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KODAI MATH. J.
39 (2016), 455–468

ON COMBINATORIAL CRITERIA FOR NON-DEGENERATE SINGULARITIES

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Abstract

In this article we give a sufficient and necessary condition for a Kouchnirenko non-degenerate holomorphic function to have an isolated singularity at 0 in terms of its support. As a corollary we give some useful sufficient conditions for singularity to be isolated.

1. Introduction

Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be the germ of a holomorphic function. One of the problems in the theory of singularities is to check effectively that f has an isolated singularity at 0. Many authors deal with this problem in various context. For instance, by the local Nullstellensatz, f has an isolated singularity at 0 if and only if the Milnor number $\mu(f)$ is finite. Similarly, the Łojasiewicz exponent $Ł_0(f)$ is finite if and only if f has an isolated singularity at 0 (for definitions see Preliminaries).

Kouchnirenko in [9] gave for a set $M \subset \mathbf{N}^n$ a necessary and sufficient condition that f with $\operatorname{supp} f \subset M$, has an isolated singularity at 0 (see Theorem 2.8). Other authors: Wall [22], Orlik and Randell [16], Shcherbak [21] obtained similar results. One can find more historical comments on this topic in [15] and [7].

The quasihomogeneous case was considered by the authors mentioned above as well as by Saito ([19], [20]), Kreuzer and Skarke [10], Hertling and Kurbel [7].

In this paper we examine the problem in the class of non-degenerate holomorphic functions. As the main result we prove that a non-degenerate function (see Preliminaries for the definition) with the support satisfying a combinatorial condition has an isolated singularity at 0 (Theorem 3.1). As a corollary we give

Mathematics Subject Classification. 32S05.

Key words and phrases. Isolated singularity, non-degeneracy in the Kouchnirenko sense, Milnor number, Newton number.

The paper was partially supported by the Polish National Science Centre (NCN), Grant No 2012/07/B/ST1/03293.

Received July 29, 2014; revised September 18, 2015.

some useful sufficient conditions for a holomorphic function to have an isolated singularity at 0 (Corollary 3.16). We also prove that Kouchnirenko condition for M is equivalent to the finiteness of the Newton number of λ_M (Corollary 3.12). It was announced already by Kouchnirenko [8, Remarque 1.13 (ii)] but without a proof. C. T. C. Wall considered different type of non-degeneracy from the Kouchnirenko one. He got similar results for his non-degeneracy to the ones obtained in this paper (see Lemma 1.2 and Theorem 1.6 in [23]).

We also explain some details concerning non-convenient singularities. Kouchnirenko in his celebrated paper gave the formula for the Milnor number only in the convenient case [8, Théorème I (ii)]. Consequently, many authors cited this formula only in this case. For example Damon and Gaffney wrote "Note that Kouchnirenko only carries out his analysis for fit germs" [4, Section 2] and Wall wrote "Although Kouchnirenko gives rather general definition of non-degeneracy, his main results are proved only for function satisfying an additional condition called (in French) 'commode'" [23]. However, Kouchnirenko did prove his formula also for non-convenient functions (see [8, Section 3]). Therefore, we explicitly give the formula for the Milnor number (Corollary 3.10) without the assumption that function is convenient. Kouchnirenko proved this formula for non-convenient functions using [8, Théorème 3.7]. We will use Lemma 3.8 instead.

In Appendix we give the effective bound for the constant C of Lemma 3.6. This bound is expressed in terms of the Łojasiewicz exponent. This invariant may be effectively computed using e.g. Gröbner basis techniques (see [18]) or may be estimated (see [5]).

2. Preliminaries

Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be a non-zero holomorphic function in an open neighborhood of $0 \in \mathbf{C}^n$. We say that f *has a singularity at* 0 if $f(0) = 0$, $\nabla f(0) = 0$, where $\nabla f = (f_{z_1}, \dots, f_{z_n})$. It is equivalent to the condition $\operatorname{ord} f \geq 2$, where $\operatorname{ord} f$ means the order of f at 0. We say that f has *an isolated singularity at* 0 if f has an isolated critical point at the origin i.e., additionally $\nabla f(z) \neq 0$ for $z \neq 0$ near 0. We denote $\mathbf{N} = \{0, 1, 2, \dots\}$ and $\mathbf{R}_+ = \{x \in \mathbf{R} : x \geq 0\}$. Let $\sum_{v \in \mathbf{N}^n} a_v z^v$ be the Taylor expansion of f at 0. We define the set $\operatorname{supp} f$ by $\operatorname{supp} f = \{v \in \mathbf{N}^n : a_v \neq 0\}$ and call it *the support of* f. Let $w_1, \dots, w_n, d$ be positive rational numbers. A polynomial $f \in \mathbf{C}[z_1, \dots, z_n]$ is called *quasihomogeneous of type* $(w_1, \dots, w_n; d)$ if

$$\sum_{i=1}^{n} v_i w_i = d \quad \text{for any } v \in \operatorname{supp} f.$$

