S.-T. YIN, Q. HE AND D.-X. ZHENG
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SOME NEW LOWER BOUNDS OF THE FIRST EIGENVALUE ON FINSLER MANIFOLDS

Song-Ting Yin, Qun He* and Da-Xiao Zheng

Abstract

We establish some unified lower bounds for the first Neumann and closed eigenvalues of the Finsler-Laplacian on compact Finsler manifolds under the weighted Ricci curvature conditions, which extend some recent theorems on the first eigenvalue of the Riemannian-Laplacian. Moreover, the Lichnerowicz type lower bound for the first Dirichlet eigenvalue of the Finsler-Laplacian is also obtained.

Introduction

For a closed Riemannian n-manifold (M, g) with the Ricci curvature satisfying $\mathrm{Ric} \geq (n-1)k (k > 0)$, Lichnerowicz ([9]) gave the lower bound of the first eigenvalue of Laplacian

$$\lambda_1(M) \geq nk.$$

When $k = 0$, using the gradient estimate, Li-Yau ([8]) obtained

$$\lambda_1(M) \geq \frac{\pi^2}{2d^2},$$

where d is the diameter of M. By utilizing the method of barrier function, Zhong-Yang ([27]) further proved that

$$\lambda_1(M) \geq \frac{\pi^2}{d^2},$$

Afterwards, Hang-Wang ([7]) showed that $\lambda_1(M) > \frac{\pi^2}{d^2}$ if $n > 1$. If $k < 0$, it is proved by Li-Yau ([8]) that

$$\lambda_1(M) \geq \frac{1}{(n-1)d^2} \exp\{-[1 + \sqrt{-4(n-1)^2 d^2 k}]\}.$$

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*Corresponding author.
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The estimate above was improved by Yang ([22]) to the following

$$\lambda_1(M) \geq \frac{\pi^2}{d^2} \exp\{-\max\{\sqrt{n-1}, \sqrt{2}\}\sqrt{-(n-1)kd^2}\}.$$

For the first closed and Neumann eigenvalues, Bakry-Qian ([1]) and Chen-Wang ([4]) put these two lower bound estimates in the same framework, and gave estimates for the first eigenvalue of very general elliptic symmetric operators, via diameter and Ricci curvature.

It is an important and interesting problem to find a unified lower bound of the first eigenvalue for Laplace operator. For this, Peter Li ([23]) conjectured that for a compact Riemannian manifold (M, g) satisfying $\mathrm{Ric} \geq K > 0$, the first eigenvalue $\lambda_1(M) \geq \frac{\pi^2}{d^2} + K$. In this direction, Yang ([23]) proved that

$$\lambda_1(M) \geq \frac{\pi^2}{d^2} + \frac{K}{4}.$$

Later, this lower bound was improved by Ling ([10]) to

$$\lambda_1(M) \geq \frac{\pi^2}{d^2} + \frac{31}{100}K.$$

When $K < 0$, Yang ([22]) conjectured that $\lambda_1(M) \geq \frac{\pi^2}{d^2} + \frac{K}{2}$. In 2007, Ling ([11]) proved Yang's conjecture. Generally, for a closed Riemannian manifold with $\mathrm{Ric} \geq K (K \in R)$, Chen-Wang ([4]) gave that

$$\lambda_1(M) \geq \frac{\pi^2}{d^2} + \frac{K}{2}.$$

There are many generalizations of the above lower bound estimations. Qian-Zhang-Zhu ([15]) extended them into Alexandrov spaces. Lott-Villani ([12]) generalized Lichnerowicz's estimate to metric measure spaces with curvature-dimension conditions. Wei-Wylie ([20]) and Chu-Hu ([3]) and Futaki-Li-Li ([5]) generalized them to f-Laplacian with the N-Bakry-Émery Ricci curvature conditions.

As a natural generalization of Riemannian manifolds, Finsler manifolds are differentiable manifolds of which on each tangent space one endows a Minkowskian norm instead of a Euclidean norm. There exists a Finsler-Laplacian, which is a nonlinear elliptic operator, on Finsler manifolds. In recent years, studies on the bound estimations of the first eigenvalue of the Finsler-Laplacian have taken on a new look. In [6] Shen firstly gave the Faber-Krahn type inequality and then further got the Cheng type inequality in [18]. After that, Wu-Xin ([21]) proved Mckean type inequalities, while Chen ([2]) gained the Cheeger type inequality and also improved the Cheng type inequality obtained by Shen. There is still plenty of scope for improvement. Recently, some refinements of the results above were achieved by Yin-He in [24]. As for the first

Neumann and closed eigenvalues, Wang-Xia ([19]) gave a sharp lower bound estimation on Finsler manifolds with weighted Ricci curvature conditions (see Lemma 2.1 below). In addition, Yin-He-Shen ([24][25][26]) obtained some estimates of the first eigenvalue under the variant weighted Ricci curvature conditions ($\mathrm{Ric}_N \geq (n-1)k$, $N \in (n, \infty)$ for $k > 0$ or $k < 0$ and $\mathrm{Ric}_\infty \geq 0$) and further established some rigidity theorems on Finsler manifolds.

But until now, there are no universal explicit estimations of the first eigenvalue on the Finsler manifolds with the weighted Ricci curvature $\mathrm{Ric}_N \geq K$ ($N \in [n, \infty], K \in R$). In this paper we are to answer Li's Conjecture and Yang's Conjecture in the Finsler setting. Here the weighted Ricci curvature and other concepts will be explained in Sec. 1 below.

Theorem 0.1. *Let $(M, F, d\mu)$ be a compact Finsler n-manifold without boundary or with a convex boundary. If the weighted Ricci curvature satisfies $\mathrm{Ric}_N \geq K$ for $N \in [n, \infty]$, $K \in R$, then the first closed or Neumann eigenvalue of Finsler-Laplacian*

$$\lambda_1 \geq 4s(1-s)\frac{\pi^2}{d^2} + sK := C(d, K, s),$$

where d denotes the diameter of (M, F) and $\forall s \in (0, 1)$. In particular,

$$\lambda_1 \geq \frac{\pi^2}{d^2} + \frac{K}{2}.$$

Remark. (i) For any K, we give a universal lower bound estimate which is independent of N. Moreover, we have

$$\sup_s C(d, K, s) = \begin{cases} 0, & K \in \left(-\infty, -\dfrac{4\pi^2}{d^2}\right); \\ \left(\dfrac{\pi}{d} + \dfrac{Kd}{4\pi}\right)^2, & K \in \left[-\dfrac{4\pi^2}{d^2}, \dfrac{4\pi^2}{d^2}\right]; \\ K, & K \in \left(\dfrac{4\pi^2}{d^2}, \dfrac{(N-1)\pi^2}{d^2}\right]. \end{cases}$$

When $N = \infty$ and $K = 0$, the estimate $\lambda_1 \geq \dfrac{\pi^2}{d^2}$ is sharp (see [26]). Unfortunately, our estimation gives no information for $K < -\dfrac{4\pi^2}{d^2}$. So we have to give better estimate when $K < 0$.

(ii) Theorem 0.1 generalizes the corresponding results in the Riemannian case and the method of proof is similar (see [3][5]).

(iii) λ_1 is the first Neumann eigenvalue of the Finsler-Laplacian means that $\Delta u = -\lambda_1 u$ in M with a Neumann boundary condition $\nabla u(x) \in T_x(\partial M)$ if ∂M is not empty ([19]).

