Y. WU AND Y. QI
KODAI MATH. J.
39 (2016), 410–424

DOUADY-EARLE EXTENSION OF THE STRONGLY SYMMETRIC HOMEOMORPHISM

Yan Wu and Yi Qi

Abstract

It is shown that the complex dilatation of the Douady-Earle extension of a strongly symmetric homeomorphism induces a vanishing Carleson measure on the unit disk **D**. As application, it is proved that the VMO-Teichmüller space is a subgroup of the universal Teichmüller space.

§1. Introduction

Let $\mathbf{D} = \{z : |z| < 1\}$ be the unit disk of the extended complex plane $\hat{\mathbf{C}}$ and let $\mathbf{D}^* = \hat{\mathbf{C}} \backslash \overline{\mathbf{D}}$ be the exterior of $\mathbf{D}$ and $S^1 = \partial \mathbf{D} = \partial \mathbf{D}^*$ be the unit circle.

A sense-preserving homeomorphism $h : S^1 \to S^1$ is said to be quasisymmetric if there exists some constant $M > 0$ such that

$$\frac{1}{M} \le \frac{|h(I_1)|}{|h(I_2)|} \le M$$

for all pairs of adjacent arcs I_1 and I_2 on S^1 with the same arc-length $|I_1| = |I_2| (\le \pi)$. It is well known in [4] that a sense-preserving self-homeomorphism h is quasisymmetric if and only if there exists some quasiconformal homeomorphism of $\mathbf{D}$ onto itself which has boundary values h.

Let $\mathrm{QS}(S^1)$ be the set of all quasisymmetric homeomorphisms of the unit circle S^1. Then $\mathrm{QS}(S^1)$ is a group under the composition of homeomorphisms. The universal Teichmüller space T is defined as

$$T = \mathrm{QS}(S^1)/\mathrm{M\ddot{o}b}(S^1),$$

where $\mathrm{M\ddot{o}b}(S^1)$ is the group of Möbius transformations of S^1. It is well known that the universal Teichmüller space plays a significant role in the study of Teichmüller theory. For more details we refer to the books [12, 13, 16, 18].

2010 *Mathematics Subject Classification*. Primary 30F60; Secondary 32G15.

Key words and phrases. Douady-Earle extension; strongly symmetric homeomorphism; VMO-Teichmüller space; Carleson measure.

The research is partially supported by the National Natural Science Foundation of China (Grant No. 11371045, 11301248).

Received July 9, 2015; revised October 13, 2015.

For every $h \in \mathrm{QS}(S^1)$, it is proved in [9] that there exists a quasiconformal extension of h to the unit disk, called the Douady-Earle extension, which is conformally invariant, that is,

$$E(\alpha \circ h \circ \beta) = \alpha \circ E(h) \circ \beta$$

holds for any $\alpha, \beta \in \mathrm{M\ddot{o}b}(S^1)$. Douady-Earle extension is very important in Teichmüller theory, which provides a great convenience to discuss Teichmüller spaces of Riemann surfaces on the unit disk, for instance.

A quasisymmetric homeomorphism h of S^1 is called integrably asymptotic affine [7] if it admits a quasiconformal extension into $\mathbf{D}$ such that its complex dilatation μ is square integrable in the Poincaré metric on $\mathbf{D}$, that is

$$\iint_{\mathbf{D}} \frac{|\mu(z)|^2}{(1-|z|^2)^2}\, dxdy < \infty.$$

It is proved in [7] that the complex dilatation of the Douady-Earle extension of an integrably asymptotic affine homeomorphism h is square integrable in the Poincaré metric on $\mathbf{D}$.

An asymptotically conformal mapping f of $\mathbf{D}$ is a quasiconformal homeomorphism of $\mathbf{D}$ with complex dilatation μ satisfying

$$\lim_{|z| \to 1^-} |\mu(z)| = 0.$$

A quasisymmetric homeomorphism h of S^1 is called symmetric if it admits an asymptotically conformal extension on $\mathbf{D}$. It is proved in [11] that the Douady-Earle extension of a symmetric homeomorphism is asymptotically conformal.

A quasisymmetric homeomorphism h of S^1 is said to be strongly quasisymmetric if for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|E| \le \delta |I| \Rightarrow |h(E)| \le \epsilon |h(I)|$$

where $I \subset S^1$ is an interval and $E \subset I$ is a measurable subset. It is equivalent to that [3] h admits a quasiconformal extension into $\mathbf{D}$ which complex dilatation μ induces a Carleson measure $|\mu(z)|^2/(1-|z|^2)\, dxdy$ on $\mathbf{D}$. It is shown in [8] that the complex dilatation of the Douady-Earle extension of a strongly quasisymmetric homeomorphism induces a Carleson measure. Furthermore, h is strongly quasisymmetric if and only if h is absolutely continuous and $\log h' \in \mathrm{BMO}(S^1)$, the space of integrable functions on S^1 of bounded mean oscillation (see [6, 10, 14, 20]).

A quasisymmetric homeomorphism h of S^1 is called strongly symmetric if h is absolutely continuous and $\log h' \in \mathrm{VMO}(S^1)$, the space of integrable functions on S^1 of vanishing mean oscillation (see [14, 20, 21]). The BMO-Teichmüller space and VMO-Teichmüller space are defined as the following models

$$T_b = \mathrm{SQS}(S^1)/\mathrm{M\ddot{o}b}(S^1) \quad \text{and} \quad T_v = \mathrm{SS}(S^1)/\mathrm{M\ddot{o}b}(S^1),$$

where $\mathrm{SQS}(S^1)$ and $\mathrm{SS}(S^1)$ are the sets of all strongly quasisymmetric and all strongly symmetric homeomorphisms of the unit circle S^1 respectively. The

BMO-Teichmüller space and VMO-Teichmüller space are two important subspaces of the universal Teichmüller space which are fully studied [1, 3, 5, 8, 23].