The numbers $w_1, \dots, w_n$ are called *weights of* f and the number d is called *weighted degree of* f. We define

$$\Gamma_+(f) = \operatorname{conv}\{v + \mathbf{R}_+^n : v \in \operatorname{supp} f\} \subset \mathbf{R}^n$$

and call it *the Newton diagram of* f. Let $u \in \mathbf{R}^n_+ \setminus \{0\}$. Put

$$l(u, \Gamma_+(f)) = \inf\{\langle u, v\rangle : v \in \Gamma_+(f)\},$$
$$\Delta(u, \Gamma_+(f)) = \{v \in \Gamma_+(f) : \langle u, v\rangle = l(u, \Gamma_+(f))\}.$$

We say that $S \subset \mathbf{R}^n$ is *a face of* $\Gamma_+(f)$ if $S = \Delta(u, \Gamma_+(f))$ for some $u \in \mathbf{R}^n_+ \setminus \{0\}$. The vector u is called *a vector supporting* S. It is easy to see that S is a closed and convex set and $S \subset \mathrm{Fr}(\Gamma_+(f))$, where $\mathrm{Fr}(A)$ denotes the boundary of A. One can check that a face $S \subset \Gamma_+(f)$ is compact if and only if there exists a vector supporting S which has all coordinates positive. We call the family of all compact faces of $\Gamma_+(f)$ *the Newton boundary of* f and denote it by $\Gamma(f)$. We denote by $\Gamma^k(f)$ the set of all compact k-dimensional faces of $\Gamma_+(f)$, $k = 0, \dots, n-1$. For each compact face $S \in \Gamma(f)$ we define quasihomogeneous polynomial $f_S = \sum_{v \in S} a_v z^v$. We say that f is *non-degenerate on the face* $S \in \Gamma(f)$ if the system of equations

$$\frac{\partial f_S}{\partial z_1} = \cdots = \frac{\partial f_S}{\partial z_n} = 0$$

has no solution in $(\mathbf{C}^*)^n$, where $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$. We say that f is *non-degenerate in the sense of Kouchnirenko* (shortly *non-degenerate*) if it is non-degenerate on each face of $\Gamma(f)$. We say that f is *convenient* if $\Gamma_+(f)$ has non-empty intersection with each coordinate axis. We say that f is *nearly convenient* if the distance of $\Gamma_+(f)$ to each coordinate axis does not exceed 1. Denote by $\mathcal{O}_n$ the local ring of germs of holomorphic functions in n-variables at $0 \in \mathbf{C}^n$. Let us recall that *the Milnor Number* $\mu(f)$ is defined as $\mu(f) = \dim_{\mathbf{C}} \mathcal{O}^n/(f'_{z_1}, \dots, f'_{z_n})$. Moreover, *the Newton number* $\nu(f)$ for convenient f is defined as

$$\nu(f) = n!V_n - (n-1)!V_{n-1} + \cdots + (-1)^n V_0,$$

where V_i denotes the sum of i-dimensional volumes of the intersection of the cone (with apex at 0) spanned by $\Gamma(f)$ with the coordinate subspaces of dimension i. We may also define the Newton number for non-convenient holomorphic function (see [8, Définition 1.9]). Namely, let f be non-convenient and $I \subset \{1, 2, \dots, n\}$ be a non-empty set such that $\Gamma_+(f) \cap OX_i = \emptyset$ for $i \in I$ and $\Gamma_+(f) \cap OX_i \neq \emptyset$ for $i \notin I$. We define

$$\nu(f) = \sup_{m \in \mathbf{N}} \nu\left(f + \sum_{i \in I} z_i^m\right).$$

Now, we recall some known results concerning support of holomorphic function having an isolated singularity at 0. Kouchnirenko in [9, Theorem 1] gave for a set $M \subset \mathbf{N}^n$ a necessary and sufficient condition for existence of f, $\operatorname{supp} f \subset M$, having an isolated singularity at 0. In addition, one can deduce from his reasoning that if M satisfies this condition, every holomorphic function f, $\operatorname{supp} f \subset M$, $\operatorname{ord} f \geq 2$, with generic coefficients has an isolated singularity at 0. Before giving his result we state definitions.

Let $M \subset \mathbf{N}^n$. Define the sets $M_i = \{v \in \mathbf{N}^n : v + e_i \in M\}$, where $e_1, \ldots, e_n$, is the standard basis in $\mathbf{R}^n$. Notice that if we take $\lambda_M(z) = \sum_{m \in M} z^m$ then $M_i = \operatorname{supp} \partial\lambda_M/\partial z_i$ for every $i = 1, 2, \ldots, n$.

Let $I \subset \{1, \ldots, n\}$ be a non-empty set. Set

$$OX_I = \{x \in \mathbf{R}^n : x_i = 0 \text{ for } i \notin I\}.$$

We may notice that OX_I is the hyperplane spanned by the axes OX_i, $i \in I$. We say that M satisfies *the Kouchnirenko condition for* I if there exist at least $|I|$ non-empty sets among the sets $M_1 \cap OX_I, M_2 \cap OX_I, \ldots, M_n \cap OX_I$. We say that M *satisfies the Kouchnirenko condition* if M satisfies the Kouchnirenko condition for every $I \subset \{1, 2, \ldots, n\}$.

