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RICCI TENSORS ON THREE-DIMENSIONAL ALMOST COKÄHLER MANIFOLDS

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Abstract

Let M^3 be a three-dimensional almost coKähler manifold such that the Ricci curvature of the Reeb vector field is invariant along the Reeb vector field. In this paper, we obtain some classification results of M^3 for which the Ricci tensor is η-parallel or the Riemannian curvature tensor is harmonic.

1. Introduction

In the last several decades, the study of almost contact geometry has been an interesting research field both from pure mathematical and physical viewpoints. One important class of differentiable manifolds in the framework of almost contact geometry is known as the coKähler manifolds, which were first introduced by Blair [1] and studied by Blair [2], Goldberg and Yano [7] and Olszak et al. [5, 11]. We point out here that the coKähler manifolds in this paper are just the cosymplectic manifolds shown in the above early literatures. The new terminology was recently adopted by many authors mainly due to Li [8], in which the author gave a topology construction of coKähler manifolds via Kähler mapping tori. According to Li's work, we see that the coKähler manifolds are really odd dimensional analogues of Kähler manifolds. We also refer the readers to a recent survey by Cappelletti-Montano et al. [3] and many references therein regarding geometric and topological results on coKähler manifolds.

As a generalization of coKähler manifolds and an analogy of almost Kähler manifolds, the almost coKähler manifolds were widely studied by many authors recently. In particular, D. Perrone in [12] obtained a complete classification of homogeneous almost coKähler manifolds of dimension three and also gave a local characterization of such manifolds under a condition of local symmetry. Recently, the present author in [15] proved that on a three-dimensional almost coKähler manifold the conditions of local symmetry and ϕ-symmetry are

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equivalent. Also, D. Perrone in [13] characterized the minimality of the Reeb vector field of three-dimensional almost coKähler manifolds. In addition, a new local classification of three-dimensional almost coKähler manifolds under the condition "$\nabla_\xi h = 2f\phi h$ and $\|\mathrm{grad}(\lambda)\|$ is a non-zero constant, where f is a smooth function and λ denotes a positive eigenvalue function of $h := \frac{1}{2}\mathscr{L}_\xi\phi$" was also provided by Erken and Murathan [6].

In this paper, we aim to study a three-dimensional almost coKähler manifold M^3 such that the Ricci curvature of the Reeb vector field is invariant along the Reeb vector field (this is equivalent to $\nabla_\xi h = 2f\phi h$, where f denotes a smooth function). Some examples of such manifolds were also provided in Section 3. If, in addition, the Ricci tensor of M^3 is of Codazzi-type (this is equivalent to that the Riemannian curvature tensor of M^3 is harmonic), we prove that M^3 is locally isometric to the product $\mathbf{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature c ($c = 0$ means that M^3 is locally the flat Euclidean space $\mathbf{R}^3$). We also prove that if the Ricci tensor of M^3 is η-parallel, then either M^3 is locally the product $\mathbf{R} \times N^2(c)$ or M^3 is locally isometric to a Lie group equipped with a left invariant almost coKähler structure. Some applications and corollaries of our main results are also provided.

2. Preliminaries

On a $(2n+1)$-dimensional smooth manifold M^{2n+1} if there exist a $(1,1)$-type tensor field ϕ, a global vector field ξ and a 1-form η such that

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

where id denotes the identity endomorphism, then we say that M^{2n+1} admits an *almost contact structure* which is denoted by the triplet (ϕ, ξ, η) and ξ is called the *characteristic* or the *Reeb vector field*. It follows from relation (2.1) that $\phi(\xi) = 0$, $\eta \circ \phi = 0$ and $\mathrm{rank}(\phi) = 2n$. We denote by $(M^{2n+1}, \phi, \xi, \eta)$ a smooth manifold M^{2n+1} endowed with an almost contact structure, which is called an *almost contact manifold*. We define an almost complex structure J on the product manifold $M^{2n+1} \times \mathbf{R}$ by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right), \tag{2.2}$$

where X denotes the vector field tangent to M^{2n+1}, t is the coordinate of $\mathbf{R}$ and f is a smooth function defined on the product $M^{2n+1} \times \mathbf{R}$.

An almost contact structure is said to be *normal* if the above almost complex structure J is integrable, i.e., J is a complex structure. According to Blair [2], the normality of an almost contact structure is expressed by $[\phi, \phi] = -2\, d\eta \otimes \xi$, where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ defined by

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$$

for any vector fields X, Y on M^{2n+1}.

If on an almost contact manifold there exists a Riemannian metric g satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for any vector fields X and Y, then g is said to be *compatible* with the given almost contact structure. In general, an almost contact manifold equipped with a compatible Riemannian metric is said to be an *almost contact metric manifold* and is denoted by $(M^{2n+1}, \phi, \xi, \eta, g)$. The *fundamental* 2-*form* Φ on an almost contact metric manifold M^{2n+1} is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X and Y.

In this paper, by an *almost coKähler manifold* we mean an almost contact metric manifold such that both the 1-form η and 2-form Φ are closed (see [3]). In particular, an almost coKähler manifold is said to be a *coKähler manifold* (see [8]) if the associated almost contact structure on it is normal, which is also equivalent to $\nabla\phi = 0$, or equivalently, $\nabla\Phi = 0$. Notice that (almost) coKähler manifolds are just the (almost) cosymplectic manifolds studied in [1, 2, 5, 7, 11]. The simplest example of (almost) coKähler manifolds is the Riemannian product of a real line or a circle and a (almost) Kähler manifold. However, there do exist some examples of (almost) coKähler manifolds which are not globally the product of a (almost) Kähler manifold and a real line or a circle (see, for example, Dacko [11, Section 3]).