We should remark that, in the Finsler setting, the gradient and the Laplacian are both nonlinear operators (see Section 1 below). In general, the following DO NOT hold:

$$\nabla(u+v) = \nabla u + \nabla v, \quad \nabla(uv) = u\nabla v + v\nabla u,$$
$$\Delta(u+v) = \Delta u + \Delta v, \quad \Delta(uv) = u\Delta v + v\Delta u + 2\langle \nabla u, \nabla v\rangle.$$

This creates many difficulties in the calculations. To overcome it, we use the weighted-linear operators, such as the weighted gradient and the weighted Finsler Laplacian. With them, we can convert some nonlinear problems into the linear ones and then do some calculations as in the Riemannian case. It is shown in [6] that the eigenfunction u of the Finsler Laplacian has only the regularity $u \in C^{1,\alpha}(M) \cap C^{\infty}(M_u)$, where $M_u := \{du \neq 0\}$. The lack of the good regularity forces us to avoid the point x where $du(x) = 0$ and address the issue on M_u. For example, we can not compute Δu in $M \backslash M_u$.

It is well known that Riemannian metrics are reversible metrics, but in general, Finsler metrics are not reversible. This means that if u is an eigenfunction of the Finsler-Laplacian, then $-u$ is not necessarily so. In this case, it is an issue to construct an auxiliary function when coping with the gradient estimate.

Finally, and most importantly, we know that Yau's gradient estimate belongs to the linear problem, the gradient and Laplacian, however, are both nonlinear operators in Finsler manifolds. Besides, for any function f, if $df(x) = 0$, then Δf generally has no definition at the point x. Thus we can not use the Finsler Laplacian Δ to adopt the maximum principle and obtain the gradient estimate, which is the key method in Riemannian geometry. To establish a gradient estimate in the Finsler setting, we have to utilize the maximum principle for the weighted Finsler Laplacian $\Delta^{\nabla u}$ and guarantee that the test function constructed attains its maximum in M_u.

By the above skill, we may reform the techniques used in the Riemannian case to solve the corresponding problems in Finsler geometry. In view of the Bochner-Weitzenböck formula obtained by Ohta-Sturm ([14]), we then prove the following results.

THEOREM 0.2. *Let $(M, F, d\mu)$ be a compact Finsler n-manifold without boundary or with a convex boundary. If the weighted Ricci curvature satisfies $\mathrm{Ric}_N \geq K (K < 0)$ for $N \in (n, \infty)$, then the first closed or Neumann eigenvalue of Finsler-Laplacian*

$$\lambda_1 \geq \frac{-C_N^2 \pi^2 K}{32} (e^{C_N d\sqrt{-K}/4} - 1)^{-2},$$

where $C_N = \max\{\sqrt{N-1}, \sqrt{2}\}$. In particular, we have the following estimations.

(1) *If $-4\pi^2/d^2 \leq K < 0$, then*

$$\lambda_1 \geq \left(\frac{\pi^2}{d^2} + \frac{d^2 K^2}{16\pi^2}\right) e^{-C_N d\sqrt{-K}/2}.$$

(2) *If* $K \le -4\pi^2/d^2$, *then*

$$\lambda_1 \ge \frac{2\pi^2}{d^2} e^{-C_N d\sqrt{-K}/2}.$$

Remark. (i) From Theorem 0.2, we obtain the universal lower bound

$$\lambda_1 \ge \frac{\pi^2}{d^2} e^{-C_N d\sqrt{-K}/2}$$

for any compact Finsler manifold with $\mathrm{Ric}_N \ge K (K < 0)$.

(ii) Theorem 0.2 is a generalization of the results by Yang ([22]) for Riemannian Laplacian and Chu-Hu ([3]) for Riemannian f-Laplacian.

For a compact Riemannian n-manifold (M, g) with Ricci curvature $\mathrm{Ric} \ge (n-1)k > 0$, if the non-empty boundary ∂M has nonnegative mean curvature with respect to the outward normal vector, Reilly ([16]) proved that the first Dirichlet eigenvalue of the Laplacian also satisfies $\lambda_1 \ge nk$.

The second objective of this paper is to extend Reilly type estimate for the first Dirichlet eigenvalue to the Finsler setting. Since the first eigenfunctions are not $C^\infty(M)$, we adopt approximation method to build the integral inequality. The key issue is how to choose a suitable convexity of the boundary. On the base of the suitable convexity, we can give a proper estimate on the integral inequality, and then obtain the following theorem.

Theorem 0.3. *Let $(M, F, d\mu)$ be a compact reversible Finsler n-manifold with a mean convex boundary. If S curvature vanishes and the Ricci curvature satisfies* $\mathrm{Ric} \ge (n-1)k > 0$, *then the first Dirichlet eigenvalue of Finsler-Laplacian*

$$\lambda_1 \ge nk.$$

The paper is organized as follows. In Section 1, the related fundamentals of Finsler geometry such as Finsler metric, weighted Ricci curvature, gradient vector, Finsler-Laplacian and some lemmas are briefly introduced. The main results will be proved in Sections 2, and 3, respectively.

1. Preliminaries

Let M be a smooth n-manifold and $\pi : TM \to M$ be the natural projection from the tangent bundle TM. Let (x, y) be a point of TM with $x \in M$, $y \in T_xM$, and let (x^i, y^i) be the local coordinates on TM with $y = y^i \partial/\partial x^i$. A *Finsler metric* on M is a function $F : TM \to [0, +\infty)$ satisfying the following properties:

(i) *Regularity*: $F(x, y)$ is smooth in $TM \backslash 0$;

(ii) *Positive homogeneity*: $F(x, \lambda y) = \lambda F(x, y)$ for $\lambda > 0$;

(iii) *Strong convexity*: The fundamental quadratic form

$$g := g_{ij}(x, y)\, dx^i \otimes dx^j, \quad g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}$$

is positively definite.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a vector field. Then the *covariant derivative* of X by $v \in T_xM$ with reference vector $w \in T_xM \backslash 0$ is defined by

$$D_v^w X(x) := \left\{ v^j \frac{\partial X^i}{\partial x^j}(x) + \Gamma^i_{jk}(w) v^j X^k(x) \right\} \frac{\partial}{\partial x^i},$$

where Γ^i_{jk} denote the coefficients of the Chern connection.

Given two linearly independent vectors $V, W \in T_xM \backslash 0$, the flag curvature is defined by

$$K(V, W) := \frac{g_V(R^V(V, W)W, V)}{g_V(V, V)g_V(W, W) - g_V(V, W)^2},$$

where R^V is the *Chern curvature*

$$R^V(X, Y)Z = D^V_X D^V_Y Z - D^V_Y D^V_X Z - D^V_{[X, Y]} Z.$$

Then the Ricci curvature for (M, F) is defined as

$$\mathrm{Ric}(V) = \sum_{i=1}^{n-1} K(V, e_i),$$

where $e_1, \ldots, e_{n-1}, \frac{V}{F(V)}$ form an orthonormal basis of T_xM with respect to g_V.

For a given volume form $d\mu = \sigma(x)\, dx$ and a vector $V \in T_xM \backslash 0$, the distortion of $(M, F, d\mu)$ is defined by

$$\tau(V) := \ln \frac{\sqrt{\det(g_{ij}(V))}}{\sigma}.$$

To measure the rate of changes of the distortion along geodesics, we define

$$S(V) := \frac{d}{dt}[\tau(\dot{c}(t))]_{t=0},$$

where $c(t)$ is the geodesic with $\dot{c}(0) = V$. S is called the *S-curvature*.

Now we introduce the weighted Ricci curvature on Finsler manifolds, which was defined by Ohta. In the present paper, we reform it as follows:

Definition 1.1 ([13]). Let $(M, F, d\mu)$ be a Finsler n-manifold. Given a vector $V \in T_xM$, let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a geodesic with $\gamma(0) = x$, $\dot{\gamma}(0) = V$. Define

$$\dot{S}(V) := F^{-2}(V)\frac{d}{dt}[S(\gamma(t), \dot{\gamma}(t))]_{t=0},$$

where $S(V)$ denotes the S-curvature at (x, V).