Remark 2.1. If M satisfies the Kouchnirenko condition, it may happen that λ_M does not have an isolated singularity at 0. For example let $\lambda_M(z) = (z_1 + z_2)(z_1 + z_3)$. It is easy to check that λ_M does not have an isolated singularity at 0 and that is degenerate on the face $S = \operatorname{conv}\{\operatorname{supp}(\lambda_M)\}$.

Example 2.2. a) Let $f(z_1, z_2) = z_1^2 + z_1 z_2$. We shall show that $\operatorname{supp} f$ satisfies the Kouchnirenko condition. Put $M = \operatorname{supp} f$. Then $M_1 = \{(0, 1), (1, 0)\}$, $M_2 = \{(1, 0)\}$. If $I = \{1, 2\}$ we easily check that M satisfies the Kouchnirenko condition for I. If $I = \{1\}$ or $I = \{2\}$, then $M_1 \cap OX_1 \neq \emptyset$.

b) Let $f(z_1, z_2, z_3) = z_1(z_1 + z_2 + z_3)$. We shall show that $\operatorname{supp} f$ does not satisfy the Kouchnirenko condition. Indeed, take $I = \{2, 3\}$ then $|I| = 2$ but $M_1 \cap OX_I \neq \emptyset$ and $M_2 \cap OX_I = M_3 \cap OX_I = \emptyset$.

Now, we explain the Kouchnirenko condition for I in the extreme cases $|I| = 1$ and $|I| = n$. It is easy to check the following property.

PROPERTY 2.3. *Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be a holomorphic function which has a singularity at* 0. *The following holds*:

(i) $\operatorname{supp} f$ *satisfies the Kouchnirenko condition for every* $I = \{i\}$, $i = 1, 2, \ldots, n$ *if and only if f is nearly convenient*,

(ii) $\operatorname{supp} f$ *satisfies the Kouchnirenko condition for* $I = \{1, 2, \ldots, n\}$ *if and only if* $f'_{z_i} \neq 0$, $i = 1, 2, \ldots, n$.

The next simple property shows that the Kouchnirenko condition for $\operatorname{supp} f$ implies that the Newton diagram of holomorphic function f which defines an isolated singularity at 0, has non-empty intersection with every $(n-1)$-dimensional coordinate hyperplane in $\mathbf{R}^n$, $n \geq 3$.

PROPERTY 2.4. *Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$, $n \geq 3$, be a holomorphic function which has a singularity at* 0. *If* $\operatorname{supp} f$ *satisfies the Kouchnirenko condition then* $\Gamma_+(f) \cap OX_I \neq \emptyset$ *for every set* $I \subset \{1, 2, \ldots, n\}$, $|I| = n - 1$.

The following two propositions, which are easy consequences of the definition, give conditions equivalent to the Kouchnirenko condition for supp f in terms of the Newton diagram of f in two and three variables.

PROPOSITION 2.5. *Let* $f : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ *be a holomorphic function which has a singularity at* 0. *Then the following conditions are equivalent*:
(i) f *is nearly convenient*,
(ii) supp f *satisfies the Kouchnirenko condition.*

PROPOSITION 2.6. *Let* $f : (\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$ *be a holomorphic function which has a singularity at* 0. *Then the following conditions are equivalent*:
(i) f *is nearly convenient and* $\Gamma_+(f) \cap OX_{\{i,j\}} \neq \emptyset$ *for every* $i, j \in \{1, 2, 3\}$, $i \neq j$,
(ii) supp f *satisfies the Kouchnirenko condition.*

There are some combinatorial conditions equivalent to the Kouchnirenko condition. Hertling and Kurbel collected such conditions for a quasihomogeneous polynomial in [7, Lemma 2.1] but their lemma is also true without the assumption of quasihomogeneity.

LEMMA 2.7. *Let* $M \subset \mathbf{N}^n$ *and* ord $\lambda_M \geq 2$. *Set* $S_I = \{k : M_k \cap OX_I \neq \emptyset\}$, $I \subset \{1, 2, \dots, n\}$. *Then the following conditions are equivalent*:
(K) $\#I \leq \#S_I$ *for* $\forall I$ (*the Kouchnirenko condition for* M)
(K′) $\#I \leq \#S_I$ *for* $\forall I$ *with* $1 \leq \#I \leq \dfrac{n+1}{2}$
(C1) $[M \cap OX_I = \emptyset \Rightarrow \#I \leq \#(S_I \backslash I)]$ *for* $\forall I$
(C1′) $[M \cap OX_I = \emptyset \Rightarrow \#I \leq \#(S_I \backslash I)]$ *for* $\forall I$ *with* $1 \leq \#I \leq \dfrac{n+1}{2}$
(C2) *If* $\#J < \#I$, *then* $S_I \backslash J \neq \emptyset$.

Proof. The proof is the same as the proof of Lemma 2.1 in [7]. ∎

Now, we recall Theorem 1 in [9].

THEOREM 2.8 ([9, Theorem 1]). *Let* $M \subset \mathbf{N}^n$ *and* ord $\lambda_M \geq 2$. *Then the following conditions are equivalent*:
(ISe) *there exists an isolated singularity* $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ *such that* supp $f \subset M$,
(K) M *satisfies the Kouchnirenko condition.*

As a direct consequence of Theorem 2.8 we get the following corollary.