The purpose of this paper is to study the Douady-Earle extensions of strongly symmetric homeomorphisms. It is obtained that h is a strongly symmetric homeomorphism if and only if h admits a quasiconformal extension into $\mathbf{D}$ which complex dilatation μ induces a vanishing Carleson measure $|\mu(z)|^2/(1-|z|^2)\,dxdy$ on $\mathbf{D}$. Moreover, it is proved that the complex dilatation of the Douady-Earle extension of h properly induces this vanishing Carleson measure. As application, it is gotten that the VMO-Teichmüller space T_v is a subgroup of the universal Teichmüller space T.

§2. Preliminaries

In this section, we recall some notions and basic results on BMO-functions, A_∞ weight functions and Carleson measures which will be needed in this paper. For more details we refer to [6, 10, 14].

$\mathrm{BMO}(S^1)$ is the space of all integrable functions on S^1 of bounded mean oscillation (see [6, 10, 14, 20]). An integrable function $u \in L^1(S^1)$ is said to be of bounded mean oscillation if

$$\|u\|_{\mathrm{BMO}} = \sup_I \frac{1}{|I|}\int_I |u-u_I|\,d\theta < \infty,$$

where I is any arc on S^1, $|I|$ is the length of I and $u_I = \frac{1}{|I|}\int_I u\,d\theta$ is the average of u over I. $\mathrm{VMO}(S^1)$ is the subspace of $\mathrm{BMO}(S^1)$ which consists of all vanishing mean oscillation functions. A function $u \in \mathrm{BMO}(S^1)$ is said to be of vanishing mean oscillation if

$$\lim_{|I|\to 0} \frac{1}{|I|}\int_I |u-u_I|\,d\theta = 0.$$

Let $\mu = \omega(x)\,dx$ be a positive Borel measure on $\mathbf{R}$, finite on compact sets. $\omega(x)$ is called an A_∞ weight function [14], denoted by $\omega \in A_\infty$, if

$$\mu(E)/\mu(I) \le C(|E|/|I|)^\alpha$$

holds for any interval I and any Borel subset E of I, where $C > 0$ and $\alpha > 0$ are constants independent of E and I. Let $h \in \mathrm{SS}(S^1)$, then h is strongly quasisymmetric, and consequently $h' \in A_\infty$ (see [14]).

For every $\omega \in A_\infty$, it holds the reverse Hölder inequality [6]. So there exists a constant $c > 0$ and $p > 1$ such that

$$\frac{1}{|I|}\int_I \omega^p(x)\,dx \le c\left(\frac{1}{|I|}\int_I \omega(x)\,dx\right)^p. \tag{2.1}$$

for every interval I in $\mathbf{R}$.

The Carleson sector $S(I)$, based on I, is defined by

$$S(I) = \left\{ z = re^{i\theta} : 1 - \frac{|I|}{2\pi} \le r < 1, e^{i\theta} \in I \right\}.$$

A positive Borel measure λ on $\mathbf{D}$ is called a bounded Carleson measure if there exists a positive constant C such that

$$\lambda(S(I)) \le C|I|$$

We say that λ is a vanishing Carleson measure if

$$\lambda(S(I)) = o(|I|), \quad |I| \to 0.$$

For a positive measure λ on $\mathbf{D}^*$, replacing $S(I)$ in the above definition by the following Carleson sector:

$$S^*(I) = \left\{ z = re^{i\theta} : 1 < r \le 1 + \frac{|I|}{2\pi}, e^{i\theta} \in I \right\},$$

We similarly obtain the definition of a bounded or vanishing Carleson measure on $\mathbf{D}^*$. Denote by $CM(\Omega)$ and $CM_0(\Omega)$ the set of all bounded Carleson measures and vanishing Carleson measures on Ω, respectively.

We need a lemma in [23] for Carleson measure.

Lemma 2.1. *For a positive measure λ on $\mathbf{D}$, set*

$$\tilde{\lambda}(z) = \iint_{\mathbf{D}} \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{w}z|^4} \lambda(w)\, dudv$$

Then $\tilde{\lambda}$ is a bounded or vanishing Carleson measure if λ is a bounded or vanishing Carleson measure on $\mathbf{D}$.

The Douady-Earle extension $w = E(h)(z)$ is defined by the equation

$$F(z, w) = \frac{1}{2\pi} \int_{S^1} \frac{h(t) - w}{1 - \overline{w}h(t)} \frac{1 - |z|^2}{|z - t|^2} |dt| = 0.$$

For $h \in \mathrm{QS}(S^1)$, let $v(h)$ denote the Beltrami coefficient of the inverse mapping of the Douady-Earle extension $E(h)$, and v denote the Beltrami coefficient of a quasiconformal extension of h^{-1}. Then we have the following result (for details, see [15]).

Lemma 2.2. *There exists a constant $C(h)$ such that $\forall w \in \mathbf{D}$*

$$\frac{|v(h)(w)|^2}{1 - |v(h)(w)|^2} \le C(h) \iint_{\mathrm{D}} \frac{|v(\zeta)|^2}{1 - |v(\zeta)|^2} \frac{(1 - |w|^2)^2}{|1 - \bar{\zeta}w|^4} \, d\xi d\eta$$

§3. Douady-Earle extension of a strongly symmetric homeomorphism

Recall that for any $h \in \mathrm{QS}(S^1)$, there exists a unique pair of conformal mappings $f : \mathbf{D} \to f(\mathbf{D})$ and $g : \mathbf{D}^* \to \hat{\mathbf{C}} \backslash \overline{f(\mathbf{D})}$, called the normalized decomposition of h, satisfying $f(0) = f'(0) - 1 = 0$, $g(\infty) = \infty$ and $h = f^{-1} \circ g$ on S^1, respectively. Furthermore, f can be extended to a quasiconformal mapping in the whole plane with Beltami coefficient μ_f. At the same time, h is called the normalized conformal welding mapping of f. It is known that $h \in \mathrm{QS}(S^1)$ if and only if $h^{-1} \in \mathrm{QS}(S^1)$. For $h \in \mathrm{SS}(S^1)$, we have

