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A THEOREM OF LIOUVILLE TYPE FOR p-HARMONIC MAPS IN WEIGHTED RIEMANNIAN MANIFOLDS

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Abstract

Let M be a weighted Riemannian manifold with non-negative Bakry-Émery-Ricci curvature and N be a complete Riemannian manifold of non-positive sectional curvature. In this paper, the p-harmonic map $u: M \to N$ is studied, and a theorem of Liouville type is obtained.

1. Introduction

Let (M^m, g) and (N^n, h) be complete Riemannian manifolds, $\dim M = m \geq 2$, $\dim N = n$, and let $p \geq 2$. A map $u: M \to N$ is said to be p-harmonic if $u|_\Omega$ is a critical point of the p-energy

$$E_p(u) = \frac{1}{p}\int_\Omega |du|^p \, dV_M,$$

for every compact domain $\Omega \subset M$. Here the differential du is a section of the bundle $T^*M \otimes u^{-1}TN \to M$ and $u^{-1}TN$ denotes the pull-back bundle via the map u and dV_M stands for the canonical Riemannian volume form on M. When u is C^2-regular, the Euler-Lagrange equation for the energy functional E_p is the p-harmonic maps equation [2]

$$\tau_p(u) := \operatorname{div}(|du|^{p-2}\, du) = |du|^{p-2}\tau_2(u) + (p-2)|du|^{p-3}\, du(\operatorname{grad}_g |du|) = 0$$

where $\tau_2(u) := \operatorname{div}(du)$ is the standard tension field of u. In this paper, $\triangle$, δ and $\tau(u) = \tau_2(u)$ always denote the Laplace operator, the co-differential operator and the tension field of a map u on the manifold (M^m, g). Several studies are given for harmonic maps (see [5, 7, 11, 13, 14]). For these harmonic maps, there

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are Liouville type theorems, which state that a harmonic map u is constant under some conditions.

In 1976, R. M. Schoen and S. T. Yau [11] proved the following Liouville type theorem.

THEOREM 1.1. *Let M be a complete Riemannian manifold of non-negative Ricci curvature and N be a complete Riemannian manifold of non-positive sectional curvature. Then any harmonic map $u : M \to N$ of $E_2(u) < \infty$ is constant.*

For p-harmonic maps, N. Nakauchi [9] proved the following theorem in 1998.

THEOREM 1.2. *Let M be a complete Riemannian manifold of non-negative Ricci curvature and N be a complete Riemannian manifold of non-positive sectional curvature. Then any p-harmonic map $u : M \to N$ of $E_p(u) < \infty$ is constant.*

Let $f : M \to R$ be a smooth function. A map $u : M \to N$ is said to be (weakly) f-harmonic if $u|_\Omega$ is a critical point of the f-energy

$$E_f(u) = \frac{1}{2}\int e^{-f}|du|^2 \, dV_M$$

for every compact domain $\Omega \subset M$. The study of f-harmonic maps began with A. Lichnerowicz in 1969 [6] and J. Eells and L. Lemaire in 1977 [3]. We now study the f-harmonic maps on weighted manifold and gradient Ricci solitons.

A weighted manifold, also known in the literature as smooth metric measure space, is a Riemannian manifold (M^m, g) endowed with a weighted volume form $e^{-f} \, dV_M$ and some smooth function $f : M \to R$. For a weighted manifold $(M^m, g, e^{-f} \, dV_M)$, we are interested in the Bakry-Émery Ricci tensor $\mathrm{Ric}_f^M = \mathrm{Ric}^M + \mathrm{Hess}\, f$, which was first introduced by A. Lichnerowicz in [7] and later by D. Bakry and M. Émery in [1]. Recently it has been found that this curvature tensor is strictly related with geometric objects whose importance is outstanding in mathematics. Imposing the constancy of Ric_f^M, one can introduce gradient Ricci soliton structure on the manifold, and the importance of gradient Ricci solitons is due to the fact that they correspond to self-similar solutions to Hamilton's Ricci flow and often arise as limits of dilations of singularities developed along the flow.