COROLLARY 2.9. *If* f *has an isolated singularity at* 0, *then the support of* f *satisfies the Kouchnirenko condition.*

Remark 2.10. It seems that Saito [19, Lemma 1.5] was the first to state the corollary above, since he proved that a support of holomorphic function having an isolated singularity at 0, satisfies condition (C1), which by Lemma 2.7 is equivalent to the Kouchnirenko condition. It can also be extracted from [21, Remark 3].

As a direct consequence of the corollary above and Property 2.3(i) we give the following property.

PROPERTY 2.11. *If f has an isolated singularity at* 0, *then f is nearly convenient.*

3. Main result

We begin with the main result.

THEOREM 3.1. *Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$, $n \geq 2$ be a non-degenerate holomorphic function which has a singularity at* 0. *If* $\operatorname{supp} f$ *satisfies the Kouchnirenko condition*, *then f has an isolated singularity at* 0.

We deduce the main theorem follows from the one below.

THEOREM 3.2. *Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$, $n \geq 2$ be a non-degenerate holomorphic function which has a singularity at* 0. *If $v(f)$ is finite*, *then f has an isolated singularity at* 0.

In fact, in Corollary 3.12 below we will show that Kouchnirenko condition for $\operatorname{supp} f$ is equivalent to the finiteness of $v(f)$. This together with Theorem 3.2 gives Theorem 3.1.

Remark 3.3. Theorem 3.1 was given by Lenarcik [11, Property 3.2] in the case $n = 2$ and by the second author in [15, Theorem 5.4] in the case $n \leq 3$. It also confirms Conjecture 5.5 stated in [15].

Remark 3.4. By Theorem 3.1 and Corollary 2.9 we see that in the class of non-degenerate function having a singularity at 0 the Kouchnirenko condition for $\operatorname{supp} f$ is equivalent that f defines an isolated singularity at 0.

Now, we give some lemmas and propositions needed in the proof of Theorem 3.2. The following proposition was discovered independently by many authors, see for example [8, Théorème I (ii)], [13, Remark 2.7], [3, Proposition 4.4], [15, Corollary 5.8].

PROPOSITION 3.5. *If f is convenient and non-degenerate*, *then f has an isolated singularity at* 0.

The lemma below says that the Newton number of a non-convenient holomorphic function is independent of the way we make it convenient. More precisely, we have the following lemma.

LEMMA 3.6. *Let $f:(\mathbf{C}^n,0)\to(\mathbf{C},0)$, $n\geq 2$ be a holomorphic function. Assume that $\nu(f)$ is finite. Let $I=\{i_1,\dots,i_k\}\subset\{1,2,\dots,n\}$ be a non-empty set such that $\Gamma_+(f)\cap OX_i=\emptyset$ for $i\in I$ and $\Gamma_+(f)\cap OX_i\neq\emptyset$ for $i\notin I$. Then there exists $C\geq 2$ such that*

$$\nu\left(f+\sum_{i\in I} z_i^{m_i}\right)=\nu(f)\quad \text{for every } m_i\geq C,\ i\in I$$

Proof. Without loss of generality we may assume that $I=\{1,2,\dots,k\}$. Put $f_m=f+\sum_{i=1}^k z_i^m$, $m\geq 2$. By assumption $\nu(f)=\sup_{m\in\mathbf{N}}\nu(f_m)<\infty$. Since $\Gamma_+(f_{m+1})\subset\Gamma_+(f_m)$, by monotonicity of the Newton number (see for example [6]) we have $\nu(f_m)\leq\nu(f_{m+1})$. Therefore the sequence $\nu(f_m)$ is convergent. Since $\nu(f_m)\in\mathbf{N}$, we get that there exists C such that

$$\nu(f_m)=\nu(f)\quad \text{for } m\geq C. \tag{1}$$

Take $m_1,\dots,m_k\geq C$. Set $m_{\max}:=\max\{m_1,\dots,m_k\}$, $m_{\min}:=\min\{m_1,\dots,m_k\}$. From the inclusion

$$\Gamma_+\left(f+\sum_{i=1}^k z_i^{m_{\max}}\right)\subset\Gamma_+\left(f+\sum_{i=1}^k z_i^{m_i}\right)\subset\Gamma_+\left(f+\sum_{i=1}^k z_i^{m_{\min}}\right),$$

and monotonicity of the Newton number and (1) we infer

$$\nu\left(f+\sum_{i=1}^k z_i^{m_i}\right)=\nu(f). \qquad \blacksquare$$

The next lemma allows us to make f both convenient and non-degenerate.

