

NOTES ON THE VALUE DISTRIBUTION OF $ff^{(k)} - b$

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Abstract

Let f denote a transcendental meromorphic function with $N(r, f) = S(r, f)$ and k be an integer. By using methods different from others, we have been able to derive several new results and pose some new conjectures that relate to the yet to be resolved conjecture concerning the quantitative estimates on the zeros of $ff^{(k)} - b$, for a non-vanishing small function b .

1. Introduction and main results

Let f denote a transcendental meromorphic function. We assume the reader is familiar with the fundamental results of Nevanlinna theory and its standard notation such as $m(r, f)$, $N(r, f)$, $T(r, f)$, $S(r, f)$ and etc, see e.g., [7, 24]. Recall that a nonconstant meromorphic function α is to be called a small function of f if $T(r, \alpha) = S(r, f) (= o(1)T(r, f))$ as $r \rightarrow \infty$, possibly outside a set of r values of finite linear measure. Moreover, a polynomial in f and its derivatives with small functions of f being the coefficients is called a differential polynomial in f , and $P_n(f)$ will be used to denote a differential polynomial in f with the total degree in f and its derivatives $\leq n$. In addition, we will use the notation $\rho(f)$ and $\lambda(f)$ to denote the order and exponent of convergence of zeros of f , respectively.

Earlier in 1959, Hayman [8] obtained the following result which is a prototype of the studies of the zeros of certain special types of differential polynomials.

THEOREM A. *Let f be a transcendental meromorphic function, $n(\geq 3)$ be an integer. Then $f^n f'$ assumes all finite values, except possibly zero, infinitely many times.*

Later, Hayman [9] conjectured that Theorem A remains to be valid when $n = 1$ and 2. Then Mues [19] confirmed the case when $n = 2$ and Bergweiler-Eremenko [2] and Chen-Fang [3] confirmed the case when $n = 1$, independently.

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It's clear now that distributions of zeros of differential polynomials $P(f)$ of the forms: $P(f) = f^n f^{(k)} - b$, with $n \geq 1$, $k = 1$ and b a nonzero constant have been dealt with. We shall proceed to study similar and unresolved problems for such differential polynomials when $n = 1$ and $k \geq 2$, as well as for more general differential polynomials when $n \geq 2$.

Before proceeding further, first we recall the following results:

THEOREM B ([1]). *If f is a transcendental meromorphic function of finite order and a is a non-vanishing polynomial, then $ff' - a$ has infinitely many zeros.*

CONJECTURE 1.1 ([22]). *Let f be a transcendental meromorphic function, k an integer ≥ 2 and b a nonzero complex number. Then $ff^{(k)} - b$ has infinitely many zeros.*

THEOREM C ([21]). *Let f be a transcendental entire function of finite order. Then there exists at most one integer $k \geq 2$, such that $ff^{(k)}$ has a nonzero exceptional value.*

Now regarding the conjecture, by using methods different from others (see, e.g., [6], [17], [20] and [25]), we are going to prove our first result, which resolves Conjecture 1.1 partly and improves Theorem C.

THEOREM 1.1. *Let f be a transcendental meromorphic function such that $N(r, f) = S(r, f)$, p and q be non-vanishing small functions of f . Then $pff^{(k)} - q$ and $pff^{(l)} - q$ at least one has infinitely many zeros for integers l and k with $l > k \geq 2$.*

The following corollary arises directly from an immediate consequence of Theorem 1.1.

COROLLARY 1.1. *Let α, β be entire functions, and p, q, R_1, R_2 be non-vanishing rational functions. Then the system $pff^{(k)} - q = R_1 e^\alpha$, $pff^{(l)} - q = R_2 e^\beta$ has no transcendental meromorphic solutions for integers l and k with $l > k \geq 2$.*

For $n \geq 2$, and as supplementary result to that of Theorems B, C and some other related results before (see, e.g., [5], [12], [16] and [18]), we shall be able to prove the following result.