It is easy to know that the f-harmonic map on manifold (M^m, g), just be the harmonic map on a weighted manifold $(M^m, g, e^{-f} \, dV_M)$. In this paper, we study the p-harmonic maps on a weighted manifold $(M^m, g, e^{-f} \, dV_M)$, that is, the map is a critical point of the (p, f)-energy

$$E_{p,f}(u) = \frac{1}{p}\int_\Omega e^{-f}|du|^p \, dV_M$$

for every compact domain $\Omega \subset M$. We said u is a (p, f)-harmonic map on M. We obtain the following general result.

Theorem 1.3. *Let $(M^m, g, e^{-f}\, dV_M)$ be an orientable, complete non-compact weighted Riemannian manifold with $\mathrm{Ric}_f^M \geq 0$, and N be a complete Riemannian manifold of non-positive sectional curvature, where $f \in C^\infty(M)$. Let $u : M \to N$ be a (p, f)-harmonic map with $E_{p,f}(u) < \infty$.*

(I) *Assume at least one of the following assumption is satisfied*

(a) *there exists a constant $C > 0$ such that $|f| \leq C$;*

(b) *f is convex and the set of its critical points is unbounded;*

(c) $\mathrm{Vol}_f(M) := \int_M e^{-f}\, dV_M = +\infty$;

(d) *there is a point $q_0 \in M$ such that $\mathrm{Ric}_f^M|_{q_0} > 0$*

(e) *there is a point $q_1 \in M$ such that $\mathrm{Ric}^M(X, X)|_{q_1} \neq 0$ for all $0 \neq X \in T_{q_1}M$.*

Then u is homotopic to a constant.

(II) *If $\mathrm{Sect}^N < 0$, then u is homotopic either to a constant or to a totally geodesic map whose image is contained in a geodesic of N.*

2. Bochner type formula

Let $(M^m, g, e^{-f}\, dV_M)$ be a weighted manifold, or a smooth metric measure space, where $f : M \to R$ is a smooth function. For every smooth function h on M, we define an operator L_f as follows:

$$L_f(h) = \triangle h - \langle df, dh\rangle,$$

where $\triangle = -(d\delta + \delta d)$ is the Laplace operator. At the same time, the Bakry-Émery-Ricci curvature is defined by the formula

$$\mathrm{Ric}_f^M = \mathrm{Ric}^M + \mathrm{Hess}\, f.$$

Lemma 2.1. *Let $(M^m, g, e^{-f}\, dV_M)$ be a weighted Riemannian manifold and N be a smooth Riemannian manifold, then a map $u : M \to N$ is (p, f)-harmonic if and only if it satisfies the Euler-Lagrange equation*

$$\begin{aligned}(2.1)\qquad \tau_{p,f}(u) &:= e^{-f}|du|^{p-2}(\tau(u) + du(\mathrm{grad}(\ln|du|^{p-2} - f)))\\ &= \mathrm{div}(e^{-f}|du|^{p-2}\, du) = -\delta(e^{-f}|du|^{p-2}\, du) = 0\end{aligned}$$

where $\tau(u) = -\delta\, du = \mathrm{div}(du)$ is the tension field of u. We call $\tau_{p,f}(u)$ the (p, f)-tension field of u.

Proof. The proof is standard computation which can be adapted from the case when $p = 2$ and $f = 0$, see for example [3]. □

In this section we give the following Bochner type formula.

Lemma 2.2. *Let $(M^m, g, e^{-f}\, dV_M)$ be a weighted Riemannian manifold and N be a smooth Riemannian manifold, and $u : M \to N$ be a C^2 map. Then*

$$L_f\left(\frac{1}{p}|du|^p\right) = |du|^{p-2}(|\nabla\, du|^2 + (p-2)|\nabla|du|\,|^2 + F_f(u)) + |du|^{p-2}\langle d(\tau(u) - du(\nabla f)), du\rangle \tag{2.2}$$

where

$$F_f(u) = \sum \langle du(\mathrm{Ric}_f^M\ e_i), du(e_i)\rangle - \sum \langle R^N(du(e_i), du(e_j))\ du(e_i), du(e_j)\rangle.$$

Proof. We start recalling the standard Bochner formula for a smooth map u [3],

$$\frac{1}{2}\triangle|du|^2 = \langle\triangle\ du, du\rangle + |\nabla\ du|^2 + F(u),$$

where

$$F(u) = \sum \langle du(\mathrm{Ric}^M\ e_i), du(e_i)\rangle - \sum \langle R^N(du(e_i), du(e_j))\ du(e_i), du(e_j)\rangle.$$