LEMMA 3.7. *Let $f:(\mathbf{C}^n,0)\to(\mathbf{C},0)$, $n\geq 2$, be a non-degenerate holomorphic function which has a singularity at 0. Assume that $\Gamma_+(f)\cap OX_i=\emptyset$ for some $i\in\{1,2,\dots,n\}$. Then there exists $C\geq 2$ such that $f_i=f+z_i^m$ is non-degenerate for every $m\geq C$.*

Proof. Let $S\in\Gamma(f)$. Since S is compact we can choose $u_S\in(0,\infty)^n$ such that $S=\Delta(u_S,\Gamma_+(f))$ (see Preliminaries). Put

$$W=\bigcup_{S\in\Gamma(f)}\{v\in\mathbf{R}_+^n:\langle u_S,v\rangle\leq l(u_S,\Gamma_+(f))\}.$$

It is easy to check that W is compact and intersects every coordinate axis. Hence we may choose $C\geq 2$ so large that the points in $\mathbf{R}_+^n$ determined by the monomials z_i^m, $m\geq C$, do not lie in W. Let $m\geq C$. We show that

$f_i = f + z_i^m$ is a non-degenerate. Let $P \in \mathbf{R}^n$ be the point determined by the monomial z_i^m. From the choice of C, we observe that P is a vertex of $\Gamma(f_i)$. By nearly-convenience of $\Gamma_+(f)$ there exists a point Q, which is at distance 1 from the axis OX_i. Hence the segment $\overline{PQ}$ is a face of $\Gamma(f_i)$ and $\Gamma^1(f_i) \neq \emptyset$. Therefore we get $\Gamma(f_i) = \Gamma(f) \cup \Delta$, where Δ is the family of the faces in $\Gamma(f_i)$ containing P as a vertex. Since f is non-degenerate, f_i is non-degenerate on each face of $\Gamma(f)$. Now take $\delta \in \Delta \cap \Gamma^k(f_i)$, $k \geq 1$. From the choice of C we get that δ is the convex hull of P and some face $\sigma \in \Gamma(f)$ such that $\dim \sigma < \dim \delta$. Therefore $(f_i)_\delta = z_i^m + f_\sigma$, where f_σ is a quasihomogeneous polynomial. Since $\dim \sigma < n - 1$, the weights of f_σ are not uniquely determined. Let $(f_i)_\delta$ be of type $(w_1, \dots, w_{i-1}, 1/m, w_{i+1}, \dots, w_n; 1)$. Take $m' > m$ and consider the polynomial $f_\sigma + z_i^{m'}$. Let $P' \in \mathbf{R}_+^n$ be the point determined by the monomial $z_i^{m'}$. Since σ is compact and $\sigma \in \Gamma(f)$, the set $\operatorname{conv}(\sigma, P')$ is also compact and there exist positive weights $(w_1', \dots, w_{i-1}', 1/m', w_{i+1}', \dots, w_n'; 1)$ of $f_\sigma + (z_i)^{m'}$. Thus f_σ is simultaneously of the types

$$(w_1, \dots, w_{i-1}, 1/m, w_{i+1}, \dots, w_n; 1) \quad \text{and} \quad (w_1', \dots, w_{i-1}', 1/m', w_{i+1}', \dots, w_n'; 1).$$

Using Euler's formula for these weights we get

$$\sum_{j \neq i} (w_j - w_j') z_j \frac{\partial f_\sigma}{\partial z_j} + \left(\frac{1}{m} - \frac{1}{m'}\right) z_i \frac{\partial f_\sigma}{\partial z_i} = 0 \tag{2}$$

Now we show that f_i is non-degenerate on δ. Suppose to the contrary that there exists $z^0 \in (\mathbf{C}^*)^n$, such that $\nabla (f_i)_\delta(z^0) = 0$. Hence $(f_\sigma)_{z_j}'(z^0) = 0$ for $j \neq i$. By (2) we get also $z_i^0 (f_\sigma)_{z_i}'(z^0) = 0$. Summing up, $\nabla f_\sigma(z^0) = 0$, which contradicts non-degeneracy of f on the face σ. ∎

Now, using induction we extend the previous lemma as follows.

Lemma 3.8. *Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$, $n \geq 2$, be a non-degenerate holomorphic function. Let $I = \{i_1, \dots, i_k\} \subset \{1, 2, \dots, n\}$ be a non-empty subset such that $\Gamma_+(f) \cap OX_i = \emptyset$ for $i \in I$ and $\Gamma_+(f) \cap OX_i \neq \emptyset$ for $i \notin I$. Then for every $C \geq 2$ there exist $m_1, \dots, m_k \geq C$ such that $f_k = f + \sum_{j=1}^k z_{i_j}^{m_j}$ is non-degenerate and convenient.*

Proof. Without loss of generality we may assume that $I = \{1, 2, \dots, k\}$. For every $j = 0, 1, \dots, k$ denote by (A_j) the assertion "For every $C \geq 2$ there exist $m_1, \dots, m_j \geq C$ such that $f_j = f + \sum_{i=1}^j z_i^{m_i}$ is non-degenerate." We show inductively (with respect to j) that (A_j) holds for every $j = 0, 1, \dots, k$. The assertion is true for $j = 0$ by the assumption. Let $j \in \{0, 1, \dots, k\}$. Suppose that (A_j) is true. We show that (A_{j+1}) is also true. Let $C \geq 2$. Since (A_j) is true there exist $m_1, \dots, m_j \geq C$ such that $f_j = f + \sum_{i=1}^j z_i^{m_i}$ is non-degenerate. By Lemma 3.7 there exists $m_{j+1} \geq C$ such that $f_{j+1} = f_j + z_{j+1}^{m_{j+1}}$ is non-degenerate. By induction (A_j) is true for every $j = 0, 1, \dots, k$. In particular, (A_k) is true. ∎

Remark 3.9. We may notice that this way we are able to prove a stronger version of the lemma above, namely it is also true for $f_k = f + \sum_{j=1}^{k} \alpha_j z_{i_j}^{m_j}$ with arbitrary $\alpha_j \neq 0$ (compare [8, Théorème 3.7]).