The *weighted Ricci curvature* of $(M, F, d\mu)$ is defined by

$$\begin{cases} \mathrm{Ric}_n(V) := \begin{cases} \mathrm{Ric}(V) + \dot{S}(V), & \text{for } S(V) = 0, \\ -\infty, & \text{otherwise}, \end{cases} \\ \mathrm{Ric}_N(V) := \mathrm{Ric}(V) + \dot{S}(V) - \dfrac{S(V)^2}{(N-n)F(V)^2}, \quad \forall N \in (n, \infty), \\ \mathrm{Ric}_\infty(V) := \mathrm{Ric}(V) + \dot{S}(V), \end{cases}$$

Let $\mathscr{L} : TM \to T^*M$ denote the *Legendre transformation*, which satisfies $\mathscr{L}(0) = 0$ and $\mathscr{L}(\lambda y) = \lambda\mathscr{L}(y)$ for all $\lambda > 0$ and $y \in TM\backslash\{0\}$. Then $\mathscr{L} : TM\backslash\{0\} \to T^*M\backslash\{0\}$ is a norm-preserving C^∞ diffeomorphism. For a smooth function $u : M \to R$, the *gradient vector* of u at $x \in M$ is defined as $\nabla u(x) := \mathscr{L}^{-1}(du(x)) \in T_xM$, which can be written as

$$\nabla u(x) := \begin{cases} g^{ij}(x, \nabla u)\dfrac{\partial u}{\partial x^j}\dfrac{\partial}{\partial x^i}, & du(x) \neq 0, \\ 0, & du(x) = 0. \end{cases}$$

Set $M_V := \{x \in M \mid V(x) \neq 0\}$ for a vector field V on M, and $M_u := M_{\nabla u}$. For a smooth vector field V on M and $x \in M_V$, we define $\nabla V(x) \in T_x^*M \otimes T_xM$ by using the covariant derivative as

$$\nabla V(v) := D_v^V V(x) \in T_xM, \quad v \in T_xM.$$

For a smooth function $u : M \to R$ and $x \in M_u$, We set $\nabla^2 u(x) := \nabla(\nabla u)(x)$. Let $\{e_a\}_{a=1}^n$ be a local orthonormal basis with respect to $g_{\nabla u}$ on M_u and put $u_{ab} = g_{\nabla u}(D_{e_a}^{\nabla u}\nabla u, e_b)$. Then we have

$$u_{ab} = u_{ba}, \quad \forall a, b.$$

Let $V = V^i\dfrac{\partial}{\partial x^i}$ be a smooth vector field on M. The *divergence* of V with respect to an arbitrary volume form $d\mu$ is defined by

$$\mathrm{div}\ V := \sum_{i=1}^n \left(\frac{\partial V^i}{\partial x^i} + V^i\frac{\partial \Phi}{\partial x^i}\right),$$

where $d\mu = e^\Phi\, dx$. Then the *Finsler-Laplacian* of u can be defined by

$$\Delta u := \mathrm{div}(\nabla u).$$

Given a vector field V such that $V \neq 0$ on M_u, the *weighted gradient vector* and the *weighted Laplacian* on the weighted Riemannian manifold (M, g_V) are defined by

$$\nabla^V u := \begin{cases} g^{ij}(V)\dfrac{\partial u}{\partial x^j}\dfrac{\partial}{\partial x^i}, & \text{in } M_u, \\ 0, & \text{in } M\backslash M_u, \end{cases} \qquad \Delta^V u := \operatorname{div}(\nabla^V u).$$

We note that $\nabla^{\nabla u} u = \nabla u$, $\Delta^{\nabla u} u = \Delta u$.

Let (M, F) be a Finsler manifold with boundary ∂M and v be the normal vector that points outward M. Here a normal vector v at $x \in \partial M$ means that for any $w \in T_x(\partial M)$, $g_v(v, w) = 0$. There are exactly two normal vectors $v, v' \in T_x M$. In general, $v' \neq -v$. The *normal curvature* $\Lambda_v(V)$ at $x \in \partial M$ in direction $V \in T_x(\partial M)$ is defined by ([17])

$$\Lambda_v(V) := g_v(v, D_{\dot\gamma}^{\dot\gamma}\dot\gamma|_x),$$

where γ is the unique local geodesic for the Finsler structure $F_{\partial M}$ on ∂M induced by F with the initial data $\gamma(0) = x$ and $\dot\gamma(0) = V$. M is said to have *convex boundary* if for any $x \in \partial M$, the normal curvature Λ at x is non-positive in any directions $V \in T_x(\partial M)$.

DEFINITION 1.2 ([26]). Let $\Omega \subset M$ be a smooth domain of a reversible Finsler manifold (M, F). The boundary $\partial\Omega$ is called mean convex if there exists (and then for all) a C^2 function ϕ satisfying

$$\begin{cases} \phi_i(x) = 0, & x \in U \cap \partial\Omega; \\ \phi_i(x) > 0, & x \in U \cap \Omega; \\ d\phi_i(x) \neq 0, & x \in U \cap \partial\Omega, \end{cases}$$

where U is a neighborhood of $x \in \partial\Omega$, such that $F(\nabla\phi)H = \sum_{a=1}^{n-1} \nabla^2\phi(e_a, e_a) \leq 0$. Here $\{e_a\}_{a=1}^n$ is a $g_{\nabla\phi}$-orthogonal frame field with $e_n = \dfrac{\nabla\phi}{F(\nabla\phi)}$.

LEMMA 1.3 ([14]). *Let (M, F) be a Finsler n-manifold. Given $u \in C^\infty(M)$, we have*

$$\Delta^{\nabla u}\left(\frac{F(\nabla u)^2}{2}\right) - D(\Delta u)(\nabla u) = |\nabla u|^2 \operatorname{Ric}_\infty(\nabla u) + |\nabla^2 u|^2_{HS(\nabla u)}$$

point-wise on M_u. Here $|\nabla^2 u|^2_{HS(\nabla u)}$ stands for the Hilbert-Schmidt norm with respect to $g_{\nabla u}$.

LEMMA 1.4 ([21]). *Let (M, F) be a Finsler n-manifold and $u : M \to R$ be a smooth function. Then on M_u we have*

$$\Delta u = \operatorname{tr}_{g_{\nabla u}}(\nabla^2 u) - S(\nabla u) = \sum_a u_{aa} - S(\nabla u),$$

where $u_{aa} = g_{\nabla u}(\nabla^2 u(e_a), e_a)$ and $\{e_a\}_{a=1}^n$ is a local $g_{\nabla u}$-orthonormal basis on M_u.

