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# FIXED POINT PROPERTY FOR A CAT(0) SPACE WHICH ADMITS A PROPER COCOMPACT GROUP ACTION

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**Abstract**

We prove that if a geodesically complete CAT(0) space $X$ admits a proper cocompact isometric action of a group, then the Izeki-Nayatani invariant of $X$ is less than 1. Let $G$ be a finite connected graph, $\mu_1(G)$ be the linear spectral gap of $G$, and $\lambda_1(G, X)$ be the nonlinear spectral gap of $G$ with respect to such a CAT(0) space $X$. Then, the result implies that the ratio $\lambda_1(G, X)/\mu_1(G)$ is bounded from below by a positive constant which is independent of the graph $G$. It follows that any isometric action of a random group of the graph model on such $X$ has a global fixed point. In particular, any isometric action of a random group of the graph model on a Bruhat-Tits building associated to a semi-simple algebraic group has a global fixed point.

## 1. Introduction

**1.1. Nonlinear spectral gaps.** Let $G = (V, E)$ be a graph, where $V$ and $E$ denote the sets of vertices and unoriented edges, respectively. Throughout this paper, we assume that every graph is simple and connected and satisfies $2 \leq |V| < \infty$. A *weight function* on $G$ is a symmetric function $m : V \times V \to [0, \infty)$ whose support equals the set $\vec{E} = \{(u, v) \in V \times V \mid \{u, v\} \in E\}$. A weight function $m$ induces a weight $m(u)$ of each vertex $u \in V$ by $m(u) = \sum_{v \in V} m(u, v)$. We use the convention that $m(V) = \sum_{v \in V} m(v)$. The pair $(G, m)$ is called a *weighted graph*. Unless we specify otherwise, we assume that every graph is equipped with the *uniform* weight function $m$ defined as

$$m(u, v) = \begin{cases} 1, & \text{if } (u, v) \in \vec{E}, \\ 0, & \text{otherwise.} \end{cases}$$

The *linear spectral gap* $\mu_1(G)$ of a weighted graph $(G, m)$ is the first positive eigenvalue of the combinatorial Laplacian $\Delta$ which acts on functions

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$f : V \to \mathbf{R}$ as

$$\Delta f(v) = f(v) - \sum_{u \in V} \frac{m(v,u)}{m(v)} f(u), \quad v \in V.$$

It can be computed variationally as

$$(1.1) \quad \mu_1(G) = \inf\left\{ \frac{\frac{1}{2}\sum_{u,v \in V} m(u,v)|f(u) - f(v)|^2}{\sum_{v \in V} m(v)|f(v) - \bar{f}|^2} \,\middle|\, f : V \to \mathbf{R} \text{ is nonconstant} \right\},$$

where $\bar{f} = \{1/m(V)\} \sum_{v \in V} \{m(x)f(x)\}$, or

$$(1.2) \quad \mu_1(G) = \inf\left\{ \frac{\sum_{u,v \in V} m(u,v)|f(u) - f(v)|^2}{\frac{1}{m(V)}\sum_{u,v \in V} m(u)m(v)|f(u) - f(v)|^2} \,\middle|\, f : V \to \mathbf{R} \text{ is nonconstant} \right\}.$$

Recently, several nonlinear analogues of $\mu_1(G)$ with respect to a general metric space $X$ were defined by considering mappings $f : V \to X$ instead of $\mathbf{R}$-valued functions. They are called *nonlinear spectral gaps* and played important roles in metric geometry and geometric group theory.

By generalizing the formula (1.1), we obtain the following definition of a nonlinear spectral gap, which was first introduced by M.-T. Wang [15] for the case where the target metric space is an Hadamard manifold. Throughout this paper, every metric space is assumed to contain at least two points.

Definition 1.1 (Wang invariant). Let $(G, m)$ be a weighted graph and $(X, d_X)$ be a complete CAT(0) space. The *Wang invariant* $\lambda_1(G, X)$ of $G$ with respect to $X$ is defined as

$$\lambda_1(G, X) = \inf\left\{ \frac{\frac{1}{2}\sum_{u,v \in V} m(u,v) d_X(f(u), f(v))^2}{\sum_{v \in V} m(v) d_X(f(v), \bar{f})^2} \,\middle|\, f : V \to X \text{ is nonconstant} \right\},$$

where $\bar{f}$ denotes the barycenter of the probability measure

$$\sum_{v \in V} \frac{m(v)}{m(V)} \operatorname{Dirac}_{f(v)}$$

on $X$. Here, $\operatorname{Dirac}_{f(v)}$ denotes the Dirac measure at $f(v) \in X$.