Proof of Theorem 3.2. If f is convenient, the assertion follows from Proposition 3.5. Suppose that f is not convenient. Let $I = \{i_1, \dots, i_k\} \subset \{1, 2, \dots, n\}$ be a set such that $\Gamma_+(f) \cap OX_i = \emptyset$ for $i \in I$ and $\Gamma_+(f) \cap OX_i \neq \emptyset$ for $i \notin I$. Without loss of generality we may assume that $I = \{1, 2, \dots, k\}$. By Lemma 3.6 there exists $C \geq 2$ such that

$$\nu\left(f + \sum_{i=1}^{k} z_i^{m_i}\right) = \nu(f) \quad \text{for every } m_i \geq C,\ i = 1, \dots, k. \tag{3}$$

By Lemma 3.8 there exist $m_1, \dots, m_k \geq \max\{C, \nu(f) + 1\}$ such that $f_k = f + \sum_{i=1}^{k} z_i^{m_i}$ is non-degenerate and convenient. By Proposition 3.5 we get that f_k has an isolated singularity at 0. Hence by (3) and Théorème I (ii) in [8], we get

$$\operatorname{ord}(f_k - f) = \min_{i=1}^{k} m_i \geq \nu(f) + 1 = \nu(f_k) + 1 = \mu(f_k) + 1. \tag{4}$$

From (4) and since f_k is $\mu(f_k) + 1$ right determined (see for example [2, Section 6.3]) we get that f and f_k are right (biholomorphically) equivalent. This implies f has an isolated singularity at 0 and

$$\mu(f) = \mu(f_k). \tag{5}$$

■

From the proof above we easily get Théorème I (ii) in [8] without the assumption that f is convenient. More precisely, we get the following corollary.

Corollary 3.10. *Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$, $n \geq 1$, be a non-degenerate holomorphic function which defines an isolated singularity at* 0. *Then $\mu(f) = \nu(f)$.*

Proof. If f is a convenient singularity, the assertion follows from Théorème I (ii) in [8]. In the opposite case, repeating the proof of Theorem 3.1 with the same notations and by (5), Théorème I (ii) in [8] and (3) we get

$$\mu(f) = \mu(\hat{f}_k) = \nu(\hat{f}_k) = \nu(f).$$

■

As a direct consequence of Theorem 3.2 and Corollary 3.10, we get the following.

Corollary 3.11. *Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$, $n \geq 1$, be a non-degenerate holomorphic function which has a singularity at* 0. *Then $\mu(f) < \infty \Leftrightarrow \nu(f) < \infty$.*

The next corollary says that Kouchnirenko condition for M is equivalent to the finiteness of the Newton number of λ_M. It was announced already by Kouchnirenko [8, Remarque 1.13 (ii)] but without proof.

Corollary 3.12. *Let* $M \subset \mathbf{N}^n$ *and* $\operatorname{ord} \lambda_M \geq 2$. *A set* M *satisfies the Kouchnirenko condition if and only if* $\nu(\lambda_M)$ *is finite.*

Proof. "$\Rightarrow$" Suppose that M satisfies the Kouchnirenko condition, then by Theorem 2.8 there exists $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$, $\operatorname{supp} f \subset M$, which has an isolated singularity at 0. Hence $\Gamma_+(f) \subset \Gamma_+(\lambda_M)$ and by monotonicity of the Newton number (see for example [6]) we have $\nu(\lambda_M) \leq \nu(f)$. On the other hand, since f has an isolated singularity at 0, we have $\nu(f) \leq \mu(f) < \infty$ by [8, Théorème 1(i)]. Summing up $\nu(\lambda_M)$ is finite.

"$\Leftarrow$" Now, suppose that $\nu(\lambda_M) < \infty$. Then by [8, Théorème 6.1] we may choose non-degenerate f with $\operatorname{supp} f = M$. Then $\nu(f) = \nu(\lambda_M) < \infty$. Therefore by Theorem 3.2 we get that f has an isolated singularity at 0. Hence, by Corollary 2.9 we get the assertion. ∎

Remark 3.13. Gwoździewicz proved monotonicity of the Newton number for convenient function using [8, Théorème I (ii)] and semi-continuity of the Milnor number. It it is easy to generalize his result to the non-convenient case using only definition of the Newton number and simple properties of Newton diagram.

The following lemma has already been announced by Kouchnirenko [8, Subsection 6.5] (without proof).