On an almost coKähler manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, we shall set $h = \frac{1}{2}\mathscr{L}_\xi\phi$ and $h' = h \circ \phi$ (notice that both h and h' are symmetric operators with respect to the metric g). Then the following formulas can be found in Olszak [11] and Perrone [12]:

$$h\xi = 0, \quad h\phi + \phi h = 0, \quad \operatorname{tr}(h) = \operatorname{tr}(h') = 0, \tag{2.4}$$

$$\nabla_\xi\phi = 0, \quad \nabla\xi = h', \quad \operatorname{div}\xi = 0, \tag{2.5}$$

$$\nabla_\xi h = -h^2\phi - \phi l, \tag{2.6}$$

$$\phi l \phi - l = 2h^2, \tag{2.7}$$

where $l := R(\cdot, \xi)\xi$ is the Jacobi operator along the Reeb vector field and the Riemannian curvature tensor R is defined by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X, Y]}Z,$$

and tr and div denote the trace and divergence operators, respectively. The well-known Ricci tensor S is defined by

$$S(X, Y) = g(QX, Y) = \operatorname{tr}\{Z \to R(Z, X)Y\},$$

where Q denotes the associated Ricci operator with respective to the metric g.

3. Three-dimensional almost coKähler manifolds

In the following, we denote by $(M^3, \phi, \xi, \eta, g)$ an almost coKähler manifold of dimension three. According to the second term of relation (2.5) we obtain that $(\mathscr{L}_\xi g)(X, Y) = 2g(h'X, Y)$, then we have

Lemma 3.1 ([7, Proposition 3]). *Any* 3*-dimensional almost coKähler manifold is coKähler if and only if* ξ *is a Killing vector field*, *or equivalently*, $h = 0$.

Following Perrone [13], let $\mathcal{U}_1$ be the open subset of M^3 on which $h \neq 0$ and $\mathcal{U}_2$ the open subset defined by $\mathcal{U}_2 = \{p \in M^3 : h = 0 \text{ in a neighborhood of } p\}$. Therefore, $\mathcal{U}_1 \cup \mathcal{U}_2$ is an open dense subset of M^3. For any point $p \in \mathcal{U}_1 \cup \mathcal{U}_2$, we find a local orthonormal basis $\{\xi, e_1, e_2 = \phi e_1\}$ of three distinct unit eigenvector fields of h in certain neighborhood of p. On $\mathcal{U}_1$ we assume that $he_1 = \lambda e_1$ and hence $he_2 = -\lambda e_2$, where λ is a positive function. Notice that λ is continuous on M^3 and smooth on $\mathcal{U}_1 \cup \mathcal{U}_2$.

Lemma 3.2 ([13, Lemma 2.1]). *On* $\mathcal{U}_1$ *we have*

$$\nabla_\xi e_1 = fe_2, \quad \nabla_\xi e_2 = -fe_1, \quad \nabla_{e_1}\xi = -\lambda e_2, \quad \nabla_{e_2}\xi = -\lambda e_1,$$

$$\nabla_{e_1} e_1 = \frac{1}{2\lambda}(e_2(\lambda) + \sigma(e_1))e_2, \quad \nabla_{e_2} e_2 = \frac{1}{2\lambda}(e_1(\lambda) + \sigma(e_2))e_1,$$

$$\nabla_{e_2} e_1 = \lambda\xi - \frac{1}{2\lambda}(e_1(\lambda) + \sigma(e_2))e_2, \quad \nabla_{e_1} e_2 = \lambda\xi - \frac{1}{2\lambda}(e_2(\lambda) + \sigma(e_1))e_1,$$

$$\nabla_\xi h = \left(\frac{1}{\lambda}\xi(\lambda)\ \mathrm{id} + 2f\phi\right)h,$$

where f *is a smooth function and* σ *is the* 1*-form defined by* $\sigma(\cdot) = S(\cdot, \xi)$.

Using the above Lemma 3.2, one obtains that the Ricci operator Q is expressed as follows (see [13, Proposition 4.1]):

$$Q = \alpha\ \mathrm{id} + \beta\eta \otimes \xi + \phi\nabla_\xi h - \sigma(\phi^2) \otimes \xi + \sigma(e_1)\eta \otimes e_1 + \sigma(e_2)\eta \otimes e_2, \tag{3.1}$$

where $\alpha = \frac{1}{2}(r + \mathrm{tr}(h^2))$, $\beta = -\frac{1}{2}(r + 3\ \mathrm{tr}(h^2))$ and r denotes the scalar curvature. Moreover, using Lemma 3.2 we have the following Poisson brackets

$$\begin{aligned} &[\xi, e_1] = (f + \lambda)e_2, \quad [e_2, \xi] = (f - \lambda)e_1, \\ &[e_1, e_2] = \frac{1}{2\lambda}(e_1(\lambda) + \sigma(e_2))e_2 - \frac{1}{2\lambda}(e_2(\lambda) + \sigma(e_1))e_1. \end{aligned} \tag{3.2}$$

Therefore, from the well-known Jacobi identity

$$[[\xi, e_1], e_2] + [[e_1, e_2], \xi] + [[e_2, \xi], e_1] = 0,$$

we get from relation (3.2) that

$$\begin{aligned} &-e_2(f + \lambda) - \xi\left(\frac{e_1(\lambda) + \sigma(e_2)}{2\lambda}\right) + \frac{e_2(\lambda) + \sigma(e_1)}{2\lambda}(f + \lambda) = 0, \\ &-e_1(f - \lambda) + \xi\left(\frac{e_2(\lambda) + \sigma(e_1)}{2\lambda}\right) + \frac{e_1(\lambda) + \sigma(e_2)}{2\lambda}(f - \lambda) = 0. \end{aligned} \tag{3.3}$$

By Lemma 3.2 and relation (3.1) we obtain

$$S(\xi,\xi) = -\mathrm{tr}(h^2) = -2\lambda^2. \tag{3.4}$$

Comparing this with the last term of Lemma 3.2 then we get

PROPOSITION 3.1. *On a three-dimensional almost coKähler manifold, the Ricci curvature of the Reeb vector field is invariant along the Reeb vector field if and only if*

$$\nabla_\xi h = 2f\phi h, \tag{3.5}$$

where f is a smooth function.

