delta(r) : d_\Delta(h(z), h(-z)) = d_\Delta(z, -z)\},$$
$$F := \{z \in \partial\Delta(r) : d_\Delta(h(z), h(0)) = d_\Delta(z, 0)\},$$

*where* $\Delta(r) := \{z \in \mathbf{C} : |z| < r\}$. *Then*

(i) $E \subset F$, *and*

(ii) *if* $E$ *contains more than two points, then* $h$ *is isometric on the convex hull of* $F$.

*Proof.* It is easily seen that, if $z_1$ and $z_2$ be points in $\Delta$ such that $d_\Delta(h(z_1), h(z_2)) = d_\Delta(z_1, z_2)$, then $h$ is isometric on the geodesic segment whose endpoints are $z_1$ and $z_2$. Therefore $E \subset F$.

Obviously, if $z \in E$, then $-z \in E$. Note that, for $z$ in $\partial\Delta(r)$, $d_\Delta(z, -z)$ is the diameter of the closed disk $\bar{\Delta}(r)$. Suppose that $E$ contains more than two points. By preceding and following $h$ by self-isometries of $\Delta$, we may assume that $h(0) = 0$, $r \in E$ and $h(r) = r$. Then we see that $h(-z) = -h(z)$ whenever $z$ is in $E$, and that $h(F) \subset \partial\Delta(r)$.

Take and fix an arbitrary point $\zeta_0$ from $E \backslash \{\pm r\}$. Then $h(\zeta_0) = \zeta_0$ or $h(\zeta_0) = \bar{\zeta}_0$, since $d_\Delta(h(\zeta_0), \pm r) = d_\Delta(h(\zeta_0), h(\pm r)) \le d_\Delta(\zeta_0, \pm r)$ and $|h(\zeta_0)| = r$. When $h(\zeta_0) = \bar{\zeta}_0$, by replacing $h$ with its complex conjugate $\bar{h}$, we may assume that $h(\zeta_0) = \zeta_0$.

Let $\zeta$ be an arbitrary point in $F$, then $d_\Delta(h(\zeta), \pm r) \le d_\Delta(\zeta, \pm r)$, $d_\Delta(h(\zeta), \pm\zeta_0) \le d_\Delta(\zeta, \pm\zeta_0)$ and $|h(\zeta)| = r$. Thus $h(\zeta) = \zeta$, consequently $h|_F = \mathrm{id}_F$. Therefore $h$ is the identity mapping on the convex hull of $F$, in particular, it is isometric. □

*Remark* 1. On the above lemma, the condition that $E$ contains more than two points is necessary. For example, if we define the mapping $h$ by

$$h(z) := \begin{cases} z, & \mathrm{Im}(z) \ge 0 \\ \bar{z}, & \mathrm{Im}(z) < 0, \end{cases}$$

then $E = \{\pm r\}$ and $F = \partial\Delta(r)$, but $h$ is not an isometry of the closed disk $\bar{\Delta}(r)$, the convex hull of $F$.

*Proof of Theorem* 4 (i). Sufficency is trivial. Suppose that $\Psi$ is isometric. Then, by Theorem 2, $h = (h_1, h_2) : \Delta \to \Delta^2$ is isometric. Since $\Delta^2 \ni (z_1, z_2) \mapsto (g_1(z_1), g_2(z_2)) \in \Delta^2$ is isometric for $g_1, g_2 \in \mathrm{Isom}(\Delta)$, we may assume that $h(0) = (0, 0)$.

We assume that neither $h_1$ nor $h_2$ is isometric, and seek a contradiction. Then there exist four points $z_{11}$, $z_{12}$, $z_{21}$, $z_{22}$ in $\Delta$ suth that

$$d_\Delta(h_1(z_{11}), h_1(z_{12})) < d_\Delta(z_{11}, z_{12}), \quad d_\Delta(h_2(z_{21}), h_2(z_{22})) < d_\Delta(z_{21}, z_{22}).$$

Choose $r < 1$ such that $\Delta(r)$ contains these four points. Let $E_j$ and $F_j$ ($j = 1, 2$) be $E$ and $F$, respectively, in Lemma 2 for each $h_j$. Then $F_1 \cup F_2 \supset E_1 \cup E_2 = \partial\Delta(r)$, since $h$ is isometric.

Neither $\partial\Delta(r)\backslash E_1$ nor $\partial\Delta(r)\backslash E_2$ is empty, since neither $h_1$ nor $h_2$ is isometric on $\bar{\Delta}(r)$. The set $E_1$ is closed in $\partial\Delta(r)$, and $E_2 \supset \partial\Delta(r)\backslash E_1$, thus $\#E_2 = \infty$. The same is true for $E_1$. By Lemma 2, neither $\partial\Delta(r)\backslash F_1$ nor $\partial\Delta(r)\backslash F_2$ is empty.

Take and fix two points $\zeta_1$, $\zeta_2$ in $\partial\Delta(r)$ such that $\zeta_j$ is not contained in $F_j$, respectively. Either $h_1$ or $h_2$ preserves the distance between $\zeta_1$ and $\zeta_2$. We may assume that $h_1$ does. Let $\alpha$ and $\beta$ be the endpoints of the connected component of $\partial\Delta(r)\backslash F_1$ to which $\zeta_1$ belongs. Since $\alpha$ and $\beta$ are in $F_1 \cap F_2$, four points $\alpha$, $\beta$, $\zeta_1$ and $\zeta_2$ are all distinct. Since four points $\alpha$, $\beta$, $\zeta_2$ and 0 are in the convex hull of $F_1$ on which $h_1$ is an isometry, and we can follow $h_1$ by an isometry which fixes 0, we may assume that $h_1$ fixes the three points $\alpha$, $\beta$, $\zeta_2$. Let $l$ be the geodesic segment connecting $\zeta_1$ and $\zeta_2$, and let $l'$ be the geodesic segment connecting $\alpha$ and $\beta$. Then $l$ and $l'$ have an intersection point, say $\zeta_3$, in $\Delta(r)$. Since $\zeta_3$ is on $l'$, $h_1$ also fixes $\zeta_3$. Thus