Proposition 3.1. *For any $h \in \mathrm{QS}(S^1)$, f, g are the above normalized decomposition of h. The following conditions are equivalent*:

(1) $h \in \mathrm{SS}(S^1)$;

(2) $h^{-1} \in \mathrm{SS}(S^1)$;

(3) *There exists a quasiconformal extension $\psi(z) : \mathbf{D} \to \mathbf{D}$ of h^{-1} whose Beltrami coefficient μ induces a vanishing Carleson measure $|\mu(z)|^2/(1 - |z|^2)\, dxdy$ on $\mathbf{D}$.*

Proof. It should be pointed out that (1) $\Leftrightarrow$ (2) is implied in [23]. For completeness, we give the proof here.

Suppose that $h \in \mathrm{SS}(S^1)$ and $h = f^{-1} \circ g$, where f, g are the normalized decomposition of h. Then $\log f' \in \mathrm{VMOA}(\mathbf{D})$, the space of analytic functions in $\mathbf{D}$ of vanishing mean oscillation (see Theorem 4.1 in [23]). It is known that $\log f' \in \mathrm{VMOA}(\mathbf{D})$ if and only if the quasicircle $\Gamma = f(S^1) = g(S^1)$ is asymptotically smooth (see Section 7.5 in [20]). Furthermore, we have $h^{-1} = g^{-1} \circ f = (rj \circ g \circ j)^{-1} \circ (rj \circ f \circ j)$, where $j(z) = \bar{z}^{-1}$ is the standard reflection of the unit circle S^1 and r is a constant such that $r(j \circ g \circ j)'(0) = 1$. So $rj \circ g \circ j$, $rj \circ f \circ j$ are the normalized decomposition of h^{-1}. Since Γ is asymptotically smooth, then $rj \circ g \circ j(S^1) = rj(\Gamma)$ is also asymptotically smooth. This means $h^{-1} \in \mathrm{SS}(S^1)$ and (1) $\Rightarrow$ (2). With similar discussion, (2) $\Rightarrow$ (1).

Now we show that (1) $\Leftrightarrow$ (3). It is known that $h \in \mathrm{SS}(S^1)$ if and only if f can be extended to a quasiconformal mapping to the whole plane, denoted also by f, whose complex dilatation μ_f satisfying $|\mu_f(z)|^2/(|z|^2 - 1)\, dxdy \in CM_0(\mathbf{D}^*)$ [23]. Defining $\varphi(z) = g^{-1} \circ f(z)$, $z \in \mathrm{D}^*$, then $\varphi(z)$ is the quasiconformal extension of h^{-1} to D^* with Beltrami coefficient $v(z) = \mu_f(z)$ and $|v(z)|^2/(|z|^2 - 1)\, dxdy \in CM_0(\mathbf{D}^*)$. By reflection, h^{-1} may be extended to a quasiconformal mapping $\psi(z)$ to $\mathbf{D}$ whose Beltrami coefficient $\mu(z)$ satisfies

$$\mu(z) = \overline{v\left(\frac{1}{\bar{z}}\right)} \frac{z^2}{\bar{z}^2}, \quad z \in \mathbf{D}.$$

For any subarc $I \in S^1(|I| \le \pi)$, let $2I$ be the subarc of S^1 with the same center of I, $|2I| = 2|I|$ and $z \in S(I)$. Then, by simple calculation, we get

$$\iint_{S(I)} \frac{|\mu(z)|^2}{1 - |z|^2}\, dxdy = \iint_{S'(I)} \frac{|v(w)|^2}{|w|^2 - 1} \frac{1}{|w|^2}\, dudv \le \iint_{S^*(2I)} \frac{|v(w)|^2}{|w|^2 - 1}\, dudv$$

where $S'(I)$ is the refection sector of $S(I)$, $S^*(2I) \subset \mathbf{D}^*$ is the Carleson sector over $2I$ on $\mathbf{D}^*$ and $S'(I) \subset S^*(2I)$.

For any given $\varepsilon > 0$, since $|v(w)|^2/(|w|^2 - 1)\, dudv \in CM_0(\mathbf{D}^*)$, there exists a $\delta > 0$ such that

$$\iint_{S^*(2I)} \frac{|v(w)|^2}{|w|^2 - 1}\, dudv < 2\varepsilon|I|$$

holds for every subarc $I \subset S^1$ with $|I| \leq \delta$. So $|\mu(z)|^2/(1 - |z|^2)\, dxdy \in CM_0(\mathbf{D})$ and $(1) \Rightarrow (3)$.

Conversely, if condition (3) holds, by quasiconformal reflection, there exists a quasiconformal extension $\phi(z) : \mathbf{D}^* \to \mathbf{D}^*$ of h^{-1} with Beltrami coefficient $\mu_\phi(z)$ satisfying $|\mu_\phi(z)|^2/(|z|^2 - 1)\, dxdy \in CM_0(\mathbf{D}^*)$. Let $\tilde{f} = g \circ \phi$, it is easy to see that $\tilde{f}$ is the quasiconformal extension of f and $|\mu_{\tilde{f}}(z)|^2/(|z|^2 - 1)\, dxdy \in CM_0(\mathbf{D}^*)$. Thus $(3) \Rightarrow (1)$. $\square$

Now we prove that the complex dilatation of the Douady-Earle extension of a strongly symmetric homeomorphism induces a vanishing Carleson measure on $\mathbf{D}$.