The Wang invariant plays a crucial role in the theory of rigidity of groups ([15], [7], [10]). By generalizing the formula (1.2), we obtain the definition of another nonlinear spectral gap which was defined by Gromov [4].

Definition 1.2. Let $(G, m)$ be a weighted graph and $(X, d_X)$ be a metric space. The *Gromov nonlinear spectral gap* $\lambda_1^{\mathrm{Gro}}(G, X)$ is defined as

$$\lambda_1^{\mathrm{Gro}}(G, X) = \inf\left\{ \frac{\sum_{u,v\in V} m(u,v) d_X(f(u), f(v))^2}{\frac{1}{m(V)} \sum_{u,v\in V} m(u)m(v) d_X(f(u), f(v))^2} \,\middle|\, f : V \to X \text{ is nonconstant} \right\}.$$

By definition, we have

$$\lambda_1(G, \mathbf{R}) = \lambda_1^{\mathrm{Gro}}(G, \mathbf{R}) = \mu_1(G)$$

for any graph $G$. Moreover, we also have

$$\lambda_1(G, \mathscr{H}) = \lambda_1^{\mathrm{Gro}}(G, \mathscr{H}) = \mu_1(G).$$

for any graph $G$ and Hilbert space $\mathscr{H}$. For a general complete CAT(0) space $X$ and a graph $G$, these two nonlinear spectral gaps have the following relation (see [10]):

$$\frac{1}{2}\lambda_1(G, X) \le \lambda_1^{\mathrm{Gro}}(G, X) \le \lambda_1(G, X). \tag{1.3}$$

**1.2. Comparison of nonlinear and linear spectral gaps.** It is a fundamental question to ask for what kind of complete CAT(0) space $X$, does there exist a constant $C_X > 0$ depending only on $X$ which satisfies

$$\lambda_1(G, X) \ge C_X \mu_1(G) \tag{1.4}$$

for any graph $G$. It is known that the existence of such a constant $C_X > 0$ implies many important conclusions including the following (A) and (B):

(A) Any isometric action of a random group of the graph model on $X$ has a global fixed point.

(B) A sequence of expanders does not embed coarsely into $X$ ([4]).

The conclusion (A) was proved by Izeki, Kondo and Nayatani in [6]. For its precise statement, see Theorem 6.2 in Section 6. Roughly, it states that if we equip a suitable probability measure with a set $\mathscr{G}$ of finitely generated groups, then, with high probability, a randomly chosen group $\Gamma \in \mathscr{G}$ is infinite and any isometric action of $\Gamma$ on $X$ has a global fixed point. This guarantees the existence of infinite groups $\Gamma$ whose isometric actions on $X$ always have global fixed points.

For the definitions of coarse embeddings and sequences of expanders, see Section 6. In [4], Gromov proved that a sequence of expanders does not embed coarsely into a Hilbert space. Since then, coarse embeddability of a sequence of expanders into a metric space has become an important obstruction of the space to be embedded coarsely into a Hilbert space. The conclusion (B) states that $X$ does not have such an obstruction to embed coarsely into a Hilbert space, and is proved easily by applying the argument of Gromov. For the detailed proof of the conclusion (B), see Theorem 4.5 of [3].

The purpose of this paper is to specify complete CAT(0) spaces $X$ which allow the existence of such constants $C_X > 0$ as above. Consequently, it will specify spaces which satisfy the above conclusions (A) and (B).

Throughout this paper, we denote by $B(p,r)$ the open ball of radius $r$ centered at $p$ and by $\bar{B}(p,r)$ the closed ball of radius $r$ centered at $p$. We use the following definition.

Definition 1.3 ([1], Chapter I.8). An isometric action of a group $\Gamma$ on a metric space $X$ is called *cocompact* if there exists a compact subset $K \subset X$ such that $X = \bigcup_{\gamma \in \Gamma} \gamma K$. An isometric action of $\Gamma$ on a metric space $X$ is called *proper* if for each $p \in X$ there exists $r > 0$ such that the set $\{\gamma \in \Gamma \mid \gamma B(p,r) \cap B(p,r) \neq \phi\}$ is finite.

We prove the following theorem.