$$\begin{aligned} d_\Delta(\zeta_1,\zeta_3) &\geq d_\Delta(h_1(\zeta_1),h_1(\zeta_3)) \geq d_\Delta(h_1(\zeta_1),h_1(\zeta_2)) - d_\Delta(h_1(\zeta_3),h_1(\zeta_2)) \\ &= d_\Delta(\zeta_1,\zeta_2) - d_\Delta(\zeta_3,\zeta_2) = d_\Delta(\zeta_1,\zeta_3). \end{aligned}$$

Hence three points $h_1(\zeta_1)$, $h_1(\zeta_2)$ and $h_1(\zeta_3)$ are on one geodesic line, and $h_1(\zeta_1) = \zeta_1 \in \partial\Delta(r)$, which contradicts $\zeta_1 \notin F_1$. □

Put $X := \{z \in \Delta : \mathrm{Im}(z) \leq 0\}$. For any point $\zeta$ in $\Delta$, there exists the unique point in $X$ nearest from $\zeta$. This coresspondance defines a mapping $h_0$ from $\Delta$ to itself. Note that if $\zeta$ is in $X$, then $h_0(\zeta) = \zeta$, and that if $\zeta$ is not in $X$, then $h_0(\zeta)$ is real and the geodesic line through $\zeta$ and $h_0(\zeta)$ is orthogonal to the real axis.

LEMMA 3. *The mapping $h_0$ is distance non-increasing.*

*Proof.* Let $\zeta$ and $\zeta'$ be any two points in $\Delta$. Suppose that $\zeta$ is not in $X$ and $\zeta'$ is in $X$. By conjugating a Möbius transformation from $\Delta$ onto itself, we may assume that $\zeta$ is on the imaginary axis. Then, by drawing the (hyperbolic) perpendicular bisector between $h_0(\zeta) = 0$ and $\zeta$, we see that

$$d_\Delta(h_0(\zeta),h_0(\zeta')) \leq d_\Delta(\zeta,\zeta'). \tag{6}$$

Suppose that $\zeta$ and $\zeta'$ are not in $X$. We may assume again that $\zeta$ is on the imaginary axis. By drawing the curve through $h_0(\zeta')$ and equidistant from the imaginary axis, we have (6). Proof of the other case is trivial. □

*Proof of Theorem* 4 (ii). By puting $h_j = 0$ for $j > 3$, it is enough to prove the case $N = 3$.

Let $\gamma$ be the Möbius transformation from $\Delta$ onto itself such that $\gamma(1) = \bar{\omega}$, $\gamma(i) = 1$, and $\gamma(-1) = \omega$, where $\omega = \exp(\pi i/3)$. Put

$$h_1 := \gamma \circ h_0 \circ \gamma^{-1}, \quad h_2 := r \circ h_1 \circ r^{-1}, \quad h_3 := r^{-1} \circ h_1 \circ r,$$

where $r(\zeta) := \omega^2\zeta$. Then the mapping $h := (h_1,h_2,h_3)$ from $\Delta$ to $\Delta^3$ is what we are seeking. In fact, put $X_1 := \gamma(X)$, $X_2 := r(X_1)$, $X_3 := r^{-1}(X_1)$, then each $h_j$ is distance non-increasing and fixes any point in $X_j$. Since $\bigcup_{j=1}^{3} X_j \times X_j = \Delta \times \Delta$, any pair of two points in $\Delta$ is in some $X_j \times X_j$. Thus $h$ is isometric, hence so is $\Psi$ by Theorem 2. □

*Acknowledgement*. The author would like to thank the referee for his/her valuable comments.

## References


[1] C. J. Earle, I. Kra and S. L. Krushkal, Holomorphic motions and Teichmüller spaces, Trans. Amer. Math. Soc. **343** (1994), 927–948.

[2] C. J. Earle and F. P. Gardiner, Geometric isomorphisms between infinite dimensional Teichmüller spaces, Trans. Amer. Math. Soc. **348** (1996), 1163–1190.

[3] C. J. Earle and Z. Li, Isometrically embedded polydisks in infinite dimensional Teichmüller spaces, J. Geom. Anal. **9** (1999), 51–71.

[4] Z. Li, Geodesics discs in Teichmüller space, Sci. China, Ser. A. **48** (2005), 1075–1082.

[5] F. P. Gardiner, Approximation of infinite dimensional Teichmüller spaces, Trans. Amer. Math. Soc. **282** (1984), 367–383.

[6] F. P. Gardiner and N. Lakic, Quasiconformal Teichmüller theory, Math. Surveys Monogr., vol. 76, Amer. Math. Soc., Providence, RI, 2000.

[7] S. Krushkal, Complex geomerty of the universal Teichmüller space. II, Georgian Math. J. **14** (2007), 483–498.

[8] H. Royden, Automorphisms and isometries of Teichmüller space, Advances in the theory of Riemann surfaces (L. V. Ahlfors et al., eds.), Ann. math. stud. **66**, Princeton University Press, 1971, 369–384.

[9] G. Yao, On nonuniqueness of geodesic disks in infinite-dimensional Teichmüller spaces, Monatsh. Math., DOI 10.1007/s00605-015-0834-4 (2015).



Hiromi Ohtake
Kyoto University of Education
1 Fukakusa-Fujinomori-cho, Fushimi-ku
Kyoto 612-8522
Japan
E-mail: ohtake@kyokyo-u.ac.jp