THEOREM 3.1. *If $h \in SS(S^1)$, that is, h is a strongly symmetric homeomorphism on S^1. Let μ be the complex dilatation of the Douady-Earle extension $\Phi = E(h)$. Then it holds that $|\mu(z)|^2/(1 - |z|^2)\, dxdy \in CM_0(\mathbf{D})$.*

In order to prove Theorem 3.1, we need some preparations.

Set $\zeta_k = e^{2k\pi i/3}$ $(k = 1, 2, 3)$. For every $w \in \mathbf{D}$, let τ be the Möbius transformation of $\mathbf{D}$ onto itself with $\tau(0) = w$ and $\tau(\zeta_2) = w/|w|$. Denote $w_k = \tau(\zeta_k)$ $(k = 1, 2, 3)$ and let J_w be the subarc of S^1 with endpoints w_1 and w_3 and containing w_2. Then we have the following lemma.

LEMMA 3.1. *Let h be a symmetric homeomorphism of S^1 and Φ be the Douady-Earle extension of h, then there exist positive constants C_1 and C_2 depending only on h, such that*

$$2(1 - |w|) \leq |J_w| \leq 2\pi(1 - |w|), \tag{3.1}$$

$$\frac{1}{C_1}\frac{|h^{-1}(J_w)|}{|J_w|} \leq \frac{1 - |\Phi^{-1}(w)|^2}{1 - |w|^2} \leq C_1\frac{|h^{-1}(J_w)|}{|J_w|} \tag{3.2}$$

and

$$\frac{(1 - |w|^2)^2}{(1 - |\Phi^{-1}(w)|^2)^2} J_{\Phi^{-1}}(w) \leq C_2. \tag{3.3}$$

Proof. Since Φ is the Douady-Earle extension of h, it is bi-Lipschitz with respect to the Poincaré metric and the Lipschitz constant $C = C(K)$ depends only on the maximal dilatation $K = K_\Phi$ of Φ [9]. Hence, Φ^{-1} is also bi-Lipschitz with respect to the Poincaré metric with the same Lipschitz constant $C = C(K)$. So,

$$\frac{1}{C(K)}\rho(w)|dw| \le \rho(\Phi^{-1}(w))|d\Phi^{-1}(w)| \le C(K)\rho(w)|dw|,$$

which implies (3.3) with $C_2 = C(K)^2$ directly.

Let $z_k = h^{-1}(w_k)$ $(k = 1, 2, 3)$ and σ be the Möbius transformation of $\mathbf{D}$ onto itself with $\sigma(\zeta_k) = z_k$ $(k = 1, 2, 3)$. Set $\Phi^* = \tau^{-1} \circ \Phi \circ \sigma$. Then Φ^* is the Douady-Earle extension of the sense-preserving quasisymmetric $\Phi^*|_{S^1} = \tau^{-1} \circ h \circ \sigma$ and can be extended to a $K = K_\Phi$-quasiconformal mapping of $\mathbf{C}$ onto itself by reflection. Thus, $\Phi^*|_{S^1}$ is η_K-quasisymmetric by Corollary 3.10.4 in [2], where

$$\eta_K(t) = \lambda(K)^{2K} \max\{t^K, t^{1/K}\}, \quad t \in [0, +\infty)$$

and

$$\lambda(K) = \sup\{|f(e^{i\theta})| : f : \mathbf{C} \to \mathbf{C} \text{ is } K\text{-q.c. and fixes } 0,\ 1, 0 \le \theta \le 2\pi\}. \tag{3.4}$$

Therefore, by Proposition 5.21 in [20], there exists a constant $r' \in (0, 1)$ which depends only on K but not on w, such that $|\Phi^*(0)| \le r' < 1$.

As Φ^* is the Douady-Earle extension of the sense-preserving quasisymmetric $\Phi^*|_{S^1} = \tau^{-1} \circ h \circ \sigma$, it is bi-Lipschitz with respect to the Poincaré metric, where the Lipschitz constant $C(K) \ge 1$ depends only on K [9]. Thus,

$$\log \frac{1 + |\Phi^{*-1}(0)|}{1 - |\Phi^{*-1}(0)|} \le C(K) \log \frac{1 + |\Phi^*(0)|}{1 - |\Phi^*(0)|}$$

This implies that

$$|\Phi^{*-1}(0)| \le r_0 < 1, \tag{3.5}$$

where r_0 is a constant depending only on K but not on the choice of w.

It is easy to see that $\tau(\zeta) = (\zeta + e^{i\alpha}w)/(e^{i\alpha} + \zeta\overline{w})$, where $\alpha = \dfrac{4\pi}{3} - \theta$ and θ is the argument of w. By a simple computation, we have

$$|w_1 - w_2| = \frac{\sqrt{3}(1 - |w|)}{|\zeta_1 + |w|\,|}, \quad |w_2 - w_3| = \frac{\sqrt{3}(1 - |w|)}{|\zeta_2 + |w|\,|},$$

and

$$|w_1 - w_3| = \frac{\sqrt{3}(1 + |w|)(1 - |w|)}{|\zeta_2 + |w|\,|\,|\zeta_1 + |w|\,|}.$$

Consequently, it is gotten that $|w_1 - w_2| = |w_2 - w_3|$ and

$$1 - |w| \le |w_1 - w_2| \le 2(1 - |w|).$$

So, $|w_1 - w_2|$, $|w_2 - w_3|$, $|w_1 - w_3|$ are all comparable with $1 - |w|$ and the constants appeared in the comparisons are universal, and

$$|J_w| \ge |w_1 - w_2| + |w_2 - w_3| \ge 2(1 - |w|).$$

By Jordan inequality,

$$|J_w| = 2|\widehat{w_1 w_2}| \le \pi |w_1 - w_2| \le 2\pi(1 - |w|).$$

Thus, (3.1) is true.

We now prove that $|z_1 - z_2|$, $|z_2 - z_3|$ and $|z_3 - z_1|$ are all comparable with $1 - |\Phi^{-1}(w)|$ and the constants appeared in the comparisons depend only on $K = K_\phi$.