THEOREM 1.2. *If f is a transcendental meromorphic function with $N(r, f) = S(r, f)$, $n \geq 2$ is an integer, then $F = a_n f^n f^{(k)} + P_{n-2}(f)$ has infinitely many zeros, and $\lambda(F) = \rho(f)$, where $P_{n-2}(f)$ is a non-vanishing differential polynomial in f with small functions as its coefficients and $\deg P_{n-2}(f) \leq n - 2$, a_n is a small function of f with $a_n \neq 0$.*

2. Some lemmas

The following two lemmas are crucial to the proofs of our main results.

LEMMA 2.1 ([4, 23]). *Let f be a transcendental meromorphic solution of*

$$f^n P(z, f) = Q(z, f),$$

where $P(z, f)$ and $Q(z, f)$ are polynomials in f and its derivatives with meromorphic coefficients $\{a_\lambda \mid \lambda \in I\}$ such that $m(r, a_\lambda) = S(r, f)$ for all $r \in I$. If the total degree of $Q(z, f)$ as a polynomial in f and its derivatives is less than or equal to n , then $m(r, P(z, f)) = S(r, f)$.

LEMMA 2.2 ([13, 14]). *Let f be a meromorphic solution of an algebraic equation*

$$(2.1) \quad P(z, f, f', \dots, f^{(n)}) = 0,$$

where P is a polynomial in $f, f', \dots, f^{(n)}$ with meromorphic coefficients small with respect to f . If a complex constant c does not satisfy equation (2.1), then

$$m\left(r, \frac{1}{f-c}\right) = S(r, f).$$

3. Proof of Theorem 1.1

We shall prove the theorem by contradiction. Suppose contrary to our assertion that $pf^{(k)} - q$ and $pf^{(l)} - q$ both have finitely many zeros for integers l and k with $l > k \geq 2$. Accordingly, there exist entire functions α, β and meromorphic functions R_1, R_2 with $N(r, R_i) + N(r, 1/R_i) = S(r, f)$ ($i = 1, 2$) such that

$$(3.1) \quad pf^{(k)} - q = R_1 e^\alpha$$

and

$$(3.2) \quad pf^{(l)} - q = R_2 e^\beta.$$

From (3.1) and the result of Milloux (see, e.g., [7]), we have

$$T(r, e^\alpha) \leq 2T(r, f) + S(r, f),$$

which shows that $T(r, \alpha) + T(r, \alpha') = S(r, f)$.

We may assume without loss of generality by (3.1) or (3.2) that $p \equiv 1$. Thus, by differentiating (3.1) and by eliminating e^α ,

$$(3.3) \quad t_1 ff^{(k)} + f'f^{(k)} + ff^{(k+1)} = t_2,$$

where $t_1 = -\left(\frac{R'_1}{R_1} + \alpha'\right)$, $t_2 = q' - \left(\frac{R'_1}{R_1} + \alpha'\right)q$.

First of all, we show that $ff^{(k)} - q$ can not be a small function of f . Otherwise, from $N(r, f) = S(r, f)$ and Lemma 2.1, we have $T(r, f^{(k)}) = S(r, f)$. A contradiction $T(r, f) = S(r, f)$ now follows by relying to the Theorem in [10] and combining it with the proof of Proposition E in [11]. Now, we claim that $t_i \neq 0, i = 1, 2$. To show this, we assume contrary to our assertion that $t_1 \equiv 0$, then there exists a nonzero constant A such that $R_1 e^z \equiv A$, which is excluded by our hypothesis of Theorem 1.1 and (3.1), hence $t_1 \neq 0$. Likewise, we can prove $t_2 \neq 0$.

In the same arguments as above, (3.2) gives

$$(3.4) \quad t_3 ff^{(l)} + f'f^{(l)} + ff^{(l+1)} = t_4,$$

where $t_3 = -\left(\frac{R'_2}{R_2} + \beta'\right), t_4 = q' - \left(\frac{R'_2}{R_2} + \beta'\right)q$.