Theorem 1.4. *Let $\mathscr{X} = \{X_1, X_2, \dots, X_n\}$ be a finite set of geodesically complete* CAT(0) *spaces such that each $X_i$ admits a proper cocompact isometric action of a group. Then, there exists a constant $C = C_{\mathscr{X}} > 0$ which depends only on $\mathscr{X}$ such that the inequality*

$$\lambda_1(G,X) \geq C\mu_1(G) \tag{1.5}$$

*holds whenever $X$ is a (finite or infinite) product of copies of spaces in $\mathscr{X}$ and $G$ is a graph. In particular, any Bruhat-Tits building $X$ associated to a semi-simple algebraic group admits the existence of a constant $C = C_X$ which satisfies the inequality* (1.5) *for any graph $G$.*

We also prove an estimate of the same type when a complete CAT(0) space $X$ is uniformly locally doubling in the following sense.

Definition 1.5. Fix $N \in [1,\infty)$. A metric space is called *doubling with doubling constant $N$* if every closed ball can be covered by at most $N$ closed balls of half the radius. We say that a metric space is *uniformly locally doubling with doubling constant $N$* if any point has a neighborhood which is doubling with doubling constant $N$.

Theorem 1.6. *For each $N \in [1,\infty)$, there exists a constant $C = C_N > 0$ such that the inequality*

$$\lambda_1(G,X) \geq C\mu_1(G) \tag{1.6}$$

*holds for every graph $G$ and a complete* CAT(0) *space $X$ which is isometric to a* (*finite or infinite*) *product of uniformly locally doubling* CAT(0) *spaces with a common doubling constant $N$.*

Theorem 1.4 and Theorem 1.6 yield that if a complete CAT(0) space $X$ satisfies the hypothesis of either theorem, $X$ satisfies the conclusions (A) and (B). We state this explicitly as Threorem 6.3 and Theorem 6.6 in Section 6. In particular, if $X$ is a Bruhat-Tits building associated to a semi-simple algebraic group, then any isometric action of a random group of the graph model on $X$ has a global fixed point.

**1.3. Relations with other results.** Naor-Silberman [9] proved that if a metric space $X$ has finite Nagata dimension, then for every $\varepsilon > 0$, there exists a constant $C_{X,\varepsilon}$ which satisfies

$$\lambda_1^{\mathrm{Gro}}(G, X) \geq C_{X,\varepsilon}\mu_1(G)^{1+\varepsilon}, \tag{1.7}$$

for every graph $G$. Moreover, Naor-Silberman [9] also proved that the weaker inequality (1.7) suffices to imply the fixed point property (A) of a random group for $X$ whenever $X$ is $p$-uniformly convex for some $p \geq 2$. Since each Bruhat-Tits building associated to a semi-simple algebraic group has finite Nagata dimension, and complete CAT(0) spaces are 2-uniformly convex, for such a building $X$, the fixed point property (A) also follows from their result.

However, an advantage of our result is that our estimate (1.4) is better than their estimate (1.7). In fact, by (1.3), we obtain the following corollaries of Theorem 1.4 and Theorem 1.6, respectively, which improve Naor-Silbermann's estimate (1.7) when the target metric space satisfies the hypothesis of either theorem.

Corollary 1.7. *If a geodesically complete* CAT(0) *space $X$ admits a proper cocompact isometric action of a group, then there exists a constant $C'_X > 0$ such that the inequality*

$$\lambda_1^{\mathrm{Gro}}(G, X) \geq C'_X\mu_1(G) \tag{1.8}$$

*holds for every graph $G$.*

Corollary 1.8. *For each $N \in [1, \infty)$, there exists a constant $C'_N > 0$ such that the inequality*

$$\lambda_1^{\mathrm{Gro}}(G, X) \geq C'_N\mu_1(G) \tag{1.9}$$

*holds for every uniformly locally doubling complete* CAT(0) *space $X$ with doubling constant $N$ and every graph $G$.*

**1.4. The Izeki-Nayatani invariant.** To obtain such a stronger estimate, we use the so-called Izeki-Nayatani invariant. Izeki and Nayatani introduced the

Izeki-Nayatani invariant $0 \le \delta(X) \le 1$ of a complete CAT(0) space $X$ in [7], and proved that

$$\lambda_1(G, X) \ge (1 - \delta(X))\mu_1(G) \tag{1.11}$$

for any complete CAT(0) space $X$ and graph $G$. For the definition of the Izeki-Nayatani invariant, see Section 3.