Let $z = \Phi^{-1}(w)$ and let $\zeta' \in S^1$ such that $\sigma(\zeta') = z/|z|$. Set

$$\sigma(\zeta) = e^{i\beta} \frac{\zeta - a}{1 - \bar{a}\zeta}, \quad \zeta \in \mathbf{D},$$

where $a \in \mathbf{D}$ and $\beta \in \mathbf{R}$ are constants determined by σ. Then

$$\frac{|z_i - z_j|}{1 - |z|} = \frac{|\sigma(\zeta_i) - \sigma(\zeta_j)|}{|\sigma(\zeta') - \sigma(\Phi^{*-1}(0))|} = \frac{|\zeta_i - \zeta_j|}{|\zeta' - \Phi^{*-1}(0)|} \frac{|1 - \bar{a}\zeta'|\,|1 - \bar{a}\Phi^{*-1}(0)|}{|1 - \bar{a}\zeta_i|\,|1 - \bar{a}\zeta_j|} \tag{3.6}$$

for $1 \le i < j \le 3$. If $\arg a \in [-\pi/3, \pi/3)$, then

$$|1 - \bar{a}\zeta_1| \ge \sqrt{3}/2 \quad \text{and} \quad |1 - \bar{a}\zeta_2| \ge \sqrt{3}/2.$$

Thus, by (3.5) and (3.6),

$$\frac{|z_1 - z_2|}{1 - |z|} \le \frac{\sqrt{3}}{1 - r_0} \cdot \frac{16}{3}. \tag{3.7}$$

Similarly, if $\arg a \in [\pi/3, \pi)$ or $[\pi, 5\pi/3)$, (3.7) is also true for replacing $|z_1 - z_2|$ by $|z_1 - z_3|$ or $|z_2 - z_3|$, respectively.

On the other hand,

$$\frac{1 - |z|}{|z_i - z_j|} \le \frac{|z_i - z|}{|z_i - z_j|} = \frac{|\zeta_i - \Phi^{*-1}(0)|}{|\zeta_i - \zeta_j|} \frac{|1 - \bar{a}\zeta_j|}{|1 - \bar{a}\Phi^{*-1}(0)|} \le \frac{4}{\sqrt{3}} \frac{1}{1 - r_0}$$

for $1 \le i < j \le 3$. Since h is a symmetric homeomorphism and $|w_1 - w_2| = |w_2 - w_3|$, then $|z_1 - z_2|$, $|z_2 - z_3|$ and $|z_3 - z_1|$ can be compared with each other and the constants in the comparisons depend only on K. Thus, all these three quantities are all comparable with $1 - |z|$ and constants in the comparisons depend only on $r_0 = r_0(K)$ but independent on w.

Therefore, there exists a constant $C \ge 1$ depending only on K such that

$$\frac{1}{C} \frac{|h^{-1}(w_1) - h^{-1}(w_3)|}{|w_1 - w_3|} \le \frac{1 - |z|}{1 - |w|} \le C \frac{|h^{-1}(w_1) - h^{-1}(w_3)|}{|w_1 - w_3|},$$

which implies (3.2) directly. The proof of Lemma 3.1 is completed. □

Now we prove the Theorem 3.1.

Proof. For every $h \in \mathrm{SS}(S^1)$, by proposition 3.1, there exists a quasiconformal extension g of h^{-1} satisfying $|\mu_g(z)|^2/(1 - |z|^2)\, dxdy \in CM_0(\mathbf{D})$. Let v

denote the Beltrami coefficient of the inverse mapping Φ^{-1} of the Douady-Earle extension Φ. By Lemma 2.2, there exists a constant $C(h)$ such that $\forall w \in \mathbf{D}$

$$\frac{|v(w)|^2}{1-|v(w)|^2} \leq C(h) \iint_{\mathrm{D}} \frac{|\mu_g(\zeta)|^2}{1-|\mu_g(\zeta)|^2} \frac{(1-|w|^2)^2}{|1-\bar{\zeta}w|^4}\, d\xi d\eta$$

Furthermore,

$$\begin{aligned}\frac{|v(w)|^2}{1-|w|^2} &\leq C(h) \iint_{\mathrm{D}} \frac{1-|v(w)|^2}{1-|\mu_g(\zeta)|^2} \frac{|\mu_g(\zeta)|^2}{1-|\zeta|^2} \frac{(1-|w|^2)(1-|\zeta|^2)}{|1-\bar{\zeta}w|^4}\, d\xi d\eta \\ &\leq \frac{C(h)}{1-\|\mu_g\|_\infty^2} \iint_{\mathrm{D}} \frac{|\mu_g(\zeta)|^2}{1-|\zeta|^2} \frac{(1-|w|^2)(1-|\zeta|^2)}{|1-\bar{\zeta}w|^4}\, d\xi d\eta\end{aligned}$$

It follows from Lemma 2.1 that $|v(w)|^2/(1-|w|^2)\, dudv \in CM_0(\mathbf{D})$. In what follows we prove that $|v(w)|^2/(1-|w|^2)\, dudv \in CM_0(\mathbf{D})$ implies $|\mu(z)|^2/(1-|z|^2)\, dxdy \in CM_0(\mathbf{D})$.

Since $h \in \mathrm{SS}(S^1)$, h is a symmetric homeomorphism [22], namely,

$$\frac{|h(I_1)|}{|h(I_2)|} = 1 + o(1)$$

holds for every pair of adjacent subarcs I_1 and I_2 in $[0, 2\pi]$ with $|I_1| = |I_2| \to 0_+$.