Next, we present several examples of three-dimensional almost coKähler manifolds satisfying condition (3.5). Firstly, from $(\mathscr{L}_\xi g)(X,Y) = 2g(h'X,Y)$ we see that relation (3.5) holds trivially on any almost coKähler manifold with ξ a Killing vector field.

Example 3.1. Let M^3 be an almost coKähler manifold of dimension 3 such that the Reeb vector field ξ belongs to the (k,μ,ν)-nullity distribution (see [5]), i.e.,

$$\begin{aligned} R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \\ + \nu(\eta(Y)h'X - \eta(X)h'Y) \end{aligned}$$

for any vector fields X, Y, Z, where we have assumed that k is a non-zero constant and μ, ν are smooth functions. From (2.4) and the above relation we know that $S(\xi,\xi) = 2k$ is a non-zero constant. Then, applying Lemma 3.2 we see that relation (3.5) holds on M^3.

Example 3.2 ([6]). Let us recall the following example constructed in [6]. On a three-dimensional manifold $M^3 = \{(x,y,z) \in \mathbf{R}^3 \mid z > 0\}$ we denote by

$$\xi = \frac{\partial}{\partial x}, \quad e = z^2\frac{\partial}{\partial x} + \left(2xz - \frac{z+y}{2z}\right)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad \phi e = \frac{\partial}{\partial y}.$$

Consider a Riemannian metric g and a $(1,1)$-type tensor field ϕ defined by

$$g = \begin{pmatrix} 1 & 0 & -\dfrac{a_1}{a_3} \\ 0 & 1 & -\dfrac{a_2}{a_3} \\ -\dfrac{a_1}{a_3} & -\dfrac{a_2}{a_3} & \dfrac{1+a_1^2+a_2^2}{a_3} \end{pmatrix} \quad \text{and} \quad \phi = \begin{pmatrix} 0 & -a_1 & \dfrac{a_1a_2}{a_3} \\ 0 & -a_2 & \dfrac{1+a_2^2}{a_3} \\ 0 & -a_3 & a_2 \end{pmatrix}$$

respectively, with respective to the local basis $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}$, where $a_1 = z^2$, $a_2 = 2xz - \frac{x+y}{2z}$ and $a_3 = 1$. The 1-form η of M^3 is defined by $\eta = dx - z^2\,dz$ and the fundamental 2-form of M^3 is given by $\Phi = dy \wedge dz$. One can check that $(M^3, \phi, \xi, \eta, g)$ is a non-coKähler almost coKähler manifold satisfying condition (3.5), i.e., $\nabla_\xi h = 2z\phi h$. For more details see [6, Example 3.6].

Example 3.3. The following example is a special case of [11, Section 3]. Assume that $\mathfrak{g}$ is a three-dimensional real solvable Lie algebra with basis $\{e_0, e_1, e_2\}$ and whose Lie brackets are given as follows:

$$[e_0, e_1] = -[e_1, e_0] = -ae_1 - be_2, \quad [e_0, e_2] = -[e_2, e_0] = -be_1 + ae_2, \quad [e_1, e_2] = 0,$$

where both a and b are assumed to be constants satisfying $a^2 + b^2 > 0$.

Let M^3 be a Lie group whose Lie algebra is $\mathfrak{g}$. We denote by $\{E_0, E_1, E_2\}$ three left invariant vector fields on M^3 extended from $\{e_0, e_1, e_2\}$. Then we have

$$[E_0, E_1] = -[E_1, E_0] = -aE_1 - bE_2, \quad [E_1, E_2] = 0,$$
$$[E_0, E_2] = -[E_2, E_0] = -bE_1 + aE_2.$$

We define a left invariant Riemannian metric g on M^3 by

$$g(E_i, E_j) = \delta_{ij} \quad \text{for any } 0 \leq i, j \leq 2.$$

We denote by ∇ the left invariant connection with respect to the Riemannian metric g. It follows that

$$\begin{aligned}
&\nabla_{E_1} E_0 = aE_1 + bE_2, \quad \nabla_{E_2} E_0 = bE_1 - aE_2, \\
&\nabla_{E_1} E_1 = -aE_0, \quad \nabla_{E_1} E_2 = \nabla_{E_2} E_1 = -bE_0, \quad \nabla_{E_2} E_2 = aE_0, \\
&\nabla_{E_0} E_1 = 0, \quad \nabla_{E_0} E_2 = 0, \quad \nabla_{E_0} E_0 = 0.
\end{aligned} \tag{3.6}$$

Finally, we define on M^3 a 1-form η and a $(1,1)$-type tensor field ϕ by

$$\eta(E_i) = \delta_{0i} \quad \text{for any } i = 0, 1, 2$$

and

$$\phi E_0 = 0, \quad \phi E_1 = E_2 \quad \text{and} \quad \phi E_2 = -E_1,$$

respectively.