On the other hand, a standard method to compare the linear spectral gap and the nonlinear spectral gap with respect to $X$ is to estimate the bi-Lipschitz distortion of $X$ into a Hilbert space. The bi-Lipschitz distortion $c_2(X)$ of a metric space $X$ into a Hilbert space is the infimum of $D > 0$ such that there exists a 1-Lipschitz mapping $f : X \to \mathscr{H}$ to a Hilbert space which satisfies

$$\frac{1}{D} d_X(x, y) \le \|f(x) - f(y)\| \le d_X(x, y)$$

for every $x, y \in X$, and we have

$$\lambda_1^{\mathrm{Gro}}(G, X) \ge \frac{1}{c_2(X)^2}\mu_1(G).$$

Although the Izeki-Natayatani invariant $\delta(X)$ can also be estimated by using the bi-Lipschitz distortion $c_2(X)$ into a Hilbert space, the present author [13] established another method to estimate it, which even does not require the existence of bi-Lipschitz embeddings of $X$ into a Hilbert space. This method enables us to obtain our estimates. We summarize this method in Section 3.

**1.5. Organization.** The paper is organized as follows. In Section 2, we briefly review some basic notions concerning CAT(0) spaces. In Section 3, we recall the definition of the Izeki-Nayatani invariant and discuss some basic properties of it. We also summarize the method obtained in [13] to estimate this invariant. In Section 4, we prove Theorem 1.4. In Section 5, we prove Theorem 1.6. To prove Theorem 1.6, we prove that the ultralimit of a sequence of doubling length spaces with a common doubling constant is also doubling with the same constant. In Section 6, we see that our results imply fixed-point theorems of random groups and non-embeddability of sequences of expanders. In Appendix, we discuss some other facts concerning the Izeki-Nayatani invariant.

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## 2. Preliminaries

In this section, we briefly recall some basic notions in metric geometry. For a detailed exposition, we refer the reader to [1], [2], and [11].

Let $(X, d_X)$ be a metric space. A continuous mapping $\gamma : I \to X$ from an interval $I \subset \mathbf{R}$ to $X$ is called a *path* in $X$. When $I = [a, b]$ is a closed interval, it is called a path joining $\gamma(a)$ to $\gamma(b)$. The *length* $L(\gamma)$ of a path $\gamma : [a,b] \to X$ is defined as

$$L(\gamma) = \sup \sum_{i=1}^{k} d_X(\gamma(t_{i-1}), \gamma(t_i)),$$

where the supremum is taken over all finite subdivisions

$$a = t_0 \leq t_1 \leq \cdots \leq t_k = b.$$

A path $\gamma : [a,b] \to X$ is called *arc-length parametrized* if $L(\gamma|_{[a,t]}) = |t - a|$ for all $t \in [a,b]$, where $\gamma|_{[a,t]}$ is the restriction of $\gamma$ to $[a,t]$. Any path can be reparametrized to an arc-length parametrized path. $X$ is called a *length space* if the distance $d_X(p,q)$ between any two points $p,q \in X$ is equal to the infimum over the lengths of paths joining $p$ to $q$. We call a path $\gamma : I \to X$ a *geodesic* if it is an isometric embedding of the interval $I$ to $X$. A metric space is called a *geodesic space* if every pair of points is joined by a geodesic. We call a path $\gamma : I \to X$ a *local geodesic* if for every $t \in I$ there exists a neighborhood $J$ of $t$ in $I$ such that the restriction $\gamma|_J : J \to Y$ is a geodesic.

Definition 2.1. A metric space $X$ is called *geodesically complete* if it is complete and any local geodesic $\gamma : [0,a] \to X$ is a restriction of some local geodesic $\tilde{\gamma} : [0,b] \to X$ with $0 < a < b$.

A *geodesic triangle* in $X$ is a triple $\triangle = (\gamma_1, \gamma_2, \gamma_3)$ of geodesics $\gamma_i : [a_i, b_i] \to X$ such that

$$\gamma_1(b_1) = \gamma_2(a_2), \quad \gamma_2(b_2) = \gamma_3(a_3), \quad \gamma_3(b_3) = \gamma_1(a_1).$$