For every $I \subset S^1$, set $I = I_1 + I_1'$ and $2I = I_2 + I_1 + I_1' + I_2'$, where I_2, I_1, I_1', I_2' are adjacent subarcs with $|I_1| = |I_1'| = |I_2| = |I_2'|$. Then we have

$$|h(I_1 + I_2)| = 2|h(I_1)| + o(1) = |h(I)| + o(1)$$

and

$$|h(I_1' + I_2')| = 2|h(I_1')| + o(1) = |h(I)| + o(1)$$

as $|I| \to 0_+$. Thus,

$$\frac{|h(2I)|}{|h(I)|} = 2 + o(1), \quad |I| \to 0_+.$$

Furthermore, for a positive integer $N > 1$, it is not hard to verify that

$$\frac{|h(NI)|}{|h(I)|} = N + o(1), \quad |I| \to 0_+, \tag{3.8}$$

where I and NI are the subarcs of S^1 with the same center and $|NI| = N|I|$.

Let z_0 be the center of I and let $D(2I)$ be the disk centered at z_0 and $D(2I) \cap \partial\mathbf{D} = 2I$. It is easy to verify that the Carleson sector $S(I) \subset D(2I)$ for every I with $|I| < \pi$. By reflections and pre-compositing a Teichmüller shift [24] (A Teichmüller shift mapping on the unit disk $\mathbf{D}$ is the uniquely extremal mapping $T[w_1, w_2]$ which sends w_1 to w_2 and is equal to the identity on $\partial\mathbf{D}$), Φ can be extended to a K'-quasiconformal mapping $\tilde{\Phi}$ of $\mathbf{C}$ onto itself with $\tilde{\Phi}|_{\bar{\mathbf{D}}} = \Phi$, where K' depends only on Φ. Since it is clear that

$$\max_{w \in \partial\Phi(S(I)) \setminus \partial\mathbf{D}} |w - h(z_0)| \leq \max_{z \in \partial D(2I)} |\tilde{\Phi}(z) - h(z_0)|$$

and

$$\min_{z \in \partial D(2I)} |\tilde{\Phi}(z) - h(z_0)| \leq |h(2I)|,$$

so, by Teichmüller distortion theorem [17] and (3.8), we have

$$\max_{w \in \partial\Phi(S(I))} |w - h(z_0)| \leq \lambda(K')|h(2I)| \leq 3\lambda(K')|h(I)|$$

for sufficient small arc I, where $\lambda(K')$ is defined in (3.4) depending only on the maximal dilatation K'. Choose an integer N' depending only on K' with $N' \geq 6\pi\lambda(K')$. Then by the definition of the Carleson sector, we have

$$\Phi(S(I)) \subset S(N'h(I)).$$

Denote $d\lambda = |\mu(z)|^2/(1 - |z|^2)\, dxdy$ and $d\lambda' = |\nu(w)|^2/(1 - |w|^2)\, dudv$. For any given $\varepsilon > 0$, as we have just proved that λ' is a vanishing Carleson measure, there exists a $\delta' > 0$ such that

$$\lambda'(S(J)) < \frac{\varepsilon}{4}|J|$$

for every subarc $J \subset S^1$ with $|J| \leq \delta'/2$.

Let $J = N'h(I)$ be the open subarc of the same center point with $h(I)$ and $|J| = N'|h(I)|$. Then there is a $\delta_1 > 0$ such that $|J| \leq \delta'/2$ and $\Phi(S(I)) \subset S(J)$ holds for every subarc I on S^1 with $|I| < \delta_1$.

By the properties of integral,

$$\lambda(S(I)) = \iint_{S(I)} \frac{|\mu(z)|^2}{1 - |z|^2}\, dxdy = \iint_{\Phi(S(I))} \frac{1 - |w|^2}{1 - |\Phi^{-1}(w)|^2} J_{\Phi^{-1}}(w)\, d\lambda'$$
$$\leq \iint_{S(J)} \frac{1 - |w|^2}{1 - |\Phi^{-1}(w)|^2} J_{\Phi^{-1}}(w)\, d\lambda'.$$

Then, from (3.2) and (3.3) in Lemma 3.1, we have

$$\lambda(S(I)) \leq C \iint_{S(J)} \frac{|h^{-1}(J_w)|}{|J_w|}\, d\lambda', \tag{3.9}$$

where $C = C_1C_2$ is a constant depending only on K.

Let ψ be a lift of h^{-1} to the real line $\mathbf{R}$ over the obvious covering mapping. Then ψ is strictly increasing, continuous and $\psi(\theta + 2\pi) - \psi(\theta) = 2\pi$.

As $h^{-1} \in SS(S^1)$, ψ is differentiable almost everywhere in $\mathbf{R}$ and

$$(h^{-1})'(e^{i\theta}) = e^{i(\psi(\theta)-\theta)}\psi'(\theta).$$

Let $2J$ be the arc on S^1 with the same center as J and of length $2|J|$. Choose a component of the lift of $2J$, which is an open interval, and denoted by $2J$. Denote also by J the component lift of J contained in the component $2J$ and I the component lift of I contained in $\psi(J)$. Let

$$\phi(\theta) = \psi'(\theta)\chi_{2J}(\theta),$$

where χ_{2J} is the characteristic function of $2J$ on $\mathbf{R}$. Let

$$M\phi(\theta) = \sup_{\theta \in J'} \frac{1}{|J'|}\int_{J'} |\phi(t)|\, dt$$

be the Hardy-Littlewood maximal function of ϕ, where the supremum is taken over all intervals J' containing θ. Then

$$M\phi(\theta) \geq |h^{-1}(J')|/|J'| \tag{3.10}$$

holds for all subarc $J' \subset 2J$ containing θ.

By a property of Hardy-Littlewood maximal functions, $\{\theta \in \mathbf{R} : M\phi(\theta) > k\}$ is an open set for every $k > 0$. Thus,

$$\{\theta \in 2J : M\phi(\theta) > k\} = 2J \cap \{\theta \in \mathbf{R} : M\phi(\theta) > k\}$$

is open and consequently,

$$\{\theta \in 2J : M\phi(\theta) > k\} = \bigcup J_l, \tag{3.11}$$

where $\{J_l\}$ is a finite or infinite sequence of disjoint intervals contained in J.