Then

$$\begin{aligned}
L_f\left(\frac{1}{p}|du|^p\right) &= \frac{1}{p}\triangle|du|^p - \left\langle df, d\left(\frac{1}{p}|du|^p\right)\right\rangle \\
&= \frac{1}{2}|du|^{p-2}\triangle|du|^2 + (p-2)|du|^{p-2}|\nabla|du|\,|^2 - \frac{1}{2}|du|^{p-2}\langle df, d(|du|^2)\rangle \\
&= |du|^{p-2}(\langle\triangle du, du\rangle + |\nabla\ du|^2 + F(u)) \\
&\quad + (p-2)|du|^{p-2}|\nabla|du|\,|^2 - \frac{1}{2}|du|^{p-2}\langle df, d(|du|^2)\rangle \\
&= |du|^{p-2}(|\nabla\ du|^2 + (p-2)|\nabla|du|\,|^2 + F_f(u)) \\
&\quad + |du|^{p-2}(\langle\triangle\ du, du\rangle - \langle du(\nabla_{(.)}\nabla f), du\rangle) - \frac{1}{2}|du|^{p-2}\langle df, d(|du|^2)\rangle
\end{aligned}$$

where

$$\begin{aligned}
F_f(u) &= \sum \langle du(\mathrm{Ric}_f^M\ e_i), du(e_i)\rangle - \sum \langle R^N(du(e_i), du(e_j))\ du(e_i), du(e_j)\rangle. \\
&= F(u) + \sum \langle du(\nabla_{e_i}\nabla f), du(e_i)\rangle \\
&= F(u) + \langle du(\nabla_{(.)}\nabla f), du\rangle.
\end{aligned}$$

It is easy to know that

$$\triangle\ du = -(d\delta + \delta d)\ du = -d(\delta d)u = d(\tau(u)).$$

Hence the lemma is proved once we show that

$$\langle du(\nabla_{(.)}\nabla f), du\rangle) + \frac{1}{2}\langle df, d(|du|^2)\rangle = \langle d(du(\nabla f)), du\rangle. \tag{2.3}$$

Let $\{x_a\}_{a=1}^m$ be the normal coordinate chart at $q \in M$ on M and $\{\theta_A\}_{A=1}^n$ and $\{E_A\}_{A=1}^n$ orthonormal coframe and dual frame on N at $u(q)$ respectively. Moreover denote the components of the metric on M as $g_{ab} := g_M\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right)$. Then we have the following equalities in this coordinate.

$$g_{ab} = g^{ab} = \begin{cases} 1 & (a = b) \\ 0 & (a \neq b) \end{cases}, \quad \frac{\partial g^{ab}}{\partial x^c} = 0, \quad \Gamma_{ab}^c = 0 \quad \text{at } q.$$

Now, we can write

$$du = u_a^A E_A \otimes dx^a, \quad \nabla f = f^a \frac{\partial}{\partial x^a}$$

$$\nabla_{\partial/\partial x^a} \nabla f = \nabla_{\partial/\partial x^a}\left(f^b \frac{\partial}{\partial x^b}\right) = (f_a^c + f^b \Gamma_{ab}^c) \frac{\partial}{\partial x^c},$$

that is,

$$\nabla_{(.)} \nabla f = (f_a^c + f^b \Gamma_{ab}^c)\, dx^a \otimes \frac{\partial}{\partial x^c}.$$

At the given point $q \in M$, we have

$$du(\nabla_{(.)} \nabla f) = u_c^A f_a^c \, dx^a \otimes E_A.$$
$$\langle du(\nabla_{(.)} \nabla f), du \rangle = g^{ad} u_c^A u_d^A f_a^c = u_c^A u_a^A f_a^c.$$

and

$$\langle df, d(|du|^2) \rangle = \left\langle f_a \, dx^a, \frac{\partial}{\partial x^d} (g^{bc} u_c^A u_b^A)\, dx^d \right\rangle = 2 f^a u_b^A u_{ba}^A.$$

Then

$$\langle du(\nabla_{(.)} \nabla f), du \rangle + \frac{1}{2} \langle df, d(|du|^2) \rangle = u_c^A u_a^A f_c^a + f^a u_b^A u_{ba}^A.$$