For a geodesic triangle $\triangle = (\gamma_1, \gamma_2, \gamma_3)$ there is a geodesic triangle

$$\overline{\triangle} = (\overline{\gamma_1}, \overline{\gamma_2}, \overline{\gamma_3}), \quad \overline{\gamma_i} : [a_i, b_i] \to \mathbf{R}^2$$

in $\mathbf{R}^2$ such that $L(\gamma_i) = L(\overline{\gamma_i})$ for each $i$. Such a triangle $\overline{\triangle}$ is unique up to isometry of $\mathbf{R}^2$. We call it the *comparison triangle* of $\triangle$ in $\mathbf{R}^2$. A geodesic triangle $\triangle$ is said to be *thin* if

$$d_Y(\gamma_i(s), \gamma_j(t)) \leq d_\kappa(\overline{\gamma_i}(s), \overline{\gamma_j}(t))$$

whenever $i, j \in \{1,2,3\}$, $s \in [a_i, b_i]$, and $t \in [a_j, b_j]$.

Definition 2.2. A geodesic space $X$ is called a CAT(0) space if every geodesic triangle in $X$ is thin.

By definition, for any pair of points $p, q \in X$, a geodesic $\gamma : [0, d_X(p,q)] \to X$ joining $p$ to $q$ is unique whenever $X$ is a CAT(0) space. It is known that every local geodesic $\gamma : [a,b] \to X$ in a geodesically complete space is a restriction of a local geodesic $\tilde{\gamma} : \mathbf{R} \to Y$ (see [2, Corollary 9.1.28.]), and every local geodesic in a complete CAT(0) space is a geodesic (see [1, Chapter II, Proposition 1.4]). Thus, every geodesic $\gamma : [a, b] \to X$ in a geodesically complete CAT(0) space $X$ is a restriction of a geodesic $\tilde{\gamma} : \mathbf{R} \to X$.

Let $\gamma : [a, b] \to X$, $\gamma' : [a', b'] \to X$ be two geodesics in a CAT(0) space $X$ with $\gamma(a) = \gamma'(a') = p$. We define the *angle* $\angle_p(\gamma, \gamma')$ between $\gamma$ and $\gamma'$ as

$$\angle_p(\gamma, \gamma') = \lim_{t \to a,\, t' \to a'} \angle_p^0(\gamma(t), \gamma'(t')),$$

where $\angle_p^0(\gamma(t), \gamma'(t'))$ is the corresponding angle of the triangle in $\mathbf{R}^2$ whose side lengths are $d_X(p, \gamma(t))$, $d_X(\gamma(t), \gamma'(t))$ and $d_X(\gamma'(t), p)$. The existence of the limit is guaranteed by the definition of CAT(0) spaces. The law of cosines on a Euclidean space yields

$$\cos \angle_p(\gamma, \gamma') = \lim_{t \to a,\, t' \to a'} \frac{d_X(p, \gamma(t))^2 + d_X(p, \gamma'(t'))^2 - d_X(\gamma(t), \gamma'(t'))^2}{2 d_X(p, \gamma(t)) d_X(p, \gamma'(t'))}. \tag{2.1}$$

Definition 2.3. Let $(S, d_S)$ be a metric space. The cone $\mathrm{Cone}(S)$ over $S$ is the quotient of the product $S \times [0, \infty)$ obtained by identifying all points in $S \times \{0\} \subset S \times [0, \infty)$. The point represented by $(x, 0)$ for any $x \in S$ is called the *origin* of the cone and we denote this point by $o$. The cone distance $d_{\mathrm{Cone}(S)}(v, w)$ between two points $v, w \in \mathrm{Cone}(S)$ represented by $(x, t), (y, s) \in S \times [0, \infty)$ respectively, is defined by

$$d_{\mathrm{Cone}(S)}(v, w) = \sqrt{t^2 + s^2 - 2ts \cos(\min\{\pi, d_S(x, y)\})}.$$

Then $(\mathrm{Cone}(S), d_{\mathrm{Cone}(S)})$ becomes a metric space. We call this metric space the *Euclidean cone* over $(S, d_S)$.