We may assume that $|J| < \frac{\pi}{4}$. Let

$$T(J_l) = \left\{ w = re^{i\theta} : 1 - \frac{2|J_l|}{\pi} \leq r < 1, e^{i\theta} \in J_l \right\}.$$

Then,

$$\left\{ w \in S(J) : \frac{|h^{-1}(J_w)|}{|J_w|} > k \right\} \subset \bigcup T(J_l). \tag{3.12}$$

Indeed, if $w \in S(J)$ and

$$\frac{|h^{-1}(J_w)|}{|J_w|} > k, \tag{3.13}$$

then by the definition of Carleson sector, $1 - |w| < |J|/2\pi$. So by (3.1) in Lemma 3.1, we have $|J_w| < |J|$ and consequently, $J_w \subset 2J$. Thus, by (3.10) and (3.13), $e^{i\theta} := w/|w| \in \bigcup J_l$. If $w \notin \bigcup T(J_l)$, then $|J_l| < \frac{\pi}{2}(1-|w|)$ for J_l containing $w/|w|$. Thus, by (3.1), $|J_w| > |J_l|$. So, there exists a $e^{i\theta'} \in J_w \backslash \bigcup J_l$ such that $M\phi(\theta') > k$. This contradicts to (3.11). Therefore, (3.12) holds.

Since $|J_l| \le 2|J| \le \delta'$, then for the above $\varepsilon > 0$,

$$\lambda'\left(\left\{w \in S(J) : \frac{|h^{-1}(J_w)|}{|J_w|} > k\right\}\right) \le \sum_j \lambda'(T(J_l)) \le \varepsilon \sum_l |J_l|$$
$$= \varepsilon|\{\theta \in 2J : M\phi(\theta) > k\}|.$$

So, we have

$$\iint_{S(J)} \frac{|h^{-1}(J_w)|}{|J_w|}\, d\lambda' \le \varepsilon \int_{2J} M\phi\, d\theta. \tag{3.14}$$

Since $\psi'(\theta)$ belongs to the class of weights A_∞, it holds the inverse hölder inequality (2.1) for some $p > 1$ and $c > 0$, that is,

$$\frac{1}{2|J|}\int_{2J} \psi'^p\, d\theta \le c\left(\frac{1}{2|J|}\int_{2J} \psi'\, d\theta\right)^p. \tag{3.15}$$

By Hölder inequality, for $q > 1$, $1/p + 1/q = 1$, we have

$$\int_{2J} M\phi\, d\theta \le (2|J|)^{1/q}\left(\int_{2J} (M\phi)^p\, d\theta\right)^{1/p}. \tag{3.16}$$

Furthermore, by Muckenhoupt theory (see §VI.6 of [14]), there exists a constant C_p for $p > 1$, independent of ϕ, such that

$$\int_{2J} (M\phi)^p\, d\theta \le \int_{\mathbf{R}} (M\phi)^p\, d\theta \le C_p \int_{\mathbf{R}} \phi^p\, d\theta = C_p \int_{2J} \psi'^p\, d\theta. \tag{3.17}$$

From (3.15)–(3.17), we have

$$\int_{2J} M\phi(\theta)\, d\theta \le (cC_p)^{1/p} \int_{2J} \psi'(\theta)\, d\theta. \tag{3.18}$$

Combining (3.9), (3.14) and (3.18), we get

$$\lambda(S(I)) \le C'\varepsilon \int_{2J} \psi'(\theta)\, d\theta \le C'\varepsilon |h^{-1}(2J)|$$

for $|I| < \delta_1$, where $C' = C(cC_p)^{1/p}$ and $2J = 2N'h(I)$. By (3.8),

$$\frac{|h^{-1}(2J)|}{|I|} = 2N' + o(1), \quad |I| \to 0_+.$$

So for the above $\varepsilon > 0$, there exists a positive number δ with $\delta < \delta_1$ such that

$$\lambda(S(I)) \le C'(2N' + 1)\varepsilon|I|.$$

holds for every subarc I on S^1 with $|I| < \delta$. Hence $|\mu(z)|^2/(1 - |z|^2)\, dxdy \in CM_0(\mathbf{D})$. The proof of this Theorem is completed. □

An application of Theorem 3.1

As an application of Theorem 3.1, we prove the following theorem.

THEOREM 4.1. *T_v is a subgroup of T.*

Proof. It is clear that the universal Teichmüller space T and the VMO-Teichmüller space T_v can be identified as the spaces of all normalized quasisymmetric and all strongly symmetric homeomorphisms of S^1 respectively. Here, a homeomorphism of S^1 is called normalized if it fixes ± 1 and i.

Let $h_1, h_2 \in T_v$ be the normalized strongly symmetric homeomorphisms and $\Phi = E(h_1)$ be the Douady-Earle extension of h_1 with the Beltrami differential μ_1. By Theorem 3.1, $|\mu_1(z)|^2/(1 - |z|^2)\, dxdy \in CM_0(\mathbf{D})$. Furthermore, by Proposition 3.1, there exists a quasiconformal extension f of h_2 with Beltrami differential μ_2 satisfying $|\mu_2(z)|^2/(1 - |z|^2)\, dxdy \in CM_0(\mathbf{D})$. Let ρ be the Beltrami differential of $f \circ \Phi^{-1}$, then for any $z \in \mathbf{D}$,

$$|\rho(\Phi(z))|^2 = \left|\frac{\mu_2(z) - \mu_1(z)}{1 - \mu_2(z)\overline{\mu_1(z)}}\right|^2 \le \frac{2(|\mu_1(z)|^2 + |\mu_2(z)|^2)}{(1 - \|\mu_1\|_\infty \|\mu_2\|_\infty)^2}.$$