Obviously, $t_3 \neq 0, t_4 \neq 0$, and $T(r, t_i) = S(r, f), i = 1, 2, 3, 4$.

Again, from equation (3.3) or (3.4), the fact that α' and β' are small functions of f , and Lemma 2.2 (where $c = 0$ is used), we would be able to conclude $m(r, 1/f) = S(r, f)$. By the Nevanlinna's first fundamental theorem, we get

$$(3.5) \quad T(r, f) = N\left(r, \frac{1}{f}\right) + S(r, f).$$

Clearly, from (3.3), we have

$$N_{(2)}\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{t_2}\right) + S(r, f) \leq T(r, t_2) + S(r, f) = S(r, f),$$

where $N_{(2)}\left(r, \frac{1}{f}\right)$, as usually, denotes the counting function of zeros of f whose multiplicities are not less than 2, which implies that the zeros of f are mainly simple zeros. Therefore, it follows by (3.5) that

$$(3.6) \quad T(r, f) = N\left(r, \frac{1}{f}\right) + S(r, f) = N_1\left(r, \frac{1}{f}\right) + S(r, f),$$

where in $N_1(r, 1/f)$ only the simple zeros of f are to be considered.

Let z_0 be a simple zero of f with $t_i(z_0) \neq 0, \infty (i = 1, 2, 3, 4)$. From (3.3) and (3.4), we have

$$(3.7) \quad (f'f^{(k)} - t_2)(z_0) = 0$$

and

$$(3.8) \quad (f'f^{(l)} - t_4)(z_0) = 0.$$

Thus, it follows by $f'(z_0) \neq 0, (3.7)$ and (3.8) that

$$(3.9) \quad (t_2 f^{(l)} - t_4 f^{(k)})(z_0) = 0.$$

Two cases will now be considered, depending on whether or not $t_2 f^{(l)} - t_4 f^{(k)}$ vanishes identically.

CASE 1. $t_2 f^{(l)} - t_4 f^{(k)} \not\equiv 0$. Set

$$(3.10) \quad h = \frac{t_2 f^{(l)} - t_4 f^{(k)}}{f}.$$

Clearly, it follows by the lemma of the logarithmic derivative and (3.6), (3.9) that $T(r, h) = S(r, f)$. On the other hand, (3.10) can be represented in the form

$$(3.11) \quad f^{(l)} = \frac{h}{t_2} f + \frac{t_4}{t_2} f^{(k)}.$$

By differentiating both sides of (3.11), we have

$$(3.12) \quad f^{(l+1)} = \left(\frac{h}{t_2}\right)' f + \frac{h}{t_2} f' + \left(\frac{t_4}{t_2}\right)' f^{(k)} + \frac{t_4}{t_2} f^{(k+1)}.$$

Substituting (3.11) and (3.12) into (3.4) will yield

$$(3.13) \quad \left\{ \frac{t_3 h}{t_2} + \left(\frac{h}{t_2}\right)' \right\} f^2 + \frac{2h}{t_2} f f' + \left\{ \frac{t_3 t_4}{t_2} + \left(\frac{t_4}{t_2}\right)' \right\} f f^{(k)} \\ + \frac{t_4}{t_2} f' f^{(k)} + \frac{t_4}{t_2} f f^{(k+1)} = t_4.$$

Again, it follows from (3.3) and (3.13) that

$$(3.14) \quad \left\{ \frac{t_3 h}{t_2} + \left(\frac{h}{t_2}\right)' \right\} f + \frac{2h}{t_2} f' + \left\{ \frac{t_3 t_4}{t_2} + \left(\frac{t_4}{t_2}\right)' - \frac{t_1 t_4}{t_2} \right\} f^{(k)} = 0.$$

In order to complete our proof of the Theorem we have to show that

$$(3.15) \quad \frac{t_3 h}{t_2} + \left(\frac{h}{t_2}\right)' \not\equiv 0 \quad \text{and} \quad \frac{t_3 t_4}{t_2} + \left(\frac{t_4}{t_2}\right)' - \frac{t_1 t_4}{t_2} \not\equiv 0.$$