For an element $v \in \mathrm{Cone}(S)$ represented by $(x, r) \in S \times [0, \infty)$ and $c > 0$, we denote by $cv$ the element represented by $(x, cr)$. We claim that

$$d_{\mathrm{Cone}(S)}(cv, cw) = c d_{\mathrm{Cone}(S)}(v, w)$$

holds for any $v, w \in \mathrm{Cone}(S)$

Definition 2.4. Let $(X, d_X)$ be a CAT(0) space, and let $p \in X$. We denote by $(S_pX)^\circ$ the quotient set of all nontrivial geodesics starting from $p$ by the equivalence relation $\sim$ defined by $\gamma \sim \gamma' \Leftrightarrow \angle_p(\gamma, \gamma') = 0$. Then the angle $\angle_p$ induces a distance on $(S_pX)^\circ$, which we denote by the same symbol $\angle_p$. The *space of directions* $S_pX$ at $p$ is the metric completion of the metric space $((S_pX)^\circ, \angle_p)$. The *tangent cone* $TC_pX$ of $X$ at $p$ is the Euclidean cone $\mathrm{Cone}(S_pX)$ over the space of directions at $p$. Define a map $\pi_p : X \to TC_pX$ by $\pi_p(q) = ([\gamma], d_X(p, q))$ where $[\gamma]$ is the equivalence class represented by the unique geodesic $\gamma$ joining $p$ and $q$.

It is easily seen that the map $\pi_p$ defined as above is 1-Lipschitz. It is also seen that each tangent cone $TC_pX$ is the metric completion of the Euclidean cone

Cone$((S_pX)^\circ)$. If we denote the canonical inclusion of $S_pX$ into $TC_pX$ by $\iota$, then it is straightforward from the definition of the metric on Euclidean cones that we have

$$\frac{2}{\pi} d_S(x, y) \leq d_T(\iota(x), \iota(y)) \leq d_S(x, y) \tag{2.2}$$

for all $x, y \in S_pX$, where $d_S$ and $d_T$ represent the distance functions of $S_pX$ and $TC_pX$, respectively.

The CAT(0) condition is preserved under taking ($\ell_2$-)product.

Definition 2.5. Let $(X_1, d_1), (X_2, d_2), \ldots$ be metric spaces with basepoints $o_1 \in X_1, o_2 \in X_2, \ldots$, respectively. The *($\ell^2$-)product* $X$ of $X_1, X_2, \ldots$ with respect to the basepoints $o_1, o_2, \ldots$ consists of all sequences $(x_n)_n$ with $x_n \in X_n$, satisfying $\sum_n d_n(o_n, x_n)^2 < \infty$, and is equipped with the metric function $d$ defined by

$$d(x, y)^2 = \sum_{n=1}^{\infty} d_n(x_n, y_n)^2$$

for any elements $x = (x_1, x_2, \ldots) \in X$ and $y = (y_1, y_2, \ldots) \in X$.

To define the Wang invariant, we need to consider a finitely supported probability measure on a complete CAT(0) space. We often write a finitely supported probability measure $\mu$ on a metric space $X$ in the form

$$\mu = \sum_{i=1}^{m} t_i \,\mathrm{Dirac}_{p_i},$$

where $\mathrm{Dirac}_{p_i}$ is the Dirac measure at $p_i \in X$ and each $t_i$ is the weight $\mu(\{p_i\})$ at $p_i$. We denote the support of a measure $\mu$ by $\mathrm{Supp}(\mu)$. When $X$ is a complete CAT(0) space, there exists a unique point on $X$ which minimizes the function $p \mapsto \int_X d_X(p, q)^2 \mu(dq) = \sum_{i=1}^m t_i d_Y(p, p_i)^2$ (see [12]). This point is called a *barycenter* of $\mu$ and denoted by $\mathrm{bar}(\mu)$.

## 3. Izeki-Nayatani invariant

In this section, we recall the definition of the Izeki-Nayatani invariant $\delta$ and its basic properties.

Definition 3.1 (Izeki-Nayatani [7]). Let $X$ be a complete CAT(0) space, and $\mathscr{P}(X)$ be the space of all finitely supported probability measures $\mu$ with $|\mathrm{Supp}(\mu)| \geq 2$ on $X$. For $\mu \in \mathscr{P}(X)$, we define $0 \leq \delta(\mu) \leq 1$ to be the infimum of

$$\frac{\|\int_X \phi(p)\mu(dp)\|^2}{\int_X \|\phi(p)\|^2 \mu(dp)}$$

over all mappings $\phi : \mathrm{Supp}(\mu) \to \mathscr{H}$ to a Hilbert space $\mathscr{H}$ such that

$$\|\phi(p)\| = d(p, \mathrm{bar}(\mu)), \tag{3.1}$$

$$\|\phi(p) - \phi(q)\| \le d(p, q) \tag{3.2}$$

for all $p, q \in \mathrm{Supp}(\mu)$. We define the *Izeki-Nayatani invariant* $\delta(X)$ of $X$ by

$$\delta(X) = \sup_