2. The first Neumann and closed eigenvalue

Before the proof of our main results, we introduce the following

LEMMA 2.1 ([19]). *Let (M^n, F, m) be an n-dimensional compact Finsler measure space, equipped with a Finsler structure F and a smooth measure m, without boundary or with a convex boundary. Assume that* $\mathrm{Ric}_N \geq K$ *for some real numbers $N \in [n, \infty]$ and $K \in R$. Let λ_1 be the first Neumann eigenvalue of the Finsler-Laplacian if ∂M is not empty. Then*

$$\lambda_1 \geq \lambda_1(K, N, d),$$

where d is the diameter of M, $\lambda_1(K, N, d)$ represents the first eigenvalue of the 1*-dimensional problem*

$$v'' - T(t)v' = -\lambda_1(K, N, d)v \quad in \ \left(-\frac{d}{2}, \frac{d}{2}\right), \quad v'\left(-\frac{d}{2}\right) = v'\left(\frac{d}{2}\right) = 0 \tag{2.1}$$

where $T(t)$ is defined by

$$T(t) = \begin{cases} \sqrt{(N-1)K} \tan\left(\sqrt{\frac{K}{N-1}}t\right), & K > 0,\ 1 < N < \infty; \\ -\sqrt{-(N-1)K} \tanh\left(\sqrt{-\frac{K}{N-1}}t\right), & K < 0,\ 1 < N < \infty; \\ 0, & K = 0,\ 1 < N < \infty; \\ Kt, & N = \infty. \end{cases}$$

Using the above Lemma and similar arguments as in the Riemaniann case, it is easy to prove

THEOREM 2.2. *Let $(M, F, d\mu)$ be a compact Finsler n-manifold without boundary or with a convex boundary. If the weighted Ricci curvature satisfies* $\mathrm{Ric}_N \geq K$ *for $N \in [n, \infty]$, $K \in R$, then the first closed or Neumann eigenvalue of Finsler-Laplacian*

$$\lambda_1 \geq 4s(1-s)\frac{\pi^2}{d^2} + sK := C(d, K, s), \tag{2.2}$$

where d denotes the diameter of (M, F) and $\forall s \in (0, 1)$. In particular,

$$\lambda_1 \geq \frac{\pi^2}{d^2} + \frac{K}{2}.$$

Proof. Differentiating both sides of (2.1), we have

$$v''' - T'v' - Tv'' = -\lambda_1(K, N, d)v'. \tag{2.3}$$

Multiplying (2.3) by $(v')^a$ for $a > 0$ and integrating it over $\left(-\frac{d}{2}, \frac{d}{2}\right)$ yields

$$\int_{-d/2}^{d/2} v'''(v')^a\,dt - \int_{-d/2}^{d/2} T'(v')^{1+a}\,dt - \int_{-d/2}^{d/2} Tv''(v')^a\,dt = -\lambda_1(K,N,d)\int_{-d/2}^{d/2}(v')^{1+a}\,dt. \tag{2.4}$$

Notice that $v' > 0$ in $\left(-\frac{d}{2}, \frac{d}{2}\right)$ (see [19] for details) and $v'\left(-\frac{d}{2}\right) = v'\left(\frac{d}{2}\right) = 0$, we get

$$\int_{-d/2}^{d/2} v'''(v')^a\,dt = -a\int_{-d/2}^{d/2}(v'')^2(v')^{a-1}\,dt = -\frac{4a}{(1+a)^2}\int_{-d/2}^{d/2}(((v')^{(a+1)/2})')^2\,dt. \tag{2.5}$$

In addition,

$$-\int_{-d/2}^{d/2} Tv''(v')^a\,dt = -\frac{1}{1+a}\int_{-d/2}^{d/2} Td(v')^{a+1} = \frac{1}{1+a}\int_{-d/2}^{d/2} T'(v')^{a+1}\,dt. \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.4), one obtains

$$\frac{4a}{(1+a)^2}\int_{-d/2}^{d/2}(((v')^{(a+1)/2})')^2\,dt = \int_{-d/2}^{d/2}\left(\lambda_1(K,N,d) - \frac{a}{a+1}T'\right)(v')^{a+1}\,dt. \tag{2.7}$$

Since $v'\left(-\frac{d}{2}\right) = v'\left(\frac{d}{2}\right) = 0$, by using Wirtinger's inequality

$$\int_{-l}^{l}\phi^2(x)\,dx \le \left(\frac{2l}{\pi}\right)^2\int_{-l}^{l}(\phi')^2(x)\,dx, \quad \forall\phi \in C^1[-l,l],\ \phi(\pm l) = 0,$$

we achieve that

$$\frac{4a}{(1+a)^2}\frac{\pi^2}{d^2}\int_{-d/2}^{d/2}(v')^{a+1}\,dt \le \int_{-d/2}^{d/2}\left(\lambda_1(K,N,d) - \frac{a}{a+1}T'\right)(v')^{a+1}\,dt \le \left(\lambda_1(K,N,d) - \frac{a}{a+1}\min_{t\in[-d/2,d/2]}T'\right)\int_{-d/2}^{d/2}(v')^{a+1}\,dt.$$

It is easy to check that $T' = K + \frac{T^2}{N-1}$. Hence,

$$\lambda_1(K,N,d) \ge \frac{4a}{(a+1)^2}\frac{\pi^2}{d^2} + \frac{a}{a+1}K.$$

Taking $s = \dfrac{a}{a+1}$ and using Lemma 2.1, we conclude

$$\lambda_1 \geq 4s(1-s)\frac{\pi^2}{d^2} + sK := C(d, K, s).$$

View the right side of above formula as a quadratic polynomial on s, and note that $d \leq \dfrac{\sqrt{N-1}\pi}{\sqrt{K}}$, $K > 0$ (see [13] for details), then it is not hard to get $\sup_s C(d, K, s)$ in the remark. □

THEOREM 2.3. *Let $(M, F, d\mu)$ be a compact Finsler n-manifold without boundary or with a convex boundary. If the weighted Ricci curvature satisfies* $\mathrm{Ric}_N \geq K (K < 0)$ *for $N \in (n, \infty)$, then the first closed or Neumann eigenvalue of Finsler-Laplacian*

$$\lambda_1 \geq \frac{-C_N^2\pi^2 K}{32}(e^{C_N d\sqrt{-K}/4} - 1)^{-2}, \tag{2.8}$$

where $C_N = \max\{\sqrt{N-1}, \sqrt{2}\}$. In particular, we have the following estimations.

(1) *If $-4\pi^2/d^2 \leq K < 0$, then*

$$\lambda_1 \geq \left(\frac{\pi^2}{d^2} + \frac{d^2K^2}{16\pi^2}\right)e^{-C_N d\sqrt{-K}/2}. \tag{2.9}$$

(2) *If $K \leq -4\pi^2/d^2$, then*

$$\lambda_1 \geq \frac{2\pi^2}{d^2}e^{-C_N d\sqrt{-K}/2}. \tag{2.10}$$

Remark. Before the proof, we give some explication. Since gradient and Laplacian are nonlinear operators in Finsler manifolds, the calculations are not so easy as in the Riemannian case. For instance, $\nabla(u+v) \neq \nabla u + \nabla v$ and $\nabla(cu) \neq c\nabla u$ for $c \in \mathbf{R}$ in general. We construct the weighted operators $\nabla^{\nabla u}$ and $\Delta^{\nabla u}$ with the reference direction ∇u. This is because these weighted operators are linear operators and moreover in M_u we have $\nabla^{\nabla u}u = \nabla u$, $\Delta^{\nabla u}u = \Delta u$. We should also point out that $\Delta^{\nabla u}$ satisfies the maximal principle on M_u but have no sense on $M \backslash M_u$. Therefore, with the weighted operators at hand, we can use the technique similar to the Riemannian case to handle the problems on M_u but need to give additional discussion on $M \backslash M_u$.