Here we state that $\{M^3, \phi, E_0, \eta, g\}$ is a non-coKähler almost coKähler manifold satisfying $h \neq 0$. For more details see Olszak [11]. Also, from the first term of relation (3.6) and relation (2.5) we obtain

$$h'E_1 = aE_1 + bE_2, \quad h'E_2 = bE_1 - aE_2.$$

Then it follows that $\operatorname{tr}(h^2) = 2(a^2 + b^2)$ is a non-zero constant. Using relation (3.4) and Proposition 3.1 we know that in this case relation (3.5) holds on M^3. In fact, using (3.6), by a simple calculation we obtain $\nabla_\xi h = 0$.

4. η-parallel Ricci tensor

We observe from Section 3 that there do exist some three-dimensional almost coKähler manifolds such that $\nabla_\xi h = 2f\phi h$, where f is a non-zero constant or vanishes. In what follows, we shall concentrate on the study of such manifolds under additional conditions. We first introduce the following

Definition 4.1. The Ricci tensor of a $(2n+1)$-dimensional almost contact metric manifold (M, ϕ, ξ, η, g) is said to be *η-parallel* if it satisfies

$$g((\nabla_X Q)Y, Z) = 0 \tag{4.1}$$

for any vector fields X, Y, Z orthogonal to ξ.

Obviously, if the Ricci tensor S of any almost contact metric manifold M is parallel (i.e., $\nabla Q = 0$) or η-Einstein (i.e., $Q = a\,\mathrm{id} + b\eta \otimes \xi$, where we assume that a is a constant and b is a function), then it is η-parallel.

Theorem 4.1. *Let M^3 be a 3-dimensional almost coKähler manifold satisfying $\nabla_\xi h = 2f\phi h$, where $f \in \mathbf{R}$. Suppose that the Ricci tensor of M^3 is η-parallel. Then either M^3 is locally isometric to the product space $\mathbf{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature c ($c = 0$ means that M^3 is locally the flat Euclidean space $\mathbf{R}^3$), or M^3 is locally isometric to a simply connected unimodular Lie group equipped with a left invariant almost coKähler structure. For the later case, we have the following classification.*

- *In case $f = 0$, then M^3 is locally isometric to the group $E(1,1)$ of rigid motions of the Minkowski 2-space.*
- *In case $f > 0$, then M^3 is locally isometric to either the universal covering $\tilde{E}(2)$ of the group of rigid motions of the Euclidean 2-space if $f > \lambda$, the Heisenberg group H^3 if $f = \lambda$ or the group $E(1,1)$ of rigid motions of the Minkowski 2-space if $f < \lambda$.*
- *In case $f < 0$, then then M^3 is locally isometric to either the universal covering $\tilde{E}(2)$ of the group of rigid motions of the Euclidean 2-space if $f < -\lambda$, the Heisenberg group H^3 if $f = -\lambda$ or the group $E(1,1)$ of rigid motions of the Minkowski 2-space if $f > -\lambda$.*

Proof. If on a 3-dimensional almost coKähler manifold M^3 the condition $\nabla_\xi h = 2f\phi h$ holds for a global constant $f \in \mathbf{R}$, then by applying Lemma 3.2 we have $\xi(\lambda) = 0$. Moreover, on $\mathscr{U}_1$ it follows from equations (3.1) and (3.5) that

$$\begin{cases} Q\xi = -2\lambda^2\xi + \sigma(e_1)e_1 + \sigma(e_2)e_2, \\ Qe_1 = \sigma(e_1)\xi + (\alpha - 2f\lambda)e_1, \\ Qe_2 = \sigma(e_2)\xi + (\alpha + 2f\lambda)e_2. \end{cases} \tag{4.2}$$

On $\mathscr{U}_1$ by applying Lemma 3.2 again we obtain the following relations.

$$(4.3)\qquad (\nabla_{\xi}Q)\xi = (\xi(\sigma(e_1)) - f\sigma(e_2))e_1 + (\xi(\sigma(e_2)) + f\sigma(e_1))e_2.$$

$$(4.4)\qquad (\nabla_{e_1}Q)e_1 = \left(e_1(\sigma(e_1)) - \frac{1}{2\lambda}\sigma(e_2)(e_2(\lambda) + \sigma(e_1))\right)\xi + e_1(\alpha - 2f\lambda)e_1 - (\lambda\sigma(e_1) + 2f(e_2(\lambda) + \sigma(e_1)))e_2.$$

$$(4.5)\qquad (\nabla_{e_2}Q)e_2 = \left(e_2(\sigma(e_2)) - \frac{1}{2\lambda}\sigma(e_1)(e_1(\lambda) + \sigma(e_2))\right)\xi - (\lambda\sigma(e_2) - 2f(e_1(\lambda) + \sigma(e_2)))e_1 + e_2(\alpha + 2f\lambda)e_2.$$

$$(4.6)\qquad (\nabla_{e_1}Q)e_2 = \left(e_1(\sigma(e_2)) + \lambda(2f\lambda - \beta) + \frac{1}{2\lambda}\sigma(e_1)(e_2(\lambda) + \sigma(e_1))\right)\xi - (\lambda\sigma(e_1) + 2f(e_2(\lambda) + \sigma(e_1)))e_1 - (2\lambda\sigma(e_2) - e_1(\alpha + 2f\lambda))e_2.$$

$$(4.7)\qquad (\nabla_{e_2}Q)e_1 = \left(e_2(\sigma(e_1)) - \lambda(2f\lambda + \beta) + \frac{1}{2\lambda}\sigma(e_2)(e_1(\lambda) + \sigma(e_2))\right)\xi - (2\lambda\sigma(e_1) - e_2(\alpha - 2f\lambda))e_1 - (\lambda\sigma(e_2) - 2f(e_1(\lambda) + \sigma(e_2)))e_2.$$