Now, suppose contrary to our assertion that $\frac{t_3 h}{t_2} + \left(\frac{h}{t_2}\right)' \equiv 0$. Then, by the definition of t_3 and on integration, we get $\frac{h}{t_2} = B R_2 e^\beta$, where B is a nonzero constant. This implies $R_2 e^\beta$ is a small function of f . Note that our previous analysis shows that $f f^{(k)} - q$ can not be a small function of f . Thus, it follows by (3.2) that $\frac{t_3 h}{t_2} + \left(\frac{h}{t_2}\right)' \not\equiv 0$. Next, we shall prove $\frac{t_3 t_4}{t_2} + \left(\frac{t_4}{t_2}\right)' - \frac{t_1 t_4}{t_2} \not\equiv 0$. Since otherwise, we have $t_3 - t_1 \equiv \frac{t_2'}{t_2} - \frac{t_4'}{t_4}$. Consequently, the expressions of t_1, t_3 suggest $\frac{R_1}{R_2} e^{\alpha-\beta} = C \frac{t_2}{t_4} := \gamma$, where C is a constant. Evidently, γ is a small function of f . Furthermore, (3.1) and (3.2) give

$$(3.16) \quad f[f^{(k)} - \gamma f^{(l)}] = (1 - \gamma)q.$$

If $\gamma \neq 1$, we may apply Lemma 2.1 to (3.16), and so we find $m(r, f^{(k)} - \gamma f^{(l)}) = S(r, f)$. Now since $N(r, f) = S(r, f)$, we see that $N(r, f^{(k)} - \gamma f^{(l)}) = S(r, f)$. Hence $T(r, f^{(k)} - \gamma f^{(l)}) = S(r, f)$. That is to say that $f^{(k)} - \gamma f^{(l)}$ is a small function of f . It is easy to verify by (3.16) that $T(r, f) = S(r, f)$, a contradiction. If $\gamma \equiv 1$, (3.16) leads $f^{(k)} \equiv f^{(l)}$. The associated characteristic equation of $f^{(k)} = f^{(l)}$ is given by

$$(3.17) \quad \lambda^k = \lambda^l.$$

Then the general solution of $f^{(k)} = f^{(l)}$ is given by

$$(3.18) \quad f(z) = c_0 + c_1z + \dots + c_{k-1}z^{k-1} + d_1e^{\lambda_1z} + \dots + d_{l-k}e^{\lambda_{l-k}z},$$

where $c_0, c_1, \dots, c_{k-1}, d_1, \dots, d_{l-k}$ are constants and $\lambda_1, \dots, \lambda_{l-k}$ are nonzero characteristic roots of (3.17) such that $\lambda_1^{l-k} = 1, \dots, \lambda_{l-k}^{l-k} = 1$.

Obviously, $\lambda_1, \dots, \lambda_{l-k}$ are distinct complex numbers. Thus, one may show easily by (3.18) that f is of order 1. Now by the arguments used in proving Theorem 1 in [21], we will be able to arrive at a contradiction. With the above discussion and (3.14) we have

$$(3.19) \quad f^{(k)} = mf + nf',$$

where

$$m = -\left\{ \frac{t_3h}{t_2} + \left(\frac{h}{t_2} \right)' \right\} / \left\{ \frac{t_3t_4}{t_2} + \left(\frac{t_4}{t_2} \right)' - \frac{t_1t_4}{t_2} \right\}, \quad n = -\frac{2h}{t_2} / \left\{ \frac{t_3t_4}{t_2} + \left(\frac{t_4}{t_2} \right)' - \frac{t_1t_4}{t_2} \right\}.$$