Proof. We firstly consider the case that $\partial M = \emptyset$. Let u be a first eigenfunction on (M, F) corresponding to the first eigenvalue λ_1. Since $\int_M u\, d\mu = -\frac{1}{\lambda_1}\int_M \Delta u\, d\mu = 0$ and noting that $-u$ is not necessarily the first eigenfunction on (M, F), we assume that

$$1 = \sup u > \inf u = -k \geq -1 \quad (\text{resp. } 1 \geq k = \sup u > \inf u = -1), \quad 0 < k \leq 1.$$

For small $\varepsilon > 0$, let

$$v = \frac{u - \frac{1}{2}(1-k)}{\frac{1}{2}(1+k)(1+\varepsilon)} \quad \left(\text{resp. } v = \frac{u + \frac{1}{2}(1-k)}{\frac{1}{2}(1+k)(1+\varepsilon)}\right).$$

Clearly, $dv = \frac{2}{(1+k)(1+\varepsilon)}\, du$. Thus by the property of Legendre transform we have

$$\nabla v = \nabla^{\nabla u} v = \frac{2}{(1+k)(1+\varepsilon)} \nabla u,$$

under which

$$\begin{cases} \Delta v = -\lambda_1 (v \pm a_\varepsilon), & a_\varepsilon = \dfrac{1-k}{(1+k)(1+\varepsilon)}, \\ \sup v = \dfrac{1}{1+\varepsilon}, & \inf v = -\dfrac{1}{1+\varepsilon}. \end{cases}$$

Let $v = \sin\theta$, then $dv = \cos\theta\, d\theta$ and

$$-\frac{1}{1+\varepsilon} \le \sin\theta \le \frac{1}{1+\varepsilon}, \quad \frac{|\nabla v|^2}{1-v^2} = |\nabla\theta|^2.$$

We note that $\theta \in \left[-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right]$, where δ satisfies $\sin\left(\frac{\pi}{2} - \delta\right) = \frac{1}{1+\varepsilon}$, and that $\nabla\theta = \nabla^{\nabla v}\theta$ since $\cos\theta > 0$. Set

$$U(\theta) := \max_{x \in M, \theta(x) = \theta} |\nabla\theta|^2 = \max_{x \in M, \theta(x) = \theta} \frac{|\nabla v|^2}{1-v^2}, \quad \theta \in \left[-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right].$$

Here and from now on, we use $|\nabla\theta|$, $|\nabla v|$ to denote $F(\nabla\theta)$, $F(\nabla v)$. Then $U(\theta)$ is continuous on $\left[-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right]$. Moreover, for any $\theta_0 \in \left(-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right)$, there exists a point $x_0 \in M$ such that $\theta(x_0) = \theta_0$ and $U(\theta_0) = |\nabla\theta|^2(x_0)$.

Now we claim that

$$|\nabla\theta| \le \sqrt{\lambda_1(1 + a_\varepsilon)} + b\sqrt{-K}\cos\theta, \quad \theta \in \left[-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right],$$

where $b = \frac{1}{2}\max\{\sqrt{N-1}, \sqrt{2}\} := \frac{1}{2}C_N$. If it is not true, one would take x_0 attaining

$$A := \max_{x \in M}\{|\nabla\theta(x)| - b\sqrt{-K}\cos\theta(x)\}.$$

and put $\theta_0 := \theta(x_0)$. Clearly,

$$A > \sqrt{\lambda_1(1 + a_\varepsilon)}. \tag{2.11}$$

If $\sqrt{U(\theta)} - b\sqrt{-K}\cos\theta$ attains its maximum at $x_0 \in M\backslash M_u$, then

$$\sqrt{U(\theta)} - b\sqrt{-K}\cos\theta \le \sqrt{U(\theta_0)} - b\sqrt{-K}\cos\theta_0 = -b\sqrt{-K}\cos\theta_0 \le \sqrt{\lambda_1(1+a_\varepsilon)}.$$

The claim holds undoubtedly. Next, we suppose that $\sqrt{U(\theta)} - b\sqrt{-K}\cos\theta$ attains its maximum at $x_0 \in M_u$. Put

$$G(x) := \left\{\frac{|\nabla v|^2}{1-v^2} - (A + b\sqrt{-K}\cos\theta)^2\right\}\cos^2\theta.$$

By the definitions of $U(\theta)$ and A, we have at x_0 that

$$\begin{cases} G(x_0) = 0, \\ \nabla^{\nabla u} G(x_0) = 0, \\ \Delta^{\nabla u} G(x_0) \le 0. \end{cases}$$

Let $e_1, e_2, \ldots, e_n = \frac{\nabla u}{|\nabla u|} = \frac{\nabla v}{|\nabla v|}$ be a local orthonormal basis with respect to $g_{\nabla u}$ on M_u. Then by simple computations on $\nabla^{\nabla u} G(x_0) = 0$ we have

$$\begin{cases} v_{an} = 0, \quad a \ne n \\ v_{nn} = -(A + b\sqrt{-K}\cos\theta)(A + 2b\sqrt{-K}\cos\theta)\sin\theta \\ \quad\;\; = \frac{-v|\nabla v|^2}{1-v^2} - b\sqrt{-K}v|\nabla v|. \end{cases} \tag{2.12}$$

Furthermore, a straightforward calculation yields

$$\begin{aligned} 0 \ge \Delta^{\nabla u} G(x_0) &= \cos^2\theta\Delta^{\nabla u}\left(\frac{|\nabla v|^2}{1-v^2} - (A + b\sqrt{-K}\cos\theta)^2\right) \\ &\quad + \left(\frac{|\nabla v|^2}{1-v^2} - (A + b\sqrt{-K}\cos\theta)^2\right)\Delta^{\nabla u}\cos^2\theta \\ &\quad + 2g_{\nabla u}\left(\nabla^{\nabla u}\left(\frac{|\nabla v|^2}{1-v^2} - (A + b\sqrt{-K}\cos\theta)^2\right), \nabla^{\nabla u}\cos^2\theta\right) \\ &= \cos^2\theta\Delta^{\nabla u}\left(\frac{|\nabla v|^2}{1-v^2} - (A + b\sqrt{-K}\cos\theta)^2\right) + 0 + 0 \\ &= \cos^2\theta\Delta^{\nabla u}\left(\frac{|\nabla v|^2}{1-v^2}\right) - \cos^2\theta\Delta^{\nabla u}((A + b\sqrt{-K}\cos\theta)^2). \end{aligned} \tag{2.13}$$

On the other hand,

$$\Delta^{\nabla u}\left(\frac{|\nabla v|^2}{1-v^2}\right) = \frac{\Delta^{\nabla u}(|\nabla v|^2)}{1-v^2} + |\nabla v|^2 \Delta^{\nabla u}\left(\frac{1}{1-v^2}\right) + 2g_{\nabla u}\left(\nabla^{\nabla u}(|\nabla v|^2), \nabla^{\nabla u}\left(\frac{1}{1-v^2}\right)\right) := A + B + C$$

where

$$A = \frac{\Delta^{\nabla u}(|\nabla v|^2)}{1-v^2},$$

$$\begin{aligned} B &= |\nabla v|^2 \Delta^{\nabla u}\left(\frac{1}{1-v^2}\right) = |\nabla v|^2 \operatorname{div}\left(\nabla^{\nabla u}\left(\frac{1}{1-v^2}\right)\right) \\ &= |\nabla v|^2 \left\{\frac{2v}{(1-v^2)^2} \operatorname{div}(\nabla^{\nabla u} v) + 2g_{\nabla u}\left(\nabla^{\nabla u} v, \nabla^{\nabla u}\left(\frac{v}{(1-v^2)^2}\right)\right)\right\} \\ &= |\nabla v|^2 \left\{\frac{2v}{(1-v^2)^2}\Delta v + \frac{2|\nabla v|^2}{(1-v^2)^2} + \frac{8v^2|\nabla v|^2}{(1-v^2)^3}\right\}, \end{aligned}$$

$$\begin{aligned} C &= 2g_{\nabla u}\left(\nabla^{\nabla u}(|\nabla v|^2), \nabla^{\nabla u}\left(\frac{1}{1-v^2}\right)\right) \\ &= \frac{8v|\nabla v|^2}{(1-v^2)^2} v_{nn} = -\frac{8v^2|\nabla v|^4}{(1-v^2)^3} - \frac{8b\sqrt{-K}v^2|\nabla v|^3}{(1-v^2)^2} \end{aligned}$$