Since the Ricci tensor is η-parallel, then we get from equations (4.1) and (4.4) that

$$(4.8)\qquad \begin{cases} e_1(\alpha - 2f\lambda) = 0, \\ \lambda\sigma(e_1) + 2f(e_2(\lambda) + \sigma(e_1)) = 0. \end{cases}$$

Similarly, we obtain from equations (4.1) and (4.5) that

$$(4.9)\qquad \begin{cases} e_2(\alpha + 2f\lambda) = 0, \\ \lambda\sigma(e_2) - 2f(e_1(\lambda) + \sigma(e_2)) = 0. \end{cases}$$

Also, comparing relations (4.6) and (4.7) with (4.1), respectively, and using relations (4.8) and (4.9) we obtain

$$(4.10)\qquad \begin{cases} 2\lambda\sigma(e_2) - e_1(\alpha + 2f\lambda) = 0, \\ 2\lambda\sigma(e_1) - e_2(\alpha - 2f\lambda) = 0. \end{cases}$$

The first term of relation (4.8) can be written as $e_1(\alpha) = 2fe_1(\lambda)$. Using this in the first term of (4.10) we obtain $\sigma(e_2) = \frac{2f}{\lambda}e_1(\lambda)$, where we have used that λ is a positive function on $\mathscr{U}_1$. Adding this relation in the second term of relation (4.9) we obtain

$$(4.11)\qquad 4f^2 e_1(\lambda) = 0.$$

Similarly, using the first term of relation (4.9) in the seccond term of (4.10) we obtain $\sigma(e_1) = -\frac{2f}{\lambda} e_2(\lambda)$. Using this in the second term of relation (4.8) we obtain

$$4f^2 e_2(\lambda) = 0. \tag{4.12}$$

Let us recall the following well-known formula

$$\operatorname{div} Q = \frac{1}{2} \operatorname{grad}(r),$$

where grad denotes the gradient operator with respect to g. By using equations (4.3)–(4.5) and relations (4.8) and (4.9) we obtain

$$\begin{aligned} \frac{1}{2} \operatorname{grad}(r) = {} & (e_1(\sigma(e_1)) + e_2(\sigma(e_2))\xi \\ & - \frac{1}{2\lambda}\sigma(e_1)(e_1(\lambda) + \sigma(e_2))\xi - \frac{1}{2\lambda}\sigma(e_2)(e_2(\lambda) + \sigma(e_1))\xi \\ & + (\xi(\sigma(e_1)) - f\sigma(e_2))e_1 + (\xi(\sigma(e_2)) + f\sigma(e_1))e_2. \end{aligned} \tag{4.13}$$

Next, we shall separate our discussions into two cases as follows.

Case I: $f = 0$. Firstly, let us suppose that $\mathscr{U}_1$ is a non-empty subset. By using condition $f = 0$ in the second terms of relations (4.8) and (4.9), respectively, and in view of $\lambda > 0$ on $\mathscr{U}_1$ we obtain that ξ is an eigenvector field of the Ricci operator. Therefore, using $\sigma(e_1) = \sigma(e_2) = 0$ in equation (4.13) we obtain that the scalar curvature is a constant. Moreover, using the first term of relation (4.8) we have $e_1(\alpha) = e_1\left(\frac{r}{2} + \lambda^2\right) = 0$. Since $\lambda > 0$ on $\mathscr{U}_1$, then we have

$$e_1(\lambda) = 0.$$

Similarly, using the first term of relation (4.9) and in view of $\lambda > 0$ we get

$$e_2(\lambda) = 0.$$

Taking into account $\xi(\lambda) = 0$ we see that λ is a constant. Since λ is continuous, thus we state that λ is a global positive constant on M^3. In this context, we obtain from relation (3.2) that

$$[\xi, e_1] = \lambda e_2, \quad [e_2, \xi] = -\lambda e_1, \quad [e_1, e_2] = 0.$$

According to Milnor [9] or Perrone [12, 13] we see that M^3 is locally isometric to a simply connected unimodular Lie group $E(1,1)$ of rigid motions of the Minkowski 2-space, equipped with a left invariant almost coKähler structure.

Next, if $\mathscr{U}_1$ is an empty subset, by Lemma 3.1 we know that M^3 is a coKähler manifold. Thus, due to $h = 0$ we get from equations (2.5) and (2.6) that $Q\xi = 0$, $l = 0$. Since $h = 0$ and $Q\xi = 0$, from (3.1) we get

$$Q = \frac{r}{2} \operatorname{id} - \frac{r}{2}\eta \otimes \xi, \tag{4.14}$$

where r denotes the scalar curvature, and this means that M^3 is an η-Einstein manifold. Taking the covariant derivative of (4.14), we obtain

$$(\nabla_X Q)Y = \frac{1}{2}X(r)Y - \frac{1}{2}X(r)\eta(Y)\xi \tag{4.15}$$

for any vector fields X, Y. Using the formula $\operatorname{div} Q = \frac{1}{2}\operatorname{grad}(r)$ and setting $Y = \xi$ in equation (4.15), we get $\xi(r) = 0$. Since the Ricci tensor is η-parallel, then it follows from equation (4.15) that $X(r) = 0$ for any vector field X orthogonal to ξ. This implies that the scalar curvature r is a constant and hence we get $\nabla Q = 0$. This is also equivalent to that M^3 is locally symmetric. Finally, from Perrone [12, Proposition 3.1], we know that any 3-dimensional locally symmetric almost coKähler manifold is coKähler and is locally isometric either the flat Euclidean space $\mathbf{R}^3$ or the Riemannian product $\mathbf{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature $c \neq 0$.