On the other hand,

$$\begin{aligned} \langle du, d(du(\nabla f)) \rangle &= \langle du, d(f^c u_c^A E_A) \rangle \\ &= \left\langle du, \frac{\partial}{\partial x^a} (f^c u_c^A E_A) \otimes dx^a \right\rangle \\ &= \langle u_d^A \, dx^d \otimes E_A, u_c^A f_a^c \, dx^a \otimes E_A \rangle \\ &\quad + \langle u_c^A \, dx^c \otimes E_A, u_{ba}^A f^a \, dx^b \otimes E_A \rangle \\ &= u_c^A u_a^A f_c^a + f^a u_b^A u_{ba}^A. \end{aligned}$$

This latter proves (2.3) and concludes the proof. ∎

3. The proof of theorem 1.3

Proof of theorem 1.3. The (2.1) can be rewritten

$$\begin{aligned}\frac{1}{2}L_f(|du|^p) &= \frac{p}{2}|du|^{p-2}(|\nabla\, du|^2 + (p-2)|\nabla|du|\,|^2 + F_f(u)) \\ &\quad + \frac{p}{2}|du|^{p-2}\langle d(\tau(u) - du(\nabla f)), du\rangle\end{aligned}$$

Let $\phi = |du|^{p/2}$, then

$$\frac{1}{2}L_f(\phi^2) = \phi L_f(\phi) + |\nabla\phi|^2,$$

It is easy to get

$$|\nabla\phi|^2 = \frac{p^2}{4}|du|^{p-2}|\nabla|du|\,|^2,$$

Using the Kato's inequality, we have

$$\begin{aligned}\phi L_f(\phi) &= \frac{1}{2}L_f(\phi^2) - |\nabla\phi|^2 \\ &= \frac{1}{2}L_f(|du|^p) - \frac{p^2}{4}|du|^{p-2}|\nabla|du|\,|^2 \\ &= \frac{p}{2}|du|^{p-2}(|\nabla\, du|^2 + (p-2)|\nabla|du|\,|^2 + F_f(u)) \\ &\quad + \frac{p}{2}|du|^{p-2}\langle d(\tau(u) - du(\nabla f)), du\rangle - \frac{p^2}{4}|du|^{p-2}|\nabla|du|\,|^2 \\ &= \frac{p}{4}|du|^{p-2}(2|\nabla\, du|^2 + (p-4)|\nabla|du|\,|^2 + 2F_f(u)) \\ &\quad + \frac{p}{2}|du|^{p-2}\langle d(\tau(u) - du(\nabla f)), du\rangle \\ &\ge \frac{p}{4}|du|^{p-2}((p-2)|\nabla|du|\,|^2 + 2F_f(u)) \\ &\quad + \frac{p}{2}|du|^{p-2}\langle d(\tau(u) - du(\nabla f)), du\rangle\end{aligned}$$

Let $B(r)$ be a ball with radius r, and w_r be a cut-off function s.t. $w_r \le 1$ on M, $w_r|_{B(r)} \equiv 1$, $w_r|_{M\setminus B(2r)} \equiv 0$ and $|\nabla w_r| \le \dfrac{2}{r}$. Since $F_f(u) \ge 0$, we have

$$\begin{aligned}&\frac{p}{2}\int_M w_r^2|du|^{p-2}(\langle d(\tau(u) - du(\nabla f))\, du\rangle)e^{-f}\, dv_g - \int_M w_r^2\phi L_f(\phi)e^{-f}\, dv_g \\ &\quad \le -\frac{p}{4}\int_M w_r^2|du|^{p-2}((p-2)|\nabla|du|\,|^2 + 2F_f(u))e^{-f}\, dv_g \le 0,\end{aligned}$$

where dv_g stands for the canonical Riemannian volume form on metric g. It is easy to know that

$$\begin{aligned}
&\int_M w_r^2 \phi L_f(\phi) e^{-f}\, dv_g \\
&\quad = \int_M w_r^2 \phi \operatorname{div}(e^{-f} d(\phi))\, dv_g \\
&\quad = \int_M w_r^2 \phi(-\delta(e^{-f} d(\phi)))\, dv_g \\
&\quad = -\int_M \langle d(w_r^2\phi), e^{-f} d(\phi)\rangle\, dv_g \\
&\quad = -\int_M w_r^2 e^{-f} |d(\phi)|^2\, dv_g - 2\int_M \langle w_r d(w_r)\phi, e^{-f} d(\phi)\rangle\, dv_g \\
&\quad \le -\int_M w_r^2 e^{-f} |d(\phi)|^2\, dv_g + 2\int_M w_r |d(w_r)| \phi |d(\phi)| e^{-f}\, dv_g \\
&\quad = -\int_M w_r^2 e^{-f} |d(\phi)|^2\, dv_g + p\int_M w_r |d(w_r)|\, |du|^{p-1} |d(|du|)| e^{-f}\, dv_g
\end{aligned}$$