On the other hand, (3.19) gives

$$(3.20) \quad f^{(k+1)} = m'f + (m + n')f' + nf''.$$

Substituting (3.19) and (3.20) into (3.3), one may find

$$(3.21) \quad (mt_1 + m')f^2 + (2m + nt_1 + n')ff' + n(f')^2 + nff'' = t_2.$$

Again, (3.11) and (3.19) imply

$$(3.22) \quad f^{(l)} = \frac{mt_4 + h}{t_2}f + \frac{nt_4}{t_2}f'.$$

By differentiating both sides of (3.22), we obtain

$$(3.23) \quad f^{(l+1)} = \left(\frac{mt_4 + h}{t_2} \right)'f + \left\{ \frac{mt_4 + h}{t_2} + \left(\frac{nt_4}{t_2} \right)' \right\}f' + \frac{nt_4}{t_2}f''.$$

Now substituting (3.22) and (3.23) into (3.4), we deduce

$$(3.24) \quad \left\{ \frac{t_3(mt_4 + h)}{t_2} + \left(\frac{mt_4 + h}{t_2} \right)' \right\}f^2 + \left\{ \frac{nt_3t_4}{t_2} + \frac{2(mt_4 + h)}{t_2} + \left(\frac{nt_4}{t_2} \right)' \right\}ff' + \frac{nt_4}{t_2}(f')^2 + \frac{nt_4}{t_2}ff'' = t_4.$$

On the other hand, (3.21) and (3.24) give

$$(3.25) \quad \left\{ \frac{t_3(mt_4 + h)}{t_2} + \left(\frac{mt_4 + h}{t_2} \right)' - \frac{t_4(mt_1 + m')}{t_2} \right\} f + \left\{ \frac{nt_3t_4}{t_2} + \frac{2(mt_4 + h)}{t_2} + \left(\frac{nt_4}{t_2} \right)' - \frac{t_4(2m + nt_1 + n')}{t_2} \right\} f' = 0.$$

Let $g := \frac{nt_3t_4}{t_2} + \frac{2(mt_4 + h)}{t_2} + \left(\frac{nt_4}{t_2} \right)' - \frac{t_4(2m + nt_1 + n')}{t_2}$. If $g \neq 0$, it is easy to see g is a small function of f . In this case, we then by (3.6) and (3.25) have $g(z_0) = 0$. Thus $T(r, f) = N_{11}(r, 1/f) + S(r, f) \leq N(r, 1/g) + S(r, f) = S(r, f)$, a contradiction. Consequently $g \equiv 0$. Furthermore, by (3.25) again, we have

$$(3.26) \quad \frac{t_3(mt_4 + h)}{t_2} + \left(\frac{mt_4 + h}{t_2} \right)' - \frac{t_4(mt_1 + m')}{t_2} \equiv 0$$

and

$$(3.27) \quad \frac{nt_3t_4}{t_2} + \frac{2(mt_4 + h)}{t_2} + \left(\frac{nt_4}{t_2} \right)' - \frac{t_4(2m + nt_1 + n')}{t_2} \equiv 0.$$

Also, using (3.26) and (3.27), then (3.24) can be rewritten as

$$(mt_1 + m')f^2 + (nt_1 + n')ff' + n(f')^2 + nff'' = t_2,$$

which, and (3.21) leads $m \equiv 0$, and hence $\frac{t_3h}{t_2} + \left(\frac{h}{t_2} \right)' \equiv 0$. This contradicts (3.15).

CASE 2. If $t_2f^{(l)} - t_4f^{(k)} \equiv 0$, by using the same arguments as in the proof of (3.13), we can conclude

$$\frac{t_3t_4}{t_2} + \left(\frac{t_4}{t_2} \right)' - \frac{t_1t_4}{t_2} \equiv 0,$$

which gives

$$(3.28) \quad t_3 - t_1 \equiv \frac{t_2'}{t_2} - \frac{t_4'}{t_4}.$$

Consequently, it follows from the expressions of t_1 , t_3 and (3.28) that $\frac{R_1}{R_2}e^{\alpha-\beta} = C\frac{t_2}{t_4}$, where C is a constant. Thus, by (3.1) and (3.2), we get $Ct_2 \equiv t_4$ and $f^{(k)} \equiv f^{(l)}$. Now by the same arguments used in Case 1, we will be able to arrive at a contradiction. This shows that $pf f^{(k)} - q$ and $pf f^{(l)} - q$ at least one has infinitely many zeros for integers l and k with $l > k \geq 2$.