Thus,

$$\begin{aligned}
\iint_{S(I)} \frac{|\rho(w)|^2}{1 - |w|^2}\, dudv &= \iint_{\Phi^{-1}(S(I))} \frac{|\rho(\Phi(z))|^2}{1 - |\Phi(z)|^2} J_\Phi(z)\, dxdy \\
&\le C \iint_{S(NJ)} \frac{|\mu_1(z)|^2 + |\mu_2(z)|^2}{1 - |z|^2} \frac{1 - |z|^2}{1 - |\Phi(z)|^2} J_\Phi(z)\, dxdy \\
&= C \iint_{S(NJ)} \frac{1 - |z|^2}{1 - |\Phi(z)|^2} J_\Phi(z) \cdot \frac{|\mu_1(z)|^2}{1 - |z|^2}\, dxdy \\
&\quad + C \iint_{S(NJ)} \frac{1 - |z|^2}{1 - |\Phi(z)|^2} J_\Phi(z) \cdot \frac{|\mu_2(z)|^2}{1 - |z|^2}\, dxdy,
\end{aligned}$$

where $S(NJ)$ is a carleson cantor containing $\Phi^{-1}(S(I))$ and $J = \Phi^{-1}(I)$.

Since $|\mu_1(z)|^2/(1 - |z|^2)\, dxdy$ and $|\mu_2(z)|^2/(1 - |z|^2)\, dxdy$ are vanishing Carleson measures on $\mathbf{D}$, similar to proof of Theorem 3.1, we have

$$\iint_{S(I)} \frac{|\eta(w)|^2}{1 - |w|^2}\, dudv = o(|I|), \quad |I| \to 0.$$

So $|\eta(w)|^2/(1-|w|^2)\,dxdy \in CM_0(\mathbf{D})$. It is obvious that $f \circ \Phi^{-1}$ is the quasiconformal extension of the normalized homeomorphism $h_2 \circ (h_1)^{-1}$. Therefore, $h_2 \circ (h_1)^{-1} \in T_v$ from Proposition 3.1 and T_v is a subgroup of T. □

References

[1] K. Astala and F. W. Gehring, Injectivity, the BMO norm and the universal Teichmüller space, J. Anal. Math. **46** (1986), 16–57.

[2] K. Astala, T. Iwaniec and G. Martin, Elliptic partial differential equations and quasiconformal mappings in the plane, Princeton Uinv. Press, New Jersey, USA, 2009.

[3] K. Astala and M. Zinsmeister, Teichmüller space and BMOA, Math. Ann. **289** (1991), 613–625.

[4] A. Beurling and L. V. Ahlfors, The boundary correspondence under quasiconformal mappings, Acta. Math. **96** (1956), 125–142.

[5] C. Bishop and P. Jones, Harmonic measure, L^2 estimates and the Schwarzian derivative, J. Anal. Math. **62** (1994), 77–113.

[6] M. D. Chen, D. G. Deng and R. L. Long, Real analysis, 2nd ed., Higher Education Press, Beijing, 2008 (in Chinese).

[7] G. Cui, Integrably asymptotic affine homeomorphisms of the circle and Teichmüller spaces, Sci. China, Ser. A. **43** (2000), 267–279.

[8] G. Cui and M. Zinsmeister, BMO Teichmüller space, Illinois J. Math. **48** (2004), 1223–1233.

[9] A. Douady and C. J. Earle, Conformally nature extension of homeomorphisms of the circle, Acta. Math. **157** (1986), 23–48.

[10] P. L. Duren, Theory of H^p spaces, Academic Press, New York and London, 1970.

[11] C. J. Earle, V. Markovic and D. Saric, Barycentric extension and the Bers embedding for asymptotic Teichmüller space, Contemporary Math. **311** (2002), 87–105.

[12] F. P. Gardiner, Teichmüller theory and quadratic differentials, Wiley-Interscience, New York, 1987.

[13] F. P. Gardiner and N. Lakic, Quasiconformal Teichmüller theory, American Mathematical Society, New York, 2000.

[14] J. B. Garnett, Bounded analytic functions, Academic Press, New York, 1981.

[15] Y. Hu and Y. Shen, On quasisymmetric homeomorphisms, Israel J. Math. **191** (2012), 209–226.

[16] O. Lehto, Univalent functions and Teichmüller space, Springer-Verlag, New York, 1986.

[17] Z. Li, Quasiconformal mappings and Teichmüller space, Peking University Press, Beijing, 2013 (in Chinese).

[18] S. Nag, The complex analytic theory of Teichmüller space, Wiley-Interscience, 1988.

[19] Ch. Pommerenke, On univalent functions, Bloch functions and VMOA, Math. Ann. **123** (1978), 199–208.

[20] Ch. Pommerenke, Boundary behaviour of conformal maps, Springer-Verlag, Berlin, 1992.

[21] D. Sarason, Functions of vanishing mean oscillation, Trans. Amer. Math. Soc. **207** (1975), 391–405.

[22] Y. Shen, Weil-Petersson Teichmüller space, arXiv:1304.3197v3.

[23] Y. Shen and H. Wei, Universal Teichmüller space and BMO, Adv. Math. **234** (2013), 129–148.

[24] K. Strebel, Point shift differentials extremal quasiconformal mappings, Ann. Acad. Sci. Fenn. A. I. Math. **23** (1998), 475–494.

Yan Wu
LMIB and School of Mathematics and Systems Science
Beihang University
Beijing, 100191
P.R. China
School of Science
Linyi University
Shandong, 276005
P.R. China
E-mail: BY1209113@buaa.edu.cn
wuyan@lyu.edu.cn

Yi Qi
LMIB and School of Mathematics and Systems Science
Beihang University
Beijing, 100191
P.R. China
E-mail: yiqi@buaa.edu.cn