At the same time, we have

$$\begin{aligned}
&\frac{p}{2}\int_M w_r^2 |du|^{p-2} (\langle d(\tau(u) - du(\nabla f)), du\rangle) e^{-f}\, dv_g \\
&\quad = \frac{p}{2}\int_M \langle d(\tau(u) - du(\nabla f)), w_r^2 e^{-f} |du|^{p-2}\, du\rangle\, dv_g \\
&\quad = \frac{p}{2}\int_M \langle \tau(u) - du(\nabla f), \delta(w_r^2 e^{-f} |du|^{p-2}\, du)\rangle\, dv_g \\
&\quad = \frac{p}{2}\int_M \langle \tau(u) - du(\nabla f), w_r^2 \delta(e^{-f} |du|^{p-2}\, du) + e^{-f} |du|^{p-2}\, du(\nabla w_r^2)\rangle\, dv_g \\
&\quad = \frac{p}{2}\int_M \langle \tau(u) - du(\nabla f), e^{-f} |du|^{p-2}\, du(\nabla w_r^2)\rangle\, dv_g \\
&\qquad (\delta(e^{-f} |du|^{p-2}\, du) = -\tau_{p,f}(u) = 0) \\
&\quad = \frac{p}{2}\int_M \langle e^{-f} (\tau(u) - du(\nabla f)), |du|^{p-2}\, du(\nabla w_r^2)\rangle\, dv_g \\
&\quad = \frac{p}{2}\int_M \langle -\delta(e^{-f}\, du), |du|^{p-2}\, du(\nabla w_r^2)\rangle\, dv_g \\
&\quad = -\frac{p}{2}\int_M \langle e^{-f}\, du, d(|du|^{p-2}\, du(\nabla w_r^2))\rangle\, dv_g
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{p}{2}\left(\int_M \langle e^{-f}\, du, (p-2)|du|^{p-3} d(|du|)\, du(\nabla w_r^2)\rangle\, dv_g \right.\\
&\qquad \left. + \int_M \langle e^{-f}\, du, |du|^{p-2} d(du(\nabla w_r^2))\rangle\, dv_g\right)\\
&= -\frac{p}{2}\left(\int_M \langle e^{-f}\, du, (p-2)|du|^{p-3} d(|du|)\, du(\nabla w_r^2)\rangle\, dv_g \right.\\
&\qquad \left. + \int_M \langle e^{-f}|du|^{p-2}\, du, d(du(\nabla w_r^2))\rangle\, dv_g\right)\\
&= -\frac{p}{2}\left(\int_M \langle e^{-f}\, du, (p-2)|du|^{p-3} d(|du|)\, du(\nabla w_r^2)\rangle\, dv_g \right.\\
&\qquad \left. + \int_M \langle \delta(e^{-f}|du|^{p-2}\, du), du(\nabla w_r^2)\rangle\, dv_g\right)\\
&= -\frac{p}{2}\int_M \langle e^{-f}\, du, (p-2)|du|^{p-3} d(|du|)\, du(\nabla w_r^2)\rangle\, dv_g\\
&\geq -p(p-2)\int_M w_r|d(w_r)|\,|du|^{p-1}|d(|du|)|e^{-f}\, dv_g
\end{aligned}
$$