This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

First of all, we show that $F = a_n f^n f^{(k)} + P_{n-2}(f)$ can not be a small function of f . Otherwise, from $N(r, f) = S(r, f)$ and Lemma 2.1, we get $m(r, f^{(k)}) = S(r, f)$ and then $T(r, f^{(k)}) = S(r, f)$. A contradiction $T(r, f) = S(r, f)$ now follows by relying to the Theorem in [10] and combining it with the proof of Proposition E in [11]. Thus, for any transcendental meromorphic function f under the condition: $N(r, f) = S(r, f)$, we obtain

$$T(r, a_n f^n f^{(k)} + P_{n-2}(f)) \neq S(r, f).$$

Accordingly, it shows that F cannot be a small function of f , and

$$(4.1) \quad \frac{F'}{F} = \frac{a'_n f^n f^{(k)} + n a_n f^{n-1} f' f^{(k)} + a_n f^n f^{(k+1)} + P'_{n-2}(f)}{a_n f^n f^{(k)} + P_{n-2}(f)}.$$

Let $\varphi := \left(a'_n - \frac{F'}{F} a_n \right) ff^{(k)} + n a_n f' f^{(k)} + a_n f' f^{(k+1)}$. Then (4.1) gives

$$(4.2) \quad f^{n-1} \varphi = \frac{F'}{F} P_{n-2}(f) - P'_{n-2}(f).$$

By (4.2) and Lemma 2.1, we have

$$(4.3) \quad m(r, \varphi) = S(r, f).$$

On the other hand, it is easy to see

$$(4.4) \quad N(r, \varphi) \leq \bar{N}\left(r, \frac{1}{F}\right) + S(r, f).$$

It follows from (4.3) and (4.4) that

$$(4.5) \quad T(r, \varphi) \leq \bar{N}\left(r, \frac{1}{F}\right) + S(r, f).$$

Again, by (4.2), we have $(n - 1)T(r, f) \leq (n - 2)T(r, f) + T(r, \varphi) + S(r, f)$, which, and (4.5) results in $T(r, f) \leq \bar{N}\left(r, \frac{1}{F}\right) + S(r, f)$. Thus $F = a_n f^n f^{(k)} + P_{n-2}(f)$ has infinitely many zeros, and $\lambda(F) \geq \rho(f)$.

Clearly, $T(r, F) \leq (n + 1)T(r, f) + S(r, f)$. Consequently, $\lambda(F) \leq \rho(f)$. Theorem 1.2 follows.

5. Some conjectures

In 2003, Langley [15] proved the following result.

THEOREM D. *Let f be meromorphic in the plane of positive order $L \leq \infty$, and assume that $N(r, f)$ has order less than L . Let b be a nonzero complex number. Then the zero sequence of $ff'' - b$ has exponent of convergence L .*

Accordingly, we have the following strong assertion:

CONJECTURE 5.1. *Let f be a transcendental meromorphic function with $N(r, f) = S(r, f)$. Then for an integer $k \geq 2$, $\lambda(ff^{(k)} - b) = \rho(f)$, where $b (\neq 0, \infty)$ is a small function of f .*

Finally, we would like to pose the following conjectures, for further studies.

CONJECTURE 5.2. *Let f be a transcendental meromorphic function with the condition $N(r, f) = S(r, f)$, $n \geq 2$ be an integer and let $F = f^n f^{(k)} + P_{n-1}(f)$, where $P_{n-1}(f) (\neq 0)$ is a differential polynomial in f with small functions as its coefficients and $\deg P_{n-1}(f) \leq n - 1$. Then*

$$N\left(r, \frac{1}{F}\right) \neq S(r, f) \quad \text{and} \quad \lambda(F) = \rho(f).$$

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