Lemma 3.14. *Let* $S \subset \mathbf{R}^n$ *be a* d*-dimensional simplex,* $d \leq n-1$, *and let* $\operatorname{vert}(S)$ *be the set of the vertices of* S. *Assume that* $0 \notin \operatorname{aff} S$, *the affine hull of* S. *Then every* $f \in \mathbf{C}[z_1, \ldots, z_n]$ *satisfying* $\operatorname{supp}(f) = \operatorname{vert}(S)$ *is non-degenerate.*

Proof. First we consider the case $d = n-1$. Let $f = \sum_{v \in \operatorname{vert}(S)} a_v z^v$, where $a_v \neq 0$. The system of equations $\{\nabla f = 0\}$ is equivalent to the system

$$\left\{ z_1 \frac{\partial f}{\partial z_1} = \cdots = z_n \frac{\partial f}{\partial z_n} = 0 \right\}$$

in $(\mathbf{C}^*)^n$. Since $z_i \dfrac{\partial f}{\partial z_i} = \sum_{v \in \operatorname{vert}(S)} a_v v_i z^v$, we see that this last system can be viewed as linear in unknowns $\{z^v\}$. This means that it has a non-zero solution in $\mathbf{C}^n$ if and only if $D := \det[a_v v_i]_{\substack{v \in \operatorname{vert}(S) \\ 1 \leq i \leq n}}$ is zero. We have

$$D = \prod_v a_v \cdot \det[v_i].$$

The assumption $0 \notin \operatorname{aff} S$ implies $\dim(\operatorname{span} S) = n$ so that the set $\operatorname{vert}(S)$ consists of n linearly independent vectors. Hence, $\det[v_i] \neq 0$ and also $D \neq 0$ as the a_v are non-zero. This means that the system $\{\nabla f = 0\}$ has no solutions in $(\mathbf{C}^*)^n$. Moreover, every choice of a face σ of the simplex S corresponds to deletion of some columns in the matrix $[a_v v_i]_{v \in \mathrm{vert}(S), 1 \le i \le n}$. Such truncated matrix still has maximal possible rank which implies that f is non-degenerate on σ. Hence, f is non-degenerate.

In the general case, one can extend the d dimensional simplex S to an $(n-1)$-dimensional one and similarly add some missing terms to the function f and in this way return to the first case. ■

Example 3.15. The assumption that $\operatorname{supp} f = \mathrm{vert}(S)$ cannot be omitted in the above lemma. Indeed, take $f(z_1, z_2) = z_1^2 + 2z_1z_2 + z_2^2$. Observe that $\operatorname{supp} f \neq \mathrm{vert}(S)$ and $f = (z_1 + z_2)^2$ is degenerate.

Also the assumption that S is a simplex cannot be omitted. Indeed, take

$$f(z_1, z_2, z_3, z_4) = z_1^2 - z_2^2 + z_1z_3^2 + z_2z_3^2.$$

We may observe that S spanned by $\operatorname{supp} f$ is not a simplex and $\operatorname{supp} f = \mathrm{vert}(S)$. Take $\phi(t) = (-t^2/2, t^2/2, t)$. Then $(\nabla f) \circ \phi = 0$, so f is degenerate.

From Lemma 3.14 and Theorem 3.1 we immediately get the following.

Corollary 3.16. *Let* $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ *be a holomorphic function which has a singularity at* 0. *Assume that all the faces* $S \in \Gamma(f)$ *are simplices and* $\operatorname{supp} f_S = \mathrm{vert}(\sigma)$, *the set of vertices of S. If* $\operatorname{supp} f$ *satisfies the Kouchnirenko condition, then* f *has an isolated singularity at* 0.

Example 3.17. Let $f(z_1, z_2, z_3) = z_1^6z_3 + z_2^4 + z_3^{12}z_2 + z_1z_3^2 + z_2z_3^4$. It is easy to check that all the faces $S \in \Gamma(f)$ are simplices and $\operatorname{supp} f_S = \mathrm{vert}(S)$. Moreover, it is easy to verify that $\operatorname{supp} f$ satisfies the Kouchnirenko condition. Hence by the corollary above we infer that f has an isolated singularity at 0. Observe that f is not convenient.

4. Appendix

Now, we find the constant C in Lemma 3.6. Namely, we prove that $C \le Ł_0(f) + 2$ (Proposition 4.2). First we give some definitions and theorems.

Let $F = (f_1, \ldots, f_n) : (\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ be a holomorphic mapping having an isolated zero at the origin. We define the number

(6) $$l_0(F) := \inf\{\alpha \in \mathbf{R}_+ : \exists C > 0\ \exists r > 0\ \forall \|z\| < r\ \|F(z)\| \ge C\|z\|^\alpha\}$$

Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be a holomorphic function which has an isolated singularity at 0. We define the number $Ł_0(f) = l_0(\nabla f)$ and we call it the *Łojasiewicz exponent* f.

First we recall the following.

Lemma 4.1 ([17, Lemma 1.4]). *Let* $F, G : (\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ *be holomorphic mappings in some neighbourhood of* $0 \in \mathbf{C}^n$. *Suppose that F has an isolated zero.*

If $\operatorname{ord}(G - F) > l_0(F)$, *then* G *has an isolated zero and*

$$l_0(G) = l_0(F), \quad i_0(G) = i_0(F),$$

where $i_0(F)$ *denotes the multiplicity* F *at* $0 \in \mathbf{C}^n$.

Now, we give a constructive version of Lemma 3.6.