Here we have used the fact $\delta(e^{-f}|du|^{p-2}\, du) = -\tau_{p,f}(u) = 0$. Now, we have

$$
\begin{aligned}
0 &\geq \frac{p}{2}\int_M w_r^2|du|^{p-2}(\langle d(\triangle u - du(\nabla f))\, du\rangle)e^{-f}\, dv_g - \int_M w_r^2\phi L_f(\phi)e^{-f}\, dv_g\\
&\geq \int_M w_r^2 e^{-f}|d(\phi)|^2\, dv_g - p(p-1)\int_M w_r|d(w_r)|\,|du|^{p-1}|d(|du|)|e^{-f}\, dv_g\\
&\geq \int_M w_r^2 e^{-f}|d(\phi)|^2\, dv_g - 2(p-1)\int_M w_r|d(w_r)|\phi|d(\phi)|e^{-f}\, dv_g\\
&\geq (1-(p-1)\varepsilon)\int_M w_r^2 e^{-f}|d(\phi)|^2\, dv_g - \frac{(p-1)}{\varepsilon}\int_M |d(w_r)|^2\phi^2 e^{-f}\, dv_g
\end{aligned}
$$

for any $0 < \varepsilon < 1$. So $d(\phi) \in L^2$. Hence by the Hölder inequality

$$
\begin{aligned}
\int_\Omega |du|^{p-1}|d(|du|)|e^{-f}\, dv_g &= \frac{2}{p}\int_\Omega \phi|d(\phi)|e^{-f}\, dv_g\\
&\leq \frac{2}{p}\left(\int_\Omega e^{-f}|d(\phi)|^2\, dv_g\right)^{1/2}\left(\int_\Omega \phi^2 e^{-f}\, dv_g\right)^{1/2} < \infty
\end{aligned}
$$

for any compact set $\Omega \subset M$. If we let $r \to \infty$, then

$$
\int_M w_r|d(w_r)|\,|du|^{p-1}|d(|du|)|e^{-f}\, dv_g \to 0.
$$

From above we get

$$\begin{aligned} 0 &\geq -\frac{p}{4}\int_M w_r^2|du|^{p-2}((p-2)|\nabla|du||^2 + 2F_f(u))e^{-f}\,dv_g \\ &= \frac{p}{2}\int_M w_r^2|du|^{p-2}(\langle d(\triangle u - du(\nabla f))\,du\rangle)e^{-f}\,dv_g - \int_M w_r^2\phi L_f(\phi)e^{-f}\,dv_g \\ &\geq \int_M w_r^2 e^{-f}|d(\phi)|^2\,dv_g \end{aligned}$$

That is

$$|du|^{p-2}((p-2)|\nabla|du||^2 + 2F_f(u)) = 0, \tag{3.1}$$

and

$$d\phi = d(|du|^{p/2}) = 0. \tag{3.2}$$

$d\phi = 0$ means $|du| = \text{const}$. Suppose $|du| = C > 0$. Then the finiteness of the (p, f)-energy $E_{p,f}(u)$ of u gives that $\mathrm{Vol}_f(M) < +\infty$. If either $|f|$ is uniformly bounded or f is convex and the set of its critical points is unbounded, then Theorems 1.3 and 5.3 in [12] implies that M has at least linear f-volume growth, giving a contradiction.

The Kato's inequality with equality holding means $\nabla\, du \equiv 0$, i.e. $u : M \to N$ is totally geodesic, which in turn gives that u is harmonic, i.e. $\tau(u) = 0$. Since $|du| = \text{const}$, $\tau_{p,f}(u) = 0$ and $\tau(u) = 0$, (2.1) becomes

$$|du|^{p-2}\,du(\nabla f) = e^f\tau_{p,f}(u) - |du|^{p-2}\tau(u) = 0. \tag{3.3}$$

Accordingly, the (3.1) reads

$$|du|^{p-2}F_f(u) = 0, \tag{3.4}$$

and by the curvature sign assumptions both

$$|du|^{p-2}\sum\langle du(\mathrm{Ric}_f^M\, e_i), du(e_i)\rangle = 0, \tag{3.5}$$

and

$$|du|^{p-2}\sum\langle R^N(du(e_i), du(e_j))\,du(e_i), du(e_j)\rangle = 0. \tag{3.6}$$

First, suppose that $|du| = C > 0$ and $\mathrm{Sect}^N < 0$, then $du(E_i)\,\|\,du(E_j)$ for all $i, j = 1, \ldots, n$ and we conclude that $u(M)$ must be contained in a geodesic of N.

On the other hand, suppose that $\mathrm{Ric}_f^M|_{q_0} > 0$ at some point $q_0 \in M$. Then necessarily $du(q_0) = 0$ which gives $du \equiv 0$.