Case II: $f \neq 0$. We consider only the non-coKähler case, that is, $\mathscr{U}_1$ is a non-empty subset. By using $f \neq 0$ in equations (4.11) and (4.12), respectively, we state that λ is a global positive constant, where we have used $\xi(\lambda) = 0$ and the fact that λ is continuous. Consequently, we get easily that $\sigma(e_2) = \frac{2f}{\lambda}e_1(\lambda) = 0$ and $\sigma(e_1) = -\frac{2f}{\lambda}e_2(\lambda) = 0$. This is equivalent to that ξ is an eigenvector field of the Ricci operator. According to [13, Theorem 3.1], we know that the Reeb vector field of 3-dimensional almost coKähler manifold is minimal if and only if ξ is an eigenvector field of the Ricci operator. Moreover, it follows from equation (4.13) that the scalar curvature is also a global constant. In this context, relation (3.2) can be rewritten as the following

$$[\xi, e_1] = (f + \lambda)e_2, \quad [e_2, \xi] = (f - \lambda)e_1, \quad [e_1, e_2] = 0.$$

Let us recall the following invariant

$$\bar{p} := \|\nabla_\xi h\| - \sqrt{2}\|h\|^2$$

defined in Perrone [13]. In case of $f > 0$, from relation (3.5) and using a simple computation we obtain that $\bar{p} = 2\sqrt{2}\lambda(f - \lambda)$. Otherwise, if $f < 0$, then we obtain $\bar{p} = -2\sqrt{2}\lambda(f + \lambda)$. Notice that both $\|\nabla_\xi h\|$ and $\|h\|$ are constants. Also, ξ is a minimal vector field. According to [13, Theorem 4.4] we state that M^3 is locally isometric to a simply connected unimodular Lie group G equipped with a left invariant almost coKähler structure. More precisely, G is the universal covering $\tilde{E}(2)$ of the group of rigid motions of the Euclidean 2-space if $\bar{p} > 0$, the Heisenberg group H^3 if $\bar{p} = 0$ or the group $E(1,1)$ of rigid motions of the Minkowski 2-space if $\bar{p} < 0$. For the coKähler case, the proof follows from *Case I*. This completes the proof. □

Remark 4.1. An explicit description of the left invariant almost coKähler structures on the Lie groups $E(1,1)$, $\tilde{E}(2)$ and H^3 are given in [12].

Let us consider a three-dimensional almost coKähler manifold M^3 satisfying the (k,μ,ν)-nullity condition (see Example 3.1) for $k,\mu \in \mathbf{R}$ and ν a function. By this nullity condition we obtain $l = -k\phi^2 + \mu h + \nu h'$ and using this in (2.7) we obtain $h^2 = k\phi^2$. Using this in (2.6) we get that $\nabla_\xi h = \mu h' - \nu h$ and hence we obtain $\nabla_\xi h^2 = -2k\nu\phi^2$. Comparing this with $h^2 = k\phi^2$ we obtain $\nu = 0$ and hence $\nabla_\xi h = -\mu\phi h$. On the other hand, by the nullity condition we see that ξ is an eigenvector field of the Ricci operator. When $k = 0$ ($\Leftrightarrow h = 0$) and the Ricci tensor is η-parallel, from Theorem 4.1 we obtain that M^3 is locally isometric to either the flat Euclidean space $\mathbf{R}^3$ or the Riemannian product $\mathbf{R} \times N^2(c)$, $c \neq 0$.

When $k < 0$ ($\Leftrightarrow h \neq 0$), from relation (3.1) we obtain the Ricci operator $Q = \left(\frac{r}{2} - k\right) \mathrm{id} + \left(3k - \frac{r}{2}\right)\eta \otimes \xi + \mu h$. In this case, from relation (4.13) we see that the scalar curvature is a constant. Thus, applying Lemma 3.2 we see that the Ricci tensor is η-parallel.

Remark 4.2. A three-dimensional non-coKähler almost coKähler manifold M^3 satisfying the (k,μ,ν)-nullity condition for $k,\mu \in \mathbf{R}$ and ν a function whose Ricci tensor is η-parallel is locally isometric to the universal covering $\tilde{E}(2)$ of the group of rigid motions of the Euclidean 2-space if $k < 0$ and either $\mu > 2\sqrt{-k}$ or $\mu < -2\sqrt{-k}$, the Heisenberg group H^3 if $k < 0$ and either $\mu = 2\sqrt{-k}$ or $\mu = -2\sqrt{-k}$ or the group $E(1,1)$ of rigid motions of the Minkowski 2-space if $k < 0$ and $-2\sqrt{-k} < \mu < 2\sqrt{-k}$.