By Lemma 1.3 and the conditions of Theorem 2.3, we get

$$\begin{aligned} \Delta^{\nabla u}(|\nabla v|^2) &= 2|\nabla v|^2 \operatorname{Ric}_\infty(\nabla v) + 2D(\Delta v)(\nabla v) + 2|\nabla^2 v|^2_{HS(\nabla v)} \\ &= 2|\nabla v|^2 \operatorname{Ric}_\infty(\nabla v) + 2D(-\lambda_1(v \pm a_\varepsilon))(\nabla v) + 2\sum_{ab} v_{ab}^2 \\ &\geq 2|\nabla v|^2 \operatorname{Ric}_\infty(\nabla v) - 2\lambda_1|\nabla v|^2 + 2\left(v_{nn}^2 + \sum_{a=1}^{n-1} v_{aa}^2\right) \\ &\geq 2|\nabla v|^2 \operatorname{Ric}_\infty(\nabla v) - 2\lambda_1|\nabla v|^2 + 2v_{nn}^2 + \frac{2}{n-1}\left(\sum_{a=1}^{n-1} v_{aa}\right)^2 \\ &= 2|\nabla v|^2 \operatorname{Ric}_\infty(\nabla v) - 2\lambda_1|\nabla v|^2 + 2v_{nn}^2 \\ &\quad + \frac{2}{n-1}(\Delta v - v_{nn} + S(\nabla v))^2 \end{aligned} \tag{2.14}$$

$$
\begin{aligned}
&\geq 2|\nabla v|^2 \operatorname{Ric}_\infty(\nabla v) - 2\lambda_1|\nabla v|^2 + 2v_{nn}^2 \\
&\quad + \frac{2}{N-1}(\Delta v - v_{nn})^2 - \frac{2S(\nabla v)^2}{N-n} \\
&= 2|\nabla v|^2 \operatorname{Ric}_N(\nabla v) - 2\lambda_1|\nabla v|^2 + 2v_{nn}^2 + \frac{2}{N-1}(\Delta v - v_{nn})^2 \\
&\geq -2(\lambda_1 - K)|\nabla v|^2 + 2v_{nn}^2 + \frac{2}{N-1}(\Delta v - v_{nn})^2
\end{aligned}
$$

Therefore we have that

$$
\begin{aligned}
\cos^2\theta\Delta^{\nabla u}\left(\frac{|\nabla v|^2}{1-v^2}\right) &\geq -2(\lambda_1 - K)|\nabla v|^2 + 2v_{nn}^2 + \frac{2}{N-1}(\Delta v - v_{nn})^2 \\
&\quad + 2|\nabla v|^2\left\{\frac{v\Delta v + |\nabla v|^2}{(1-v^2)}\right\} - \frac{8b\sqrt{-K}v^2|\nabla v|^3}{1-v^2}
\end{aligned}
\tag{2.15}
$$

at x_0. In addition,

$$
\begin{aligned}
&-\cos^2\theta\Delta^{\nabla u}((A + b\sqrt{-K}\cos\theta)^2) \\
&= -\cos^2\theta[2(A + b\sqrt{-K}\cos\theta)\Delta^{\nabla u}(A + b\sqrt{-K}\cos\theta) \\
&\qquad + 2g_{\nabla u}(b\sqrt{-K}\nabla^{\nabla u}\cos\theta, b\sqrt{-K}\nabla^{\nabla u}\cos\theta)] \\
&= -\cos^2\theta\Bigg[2b\sqrt{-K}(A + b\sqrt{-K}\cos\theta) \\
&\qquad \times\left(\frac{\lambda_1\sin\theta(\sin\theta \pm a_\varepsilon)}{\cos\theta} - \frac{|\nabla\theta|^2}{\cos\theta}\right) - 2b^2K\sin^2\theta|\nabla\theta|^2\Bigg] \\
&= 2b\sqrt{-K}(A + b\sqrt{-K}\cos\theta)(-\lambda_1\sin\theta\cos\theta(\sin\theta \pm a_\varepsilon) + \cos\theta|\nabla\theta|^2) \\
&\quad + 2b^2K\sin^2\theta\cos^2\theta|\nabla\theta|^2.
\end{aligned}
\tag{2.16}
$$

Setting $y := A + b\sqrt{-K}\cos\theta$ and substituting (2.15) (2.16) into (2.13), we have

$$
\begin{aligned}
y^2 &\leq \lambda_1(1 + a_\varepsilon) - K\cos^2\theta + by\sqrt{-K}\cos\theta - 2Ab\sqrt{-K}\cos^3\theta \\
&\quad + 2b^2K\cos^4\theta + b\sqrt{-K}\cos\theta\left|\frac{\lambda_1(1 + a_\varepsilon)}{y}\sin\theta\right| \\
&\quad - \frac{1}{N-1}\left[\frac{\lambda_1(-\sin\theta \mp a_\varepsilon)}{y} + A\sin\theta + 2b\sqrt{-K}\cos\theta\sin\theta\right]^2.
\end{aligned}
\tag{2.17}
$$

Notice that

$$
\left|\frac{\lambda_1(1 + a_\varepsilon)}{y}\sin\theta\right| \leq \sqrt{\lambda_1(1 + a_\varepsilon)} \leq A
$$

and

$$-\frac{4}{N-1}\left[\frac{\lambda_1(-\sin\theta \mp a_\varepsilon)}{y} + A\sin\theta\right] b\sqrt{-K}\cos\theta\sin\theta$$
$$= -\frac{4}{N-1}\left[\frac{\lambda_1(-\sin\theta \mp a_\varepsilon)}{y}\sin\theta + A\right] b\sqrt{-K}\cos\theta + \frac{4}{N-1}Ab\sqrt{-K}\cos^3\theta$$
$$\le \frac{4}{N-1}Ab\sqrt{-K}\cos^3\theta,$$

one gets from (2.17) that

$$\left(y - \frac{1}{2}b\sqrt{-K}\cos\theta\right)^2 \le \left(\sqrt{\lambda_1(1+a_\varepsilon)} + \frac{1}{2}b\sqrt{-K}\cos\theta\right)^2 - K\left(1 - \frac{4b^2}{N-1}\right)\cos^2\theta + \left(\frac{4}{N-1} - 2\right)Ab\sqrt{-K}\cos^3\theta - \left(\frac{4}{N-1} - 2\right)b^2K\cos^4\theta. \tag{2.18}$$

If $N \ge 3$, by choosing $b = \frac{1}{2}\sqrt{N-1}$, then (2.18) implies

$$y \le \sqrt{\lambda_1(1+a_\varepsilon)} + b\sqrt{-K}\cos\theta$$

which contradicts (2.11). If $N \le 3$, then

$$-\frac{1}{N-1}\left[\frac{\lambda_1(-\sin\theta \mp a_\varepsilon)}{y} + A\sin\theta + 2b\sqrt{-K}\cos\theta\sin\theta\right]^2$$
$$\le -\frac{1}{2}\left[\frac{\lambda_1(-\sin\theta \mp a_\varepsilon)}{y} + A\sin\theta + 2b\sqrt{-K}\cos\theta\sin\theta\right]^2.$$

By a similar argument, we can also draw the conclusion. Thus we have proved that

$$|\nabla\theta| \le \sqrt{\lambda_1(1+a_\varepsilon)} + b\sqrt{-K}\cos\theta, \quad \theta \in \left[-\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta\right],$$

where $b = \frac{1}{2}\max\{\sqrt{N-1}, \sqrt{2}\} := \frac{1}{2}C_N$.