Moreover, (3.5) can be rewrote

$$|du|^{p-2}\Big(\sum\langle du(\mathrm{Ric}^M\, e_i), du(e_i)\rangle + \langle du(\nabla_{e_i}\nabla f), du(e_i)\rangle\Big) = 0. \tag{3.7}$$

Since

$$0 = (\nabla\, du)(X, Y) = (\nabla_{du(Y)}\, du)(X) = \nabla^N_{du(Y)}\, du(X) - du(\nabla^M_Y X)$$

for all X, Y vector fields on M. (3.3) implies

$$|du|^{p-2}\langle du(\nabla^M_{e_i}\nabla f), du(e_i)\rangle = |du|^{p-2}\langle \nabla^N_{du(e_i)}\, du(\nabla f), du(e_i)\rangle = 0,$$

for each $i = 1, \dots, n$. Since $\mathrm{Ric}^M_f \geq 0$. (3.7) in particular gives

$$|du|^{p-2} \sum \langle du(\mathrm{Ric}^M\, e_i), du(e_i)\rangle = 0. \tag{3.8}$$

Hence, if there exists a point $q_1 \in M$ such that $\mathrm{Ric}^M(X, X)|_{q_1} \neq 0$ for all $0 \neq X \in T_{q_1}M$, then u is once again necessarily constant.

4. A remark

When M is an n-dimensional compact Riemannian manifold without boundary, we have the following result, which is an extension of facts in p-harmonic map case. (See Eells and Sampson [4].)

THEOREM 4.1. *Let $(M^m, g, e^{-f}\, dV_M)$ be a weighted, compact Riemannian manifold without boundary and N be a smooth Riemannian manifold, and $u : M \to N$ be a (p, f)-harmonic map.*

(a) *Assume* $\mathrm{Ric}^M_f \geq 0$ *and* $\mathrm{Sect}^N \leq 0$. *Then u is totally geodesic.*

(b) *In addition to* (a), *if* $\mathrm{Ric}^M_f > 0$ *somewhere, then u is a constant map.*

(c) *In addition to* (a), *if* $\mathrm{Sect}^N < 0$, *then u is a constant map, or u maps onto a closed geodesic in N.*

Proof. Using the Lemma 2.2, we get

$$\begin{aligned}
0 &= \int_M L_f\left(\frac{1}{p}|du|^p\right)e^{-f}\, dv_g \\
&= \int_M |du|^{p-2}(|\nabla\, du|^2 + (p-2)|\nabla|du|\,|^2 + F_f(u))e^{-f}\, dv_g \\
&\quad + \int_M |du|^{p-2}\langle d(\tau(u) - du(\nabla f)), du\rangle e^{-f}\, dv_g \\
&= \int_M |du|^{p-2}(|\nabla\, du|^2 + (p-2)|\nabla|du|\,|^2 + F_f(u))e^{-f}\, dv_g \\
&\quad + \int_M \langle d(\tau(u) - du(\nabla f)), e^{-f}|du|^{p-2}\, du\rangle\, dv_g
\end{aligned}$$

$$= \int_M |du|^{p-2}(|\nabla\, du|^2 + (p-2)|\nabla|du|\,|^2 + F_f(u))e^{-f}\, dv_g$$
$$+ \int_M \langle \tau(u) - du(\nabla f), \delta(e^{-f}|du|^{p-2}\, du)\rangle\, dv_g$$
$$= \int_M |du|^{p-2}(|\nabla\, du|^2 + (p-2)|\nabla|du|\,|^2 + F_f(u))e^{-f}\, dv_g$$
$$+ \int_M \langle \tau(u) - du(\nabla f), \tau_{p,f}(u)\rangle\, dv_g$$

Since $u : M \to N$ be a (p, f)-harmonic map, that is, $\tau_{p,f}(u) = 0$, we know

$$\int_M |du|^{p-2}(|\nabla\, du|^2 + (p-2)|\nabla|du|\,|^2 + F_f(u))e^{-f}\, dv_g \leq 0,$$

By the curvature sign assumptions, we know $F_f(u) \geq 0$, and

$$|du|^{p-2}(|\nabla\, du|^2 + (p-2)|\nabla|du|\,|^2 + F_f(u)) \geq 0.$$

Hence

$$|du|^{p-2}(|\nabla\, du|^2 + (p-2)|\nabla|du|\,|^2 + F_f(u)) = 0.$$

Analysis similar to that in the proof of Theorem 1.3 shows that the theorem holds. □

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