5. Harmonic curvature tensor

In this section, we shall present some classification results of three-dimensional almost coKähler manifolds satisfying condition (3.5) for certain constant f. It is well-known that the curvature tensor R of a Riemannian manifold is said to be *harmonic* if it is divergence free, that is, $\mathrm{div}(R) = 0$ (see Mukhopadhyay and Barua [10] and Wang and Liu [14]). Moreover, we know that the curvature tensor is harmonic if and only if the associated Ricci operator is of Codazzi-type, that is,

$$(\nabla_X Q)Y = (\nabla_Y Q)X \tag{5.1}$$

for any vector fields X, Y. Using again the well-known formula $\mathrm{div}\, Q = \frac{1}{2}\,\mathrm{grad}(r)$, from equation (5.1) we obtain the following

LEMMA 5.1. *The scalar curvature of any Riemannian manifold with harmonic curvature tensor is a constant.*

Using this lemma we obtain the following

THEOREM 5.1. *Let M^3 be a 3-dimensional almost coKähler manifold satisfying $\nabla_\xi h = 2f\phi h$, where $f \in \mathbf{R}$. If, in addition, the Riemannian curvature tensor is harmonic, then M^3 must be a coKähler manifold. More precisely, M^3 is locally isometric to the product space $\mathbf{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature c ($c = 0$ means that M^3 is locally the flat Euclidean space $\mathbf{R}^3$).*

Proof. From Lemma 3.2 and relation (3.5) we obtain that $\xi(\lambda) = 0$. Moreover, on $\mathcal{U}_1$ we see that (4.2) holds in this context. Applying Lemma 3.2 and relation (4.2) we have the following relations on $\mathcal{U}_1$.

$$(\nabla_\xi Q)e_1 = (\xi(\sigma(e_1)) - f\sigma(e_2))\xi + \xi(\alpha - 2f\lambda)e_1 - 4f^2\lambda e_2. \tag{5.2}$$

$$(\nabla_\xi Q)e_2 = (\xi(\sigma(e_2)) + f\sigma(e_1))\xi - 4f^2\lambda e_1 + \xi(\alpha + 2f\lambda)e_2. \tag{5.3}$$

$$\begin{aligned}(\nabla_{e_1} Q)\xi = {} & 2\lambda(\sigma(e_2) - 2e_1(\lambda))\xi \\ & + \left(e_1(\sigma(e_1)) - \frac{1}{2\lambda}\sigma(e_2)(e_2(\lambda) + \sigma(e_1))\right)e_1 \\ & + \left(2\lambda^3 + e_1(\sigma(e_2)) + \lambda(\alpha + 2f\lambda) + \frac{1}{2\lambda}\sigma(e_1)(e_2(\lambda) + \sigma(e_1))\right)e_2.\end{aligned} \tag{5.4}$$

$$\begin{aligned}(\nabla_{e_2} Q)\xi = {} & 2\lambda(\sigma(e_1) - 2e_2(\lambda))\xi \\ & + \left(e_2(\sigma(e_2)) - \frac{1}{2\lambda}\sigma(e_1)(e_1(\lambda) + \sigma(e_2))\right)e_2 \\ & + \left(2\lambda^3 + e_2(\sigma(e_1)) + \lambda(\alpha - 2f\lambda) + \frac{1}{2\lambda}\sigma(e_2)(e_1(\lambda) + \sigma(e_2))\right)e_1.\end{aligned} \tag{5.5}$$

Since the Riemannian curvature tensor R is harmonic, comparing equations (5.2) with (5.4), (5.3) with (5.5), (4.6) with (4.7), respectively, we obtain

$$\begin{cases} \xi(\sigma(e_1)) - f\sigma(e_2) - 2\lambda(\sigma(e_2) - 2e_1(\lambda)) = 0, \\ \xi(\alpha - 2f\lambda) - e_1(\sigma(e_1)) + \dfrac{1}{2\lambda}\sigma(e_2)(e_2(\lambda) + \sigma(e_1)) = 0, \\ 2\lambda^3 + \lambda(\alpha + 2f\lambda) + e_1(\sigma(e_2)) + \dfrac{1}{2\lambda}\sigma(e_1)(e_2(\lambda) + \sigma(e_1)) + 4f^2\lambda = 0. \end{cases} \tag{5.6}$$

$$\begin{cases} \xi(\sigma(e_2)) + f\sigma(e_1) - 2\lambda(\sigma(e_1) - 2e_2(\lambda)) = 0, \\ \xi(\alpha + 2f\lambda) - e_2(\sigma(e_2)) + \dfrac{1}{2\lambda}\sigma(e_1)(e_1(\lambda) + \sigma(e_2)) = 0, \\ 2\lambda^3 + e_2(\sigma(e_1)) + \lambda(\alpha - 2f\lambda) + \dfrac{1}{2\lambda}\sigma(e_2)(e_1(\lambda) + \sigma(e_2)) + 4f^2\lambda = 0. \end{cases} \tag{5.7}$$

and

$$(5.8)\qquad \begin{cases} \lambda\sigma(e_2) + 2f(e_1(\lambda) + \sigma(e_2)) - e_1(\alpha + 2f\lambda) = 0, \\ \lambda\sigma(e_1) - 2f(e_2(\lambda) + \sigma(e_1)) - e_2(\alpha - 2f\lambda) = 0, \\ e_2(\sigma(e_1)) + \dfrac{1}{2\lambda}(\sigma(e_2)(e_1(\lambda) + \sigma(e_2)) - \sigma(e_1)(e_2(\lambda) + \sigma(e_1))) \\ \qquad = e_1(\sigma(e_2)) + 4f\lambda^2. \end{cases}$$

On the other hand, by Lemma 5.1 we see that the scalar curvature of M^3 is a constant. Applying this and using equations (4.3)–(4.5) we obtain

$$(5.9)\qquad \begin{cases} \xi(\sigma(e_1)) + (f - \lambda)\sigma(e_2) + 2fe_1(\lambda) + e_1(\alpha - 2f\lambda) = 0, \\ \xi(\sigma(e_2)) - (f + \lambda)\sigma(e_1) - 2fe_2(\lambda) + e_2(\alpha + 2f\lambda) = 0, \\ e_1(\sigma(e_1)) + e_2(\sigma(e_2)) = \dfrac{1}{2\lambda}(\sigma(e_1)(e_1(\lambda) + \sigma(e_2)) + \sigma(e_2)(e_2(\lambda) + \sigma(e_1))). \end{cases}$$