Let $p, q \in M$ be two points such that $\theta(p) = -\frac{\pi}{2}+\delta$, $\theta(q) = \frac{\pi}{2}-\delta$. Let γ be a shortest geodesic from p to q. Denote by T the tangent vector of γ. Then

$$|\nabla\theta| = \frac{|\nabla v|}{\cos\theta} = \frac{F(\nabla v)}{\cos\theta} \ge \frac{g_{\nabla u}\left(\nabla v, \frac{T}{F(T)}\right)}{\cos\theta} = \frac{Tv}{F(T)\cos\theta} = \frac{\frac{dv}{ds}}{F(T)\cos\theta} = \frac{\frac{d\theta}{ds}}{F(T)}. \tag{2.19}$$

We restrict s to γ with $\dfrac{d\theta}{ds} \geq 0$. Then the following integration $\int d\theta$ still covers whole $\left(-\dfrac{\pi}{2}+\delta, \dfrac{\pi}{2}-\delta\right)$. Therefore, from (2.19), one gets

$$\begin{aligned} d &\geq \int_\gamma F(T)\,ds \geq \int_{-\pi/2+\delta}^{\pi/2-\delta} \frac{d\theta}{\sqrt{\lambda_1(1+a_\varepsilon)} + \frac{1}{2}C_N\sqrt{-K}\cos\theta} \\ &\geq \int_{-\pi/2+\delta}^{\pi/2-\delta} \frac{d\theta}{\sqrt{\lambda_1(1+a_\varepsilon)} + \frac{1}{2}C_N\sqrt{-K}\left(\frac{\pi}{2}-|\theta|\right)} \\ &= \frac{4}{C_N\sqrt{-K}} \log \frac{\sqrt{\lambda_1(1+a_\varepsilon)} + \frac{\pi}{4}C_N\sqrt{-K}}{\sqrt{\lambda_1(1+a_\varepsilon)} + \frac{1}{2}C_N\delta}. \end{aligned}$$

Let $\varepsilon \to 0$, then $\delta \to 0$ too. Thus we obtain

$$\lambda_1 \geq \frac{C_N^2\pi^2(-K)}{32}\left[\exp\left(\frac{C_N}{4}d\sqrt{-K}\right)-1\right]^{-2}.$$

By using the inequality $(e^x-1)^2 \leq e^{2x}-1$, $\forall x \geq 0$, we further have

$$\begin{aligned} \lambda_1 &\geq \frac{C_N^2\pi^2(-K)}{32}\left[\exp\left(\frac{C_N}{4}d\sqrt{-K}\right)-1\right]^{-2} \\ &\geq \frac{-K}{2}\left[\exp\left(\frac{C_N}{4}d\sqrt{-K}\right)-1\right]^{-2} \\ &\geq \frac{-K}{2}\left[\exp\left(\frac{C_N}{2}d\sqrt{-K}\right)-1\right]^{-1} \end{aligned}$$

which implies

$$\frac{K}{2} \geq \lambda_1\left(1-\exp\left(\frac{C_N}{2}d\sqrt{-K}\right)\right). \tag{2.20}$$

From the remark below Theorem 0.1, we know that $\lambda_1 \geq \dfrac{\pi^2}{d^2}+\dfrac{K}{2}+\dfrac{d^2K^2}{16\pi^2}$ for $K \in \left[\dfrac{-4\pi^2}{d^2}, 0\right]$. This together with (2.20) gives (2.9). If $K \leq \dfrac{-4\pi^2}{d^2}$, then from (2.20) we have

$$\frac{-2\pi^2}{d^2} \geq \lambda_1\left(1-\exp\left(\frac{C_N}{2}d\sqrt{-K}\right)\right),$$

which implies (2.10).

Now we consider the case that $\partial M \neq \emptyset$. Let u be a first Neumann eigenfunction corresponding to the first eigenvalue λ_1 and

$$G(x) := \left\{\frac{|\nabla v|^2}{1-v^2} - (A+b\sqrt{-K}\cos\theta)^2\right\}\cos^2\theta.$$

If x_0 is an interior point of M, the proof has been given above. Now we assume that $x_0 \in \partial M$. Let $\nu_{\nabla u}$ be the unit normal vector that points outward M with respect to $g_{\nabla u}$. Then

$$DG(\nu_{\nabla u})(x_0) \geq 0.$$

Since Neumann boundary condition yields $\nabla u \in T(\partial M)$, we have $Du(\nu_{\nabla u}) = g_{\nabla u}(\nu_{\nabla u}, \nabla u) = 0$. Noting that ∂M is convex and the fact that $\nabla v = \dfrac{2}{(1+k)(1+\varepsilon)}\nabla u$, one gets

$$\begin{aligned} DG(\nu_{\nabla u}) &= Dg_{\nabla u}(\nabla v, \nabla v)(\nu_{\nabla u}) = 2g_{\nabla u}(D^{\nabla u}_{\nu_{\nabla u}}\nabla v, \nabla v) \\ &= 2g_{\nabla v}(\nu_{\nabla v}, D^{\nabla v}_{\nabla v}\nabla v) \leq 0 \end{aligned}$$

at x_0. The last inequality follows from Lemma 3.1 and Lemma 3.2 in [19], which shows that it is equivalent to $g_\nu(\nu, D^{\nabla v}_{\nabla v}\nabla v) \leq 0$, where ν denotes the unit normal vector that points outwards M. The tangent derivative of G obviously vanishes due to its maximality. Therefore

$$\nabla^{\nabla u}G(x_0) = 0.$$

The rest of the proof proceeds as the case that $\partial M = \emptyset$. □

3. The first Dirichlet eigenvalue

Recall that under the weighted Ricci curvature conditions, the lower bound of the first eigenvalue on closed Finsler manifolds was obtained in [25], where the Obata type rigidity theorem was also established. In this section, we give the lower bound of the first Dirichlet (resp. Neumann) eigenvalue on compact Finsler manifolds.

THEOREM 3.1. *Let $(M, F, d\mu)$ be a compact reversible Finsler n-manifold with a mean convex boundary. If S curvature vanishes and the Ricci curvature satisfies* $\mathrm{Ric} \geq (n-1)k > 0$, *then the first Dirichlet eigenvalue of Finsler-Laplacian*

$$\lambda_1 \geq nk.$$

Proof. Let $u \in C_0^\infty(M)$ be a nonnegative function. By using Lemma 1.3 and the condition of Theorem 3.1, we have

$$\begin{aligned} \frac{1}{2}\Delta^{\nabla u}(|\nabla u|^2) &= D(\Delta u)(\nabla u) + |\nabla u|^2 \,\mathrm{Ric}_\infty(\nabla u) + |\nabla^2 u|^2_{HS(\nabla u)} \\ &\geq |\nabla u|^2 \,\mathrm{Ric}_\infty(\nabla u) + D(\Delta u)(\nabla u) + \sum_{a=1}^n u_{aa}^2 \\ &\geq |\nabla u|^2 \,\mathrm{Ric}_\infty(\nabla u) + D(\Delta u)(\nabla u) + \frac{1}{n}\left(\sum_{a=1}^n u_{aa}\right)^2 \end{aligned}$$

$$= |\nabla u|^2 \operatorname{Ric}(\nabla u) + D(\Delta u)(\nabla u) + \frac{(\Delta u)^2}{n}$$
$$\geq (n-1)k|\nabla u|^2 + D(\Delta u)(\nabla u) + \frac{(\Delta u)^2}{n}.$$

Integrating both sides of the formula above and using the divergence theorem, we have

$$\frac{1}{2}\int_{\partial M} g_v(v, \nabla^{\nabla u}|\nabla u|^2)\,d\zeta \geq \int_M \left[(n-1)k|\nabla u|^2 + D(\Delta u)(\nabla u) + \frac{(\Delta u)^2}{n}\right] d\mu. \tag{3.1}$$

where $d\zeta = v \rfloor d\mu$ denotes the induced volume form on ∂M and v denotes the unit normal vector outwards ∂M.