PROPOSITION 4.2. *Let* $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$, $n \geq 2$ *be a holomorphic function which has an isolated singularity at* 0. *Let* $I \subset \{1, 2, \ldots, n\}$ *be a non-empty set such that* $\Gamma_+(f) \cap OX_i = \emptyset$ *for* $i \in I$ *and* $\Gamma_+(f) \cap OX_i \neq \emptyset$ *for* $i \notin I$. *Let* $m_i \geq [\text{£}_0(f)] + 2$, $i \in I$. *Then*

$$\nu\left(f + \sum_{i \in I} z_i^{m_i}\right) = \nu(f) \tag{7}$$

$$\mu\left(f + \sum_{i \in I} z_i^{m_i}\right) = \mu(f) \tag{8}$$

$$\text{£}_0\left(f + \sum_{i \in I} z_i^{m_i}\right) = \text{£}_0(f) \tag{9}$$

Proof. Let $m_i \geq [\text{£}_0(f)] + 2$, $i \in I$ and $f_k = f + \sum_{i \in I} z_i^{m_i}$. We begin with the proof of (8) and (9). We get

$$\operatorname{ord}(\nabla f_k - \nabla f) \geq [\text{£}_0(f)] + 1 > \text{£}_0(f).$$

Hence by Lemma 4.1 we have $\text{£}_0(f_k) = \text{£}_0(f)$ and $\mu(f_k) = \mu(f)$.

Now we pass to the proof of (7). Since the Kouchnirenko non-degeneracy is a Zariski open condition (see for example [13, Appendix]), we may choose non-degenerate $\hat{f}$ with $\operatorname{supp} \hat{f} = \operatorname{supp} f$. Since f has an isolated singularity at 0, by Corollary 2.9 we get that $\operatorname{supp} f$ satisfies the Kouchnirenko condition. Hence $\operatorname{supp} \hat{f}$ also satisfies the Kouchnirenko condition. Therefore by Theorem 3.1 we get that $\hat{f}$ has an isolated singularity at 0. Since the Kouchnirenko non-degeneracy is a Zariski open condition we choose generic $\alpha_i \neq 0$ such that $\hat{f}_k = \hat{f} + \sum_{i \in I} \alpha_i z_i^{m_i}$ is non-degenerate. We have

$$\operatorname{ord}(\nabla \hat{f}_k - \nabla \hat{f}) \geq [\text{£}_0(\hat{f})] + 1 > \text{£}_0(\hat{f}).$$

Hence by Lemma 4.1 we have $\mu(\hat{f}_k) = \mu(\hat{f})$. Summing up by Corollary 3.10, we get

$$\nu\left(f + \sum_{i \in I} z_i^{m_i}\right) = \nu(\hat{f}_k) = \mu(\hat{f}_k) = \mu(\hat{f}) = \nu(\hat{f}) = \nu(f). \qquad \blacksquare$$

Example 4.3. The following example shows that in some cases the bound $[\text{£}_0(f)] + 2$ for m_i of Proposition 4.2 is the least possible. Indeed, take

$f(z_1, z_2) = z_2^6 + z_1^3 z_2^3 + z_1^6 z_2$. One may check that f is non-degenerate and f has an isolated singularity at 0. Using the main result of [11] we calculate $\text{Ł}_0(f) = 6.5$ and by Corollary 3.10 we get $\mu(f) = \nu(f) = 28$. Hence $[\text{Ł}_0(f)] + 2 = 8$. Now take $\hat{f} = f + z_1^7$. Using the same techniques one may calculate $\text{Ł}_0(\hat{f}) = 6$, $\mu(\hat{f}) = \nu(\hat{f}) = 27$.

Example 4.4. The following example shows that in some cases the bound $[\text{Ł}_0(f)] + 2$ for m_i of Proposition 4.2 is not the least possible. Indeed, take $f(z_1, z_2) = z_1^8 + z_1^2 z_2^2 + z_1 z_2^3$. One may check that f is non-degenerate and f has an isolated singularity at 0. Using the main result of [11] we calculate $\text{Ł}_0(f) = 7$ and by Corollary 3.10 we get $\mu(f) = \nu(f) = 13$. Hence $[\text{Ł}_0(f)] + 2 = 9$. Now take $f_N = f + z_2^N$, $N \geq 5$. Using the same techniques one may calculate $\text{Ł}_0(f_N) = 7$, $\mu(f_N) = \nu(f_N) = 13$ for every $N \geq 5$.

Remark 4.5. For non-degenerate functions Fukui [5] gave the inequality

$$\text{Ł}_0(f) \leq m_0(f) - 1,$$

where $m_0(f)$ is a combinatorial number calculated from $\Gamma_+(f)$. Also Abderrahmane [1] gave a similar estimation in terms of $\nu(f)$. In the case $n = 2, 3$ there are more exact formula for the Łojasiewicz exponent of non-degenerate holomorphic functions (see [11] and [14]).

Remark 4.6. One may also give an estimation of the constant C in Lemmas 3.7 and 3.8 in terms of $m_0(f)$. Kouchnirenko [8, Théorème 3.7] also gave an estimation of the constant C in generic version of Lemma 3.8, but his estimation is in general too large.

Acknowledgements. We would like to thank the anonymous referee for valuable comments.

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