Since $\alpha = \frac{r}{2} + \lambda^2$ (where the scalar curvature r is a constant), then it follows from the first two terms of relation (5.8) that

$$(5.10)\qquad (\lambda + 2f)\sigma(e_2) = 2\lambda e_1(\lambda)$$

and

$$(5.11)\qquad (\lambda - 2f)\sigma(e_1) = 2\lambda e_2(\lambda).$$

In what follows, we still separate our discussions into two cases as follows:

Case I: $f = 0$. Firstly, suppose that $\mathscr{U}_1$ is a non-empty subset. Obviously, since $\lambda > 0$ on $\mathscr{U}_1$, from equations (5.10) and (5.11) we have

$$\sigma(e_2) = 2e_1(\lambda) \quad \text{and} \quad \sigma(e_1) = 2e_2(\lambda).$$

Thus, by using this in the first terms of relations (5.6) and (5.7), respectively, we have

$$\xi(\sigma(e_1)) = \xi(\sigma(e_2)) = 0.$$

Furthermore, by using the above relation, $\sigma(e_2) = 2e_1(\lambda)$ and $\sigma(e_1) = 2e_2(\lambda)$ in relation (3.3) we get that λ is a global constant and ξ is an eigenvector field of the Ricci operator, where we have used that λ is continuous and $\xi(\lambda) = 0$. Thus, by using the third term of relation (5.6) (or (5.7)) we get

$$r + 6\lambda^2 = 0.$$

Using the above equation in relations (4.3)–(4.7) and (5.2)–(5.5) we obtain $\nabla Q = 0$ and hence M^3 is locally symmetric. Applying Perrone [12, Proposition 3.1] again we know that a locally symmetric almost coKähler manifold of dimension three is coKähler and this means that $h = 0$ and hence $\mathscr{U}_1$ is an empty subset, a contradiction. In case $\mathscr{U}_1$ is empty, from Lemma 3.1 we conclude that

M^3 is coKähler. In this context, we have $h = 0$ and $l = 0$ and hence by (4.14) we see that equation (4.15) holds. From Lemma 5.1 we know that the scalar curvature is a constant. Then, by equation (4.15) we obtain that M^3 is Ricci symmetric and hence also locally symmetric. As shown in proof of Theorem 4.1, from [12, Proposition 3.1] we see that M^3 is locally isometric to the product $\mathbf{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature c.

Case II: $f \neq 0$. Firstly, let us consider the subcase $\lambda = 2f \neq 0$ which is a non-zero global constant. Clearly, it follows from equation (5.10) that $\sigma(e_2) = 0$. Using this in the first term of relation (3.3) we get $\sigma(e_1) = 0$. However, it follows from the third term of relation (5.8) that $4f\lambda^2 = 0$ and hence $\lambda = 0$.

Similarly, next we consider the subcase $\lambda = -2f \neq 0$ being a non-zero global constant. By equation (5.11) we obtain $\sigma(e_1) = 0$. Using this in the second term of relation (3.3) we obtain $\sigma(e_2) = 0$. Using again the third term of relation (5.8) we obtain $\lambda = 0$.

Finally, let us consider the last subcase $\lambda^2 - 4f^2 \neq 0$ which holds on an open subset $\mathcal{U}_3 \subseteq \mathcal{U}_1$. Using equation (5.10) in the first term of relation (5.6) we have

$$\xi(\sigma(e_1)) = -\frac{6f\lambda}{\lambda + 2f} e_1(\lambda). \tag{5.12}$$

Also, using equation (5.11) in the first term of relation (5.7) we have

$$\xi(\sigma(e_2)) = \frac{6f\lambda}{\lambda - 2f} e_2(\lambda). \tag{5.13}$$

Making use of equations (5.10), (5.11) and (5.13) in the first term of relation (3.3) we obtain

$$((\lambda - 2f)^2 - 12f^2)e_2(\lambda) = 0, \tag{5.14}$$

where we have used that $\lambda > 0$ on $\mathcal{U}_1$. Obviously, assuming $(\lambda - 2f)^2 \neq 12f^2$ we obtain from equation (5.14) that $e_2(\lambda) = 0$ and hence by relation (5.11) we obtain $\sigma(e_1) = 0$. Using this in (5.12) we obtain $e_1(\lambda) = 0$. By equation (5.10) we also have $\sigma(e_2) = 0$. By the third term of relation (5.8) we obtain $\lambda = 0$ being a global constant due to $f \neq 0$. Otherwise, we conclude that $(\lambda - 2f)^2 - 12f^2 = 0$ holds. This implies that λ is a global constant. Then, by using equations (5.10) and (5.11) we have $\sigma(e_1) = \sigma(e_2) = 0$. Moreover, by the third term of relation (5.8) we get $4f\lambda^2 = 0$ and this yields again that $\lambda = 0$.

For the above several subcases, $\lambda = 0$ means that $\mathcal{U}_1$ is an empty subset and hence by Lemma 3.1 we see that M^3 is coKähler. Then the remaining proof is the same with that of *Case I*. This completes the proof. □

Remark 5.1. Since the condition of local symmetry implies that $\nabla_\xi h = 0$ and $\nabla Q = 0$, then Theorem 5.1 can be regarded as a generalization of Perrone [12, Proposition 3.1].

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