In what follows, we estimate $g_v(v, \nabla^{\nabla u}|\nabla u|^2)$ on ∂M. If $|\nabla u|(x_0) = 0$, $x_0 \in \partial M$, we draw geodesic γ such that $\gamma(0) = x_0$, $\dot\gamma(0) = w$ for $w \in T_{x_0}M$. Then

$$\frac{\partial|\nabla u|^2}{\partial w}(x_0) = \lim_{\gamma \ni x \to x_0} \frac{|\nabla u|^2(x) - |\nabla u|^2(x_0)}{d(x, x_0)}$$
$$= \lim_{\gamma \ni x \to x_0} \frac{|\nabla u|^2(x)}{d(x, x_0)}$$
$$= \lim_{\gamma \ni x \to x_0} |\nabla u|(x) \lim_{\gamma \ni x \to x_0} \frac{|\nabla u|(x)}{d(x, x_0)} = 0.$$

By arbitrariness of w, we have $\nabla^{\nabla u}|\nabla u|^2 = 0$. If $x_0 \in M_u \cap \partial M$, we can choose $\left\{e_1, \ldots, e_{n-1}, e_n = \frac{\nabla u}{|\nabla u|}\right\}$ as $g_{\nabla u}$-orthonormal basis. Then $v = -\frac{\nabla u}{|\nabla u|}$ is the unit normal vector outwards ∂M. Therefore, by Lemma 1.4, Definition 1.2 and the condition $S \equiv 0$, we get

$$g_v(v, \nabla^{\nabla u}|\nabla u|^2) = -\frac{\nabla u(g_{\nabla u}(\nabla u, \nabla u))}{|\nabla u|} = -2|\nabla u|u_{nn}$$
$$= 2|\nabla u|\left(-\Delta u + \sum_{a=1}^{n-1} u_{aa} - S(\nabla u)\right)$$
$$= 2|\nabla u|^2 H - 2|\nabla u|\Delta u \leq -2|\nabla u|\Delta u.$$

From (3.1) we have

$$-\int_{\partial M \cap M_u} |\nabla u|\Delta u\,d\zeta \geq \int_M \left[(n-1)k|\nabla u|^2 + D(\Delta u)(\nabla u) + \frac{(\Delta u)^2}{n}\right] d\mu. \tag{3.2}$$

By a classical density argument, the above formula still holds for the first Dirichlet eigenfunctions. In the following, we may as well assume u is a first Dirichlet eigenfunction relative to λ_1. Namely,

$$\begin{cases} \Delta u = -\lambda_1 u, & \text{in } M, \\ u = 0, & \text{on } \partial M. \end{cases}$$

Since $(M, F, d\mu)$ is a compact reversible Finsler n-manifold, we can suppose that $u > 0$ in M (see [6] for details). Notice that

$$(\Delta u)^2 = -\lambda_1 u \Delta u = \lambda_1 \left(|\nabla u|^2 - \frac{1}{2} \Delta^{\nabla u} u^2 \right). \tag{3.3}$$

Integrating both sides of (3.3) and using the divergence theorem and $u|_{\partial M} = 0$, one obtains

$$\int_M (\Delta u)^2 \, d\mu = \lambda_1 \int_M |\nabla u|^2 \, d\mu - \lambda_1 \int_{\partial M} u g_v(v, \nabla u) \, d\zeta = \lambda_1 \int_M |\nabla u|^2 \, d\mu,$$

Combining it with (3.2) and noting $\Delta u|_{\partial M} = 0$, we get

$$0 \geq \int_M \left[(n-1)k - \frac{n-1}{n} \lambda_1 \right] |\nabla u|^2 \, d\mu,$$

which implies $\lambda_1 \geq nk$. □

In what follows, we give the following result which can be obtained by a bit modification in the proof above.

PROPOSITION 3.2. *Let $(M, F, d\mu)$ be a compact Finsler n-manifold with a convex boundary. If the weighted Ricci curvature satisfies* $\mathrm{Ric}_N \geq (n-1)k > 0$ *for $N \in (n, \infty)$, then the first Neumann eigenvalue of Finsler-Laplacian*

$$\lambda_1 \geq \frac{n-1}{N-1} Nk.$$

Proof. By a similar argument as Theorem 3.1, and using the relationship $\mathrm{Ric}_N = \mathrm{Ric}_\infty - \dfrac{S^2}{(N-n)F^2}$ and inequality $\dfrac{(a+b)^2}{n} \geq \dfrac{a^2}{N} - \dfrac{b^2}{N-n}$ for any $N \in (n, \infty)$, we get

$$\frac{1}{2} \int_{\partial M \cap M_u} g_v(v, \nabla^{\nabla u} |\nabla u|^2) \, d\zeta \geq \int_M \left[(n-1)k |\nabla u|^2 + D(\Delta u)(\nabla u) + \frac{(\Delta u)^2}{N} \right] d\mu.$$

Since Neumann boundary condition gives $\nabla u \in T(\partial M)$, we have

$$\int_M (\Delta u)^2 \, d\mu = \lambda_1 \int_M |\nabla u|^2 \, d\mu - \lambda_1 \int_{\partial M} u g_v(v, \nabla u) \, d\zeta = \lambda_1 \int_M |\nabla u|^2 \, d\mu.$$

Therefore,

$$\frac{1}{2} \int_{\partial M \cap M_u} g_v(v, \nabla^{\nabla u} |\nabla u|^2) \, d\zeta \geq \int_M \left[(n-1)k - \frac{N-1}{N} \lambda_1 \right] |\nabla u|^2 \, d\mu.$$

Next we have only to prove $g_v(v, \nabla^{\nabla u}|\nabla u|^2) \le 0$ on $\partial M \cap M_u$ for the Neumann condition. In this case (see [19])

$$g_v(v, \nabla^{\nabla u}|\nabla u|^2) \le 0 \Leftrightarrow g_{\nabla u}(v_{\nabla u}, \nabla^{\nabla u}|\nabla u|^2) \le 0.$$

Here $v_{\nabla u}$ denotes the unit outward normal vector with respect to $g_{\nabla u}$. Since ∂M is convex, $g_v(v, D^{\nabla u}_{\nabla u}\nabla u) \le 0$. Hence,

$$\begin{aligned} g_{\nabla u}(v_{\nabla u}, \nabla^{\nabla u}|\nabla u|^2) &= Dg_{\nabla u}(\nabla u, \nabla u)(v_{\nabla u}) \\ &= 2g_{\nabla u}(D^{\nabla u}_{v_{\nabla u}}\nabla u, \nabla u) = 2g_{\nabla u}(v_{\nabla u}, D^{\nabla u}_{\nabla u}\nabla u) \\ &\le 0. \end{aligned}$$

The last inequality is due to Lemma 3.1 and Lemma 3.2 in [19], which shows that it is equivalent to $g_v(v, D^{\nabla u}_{\nabla u}\nabla u) \le 0$. □

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Song-Ting Yin
Department of Mathematics and Computer Science
Tongling University
Tongling, 244000 Anhui
P.R. China
E-mail: yst419@163.com

Qun He
Department of Mathematics
Tongji University
Shanghai, 200092
P.R. China
E-mail: hequn@mail.tongji.edu.cn

Da-Xiao Zheng
Department of Mathematics
Tongji University
Shanghai, 200092
P.R. China
E-mail: 18917240898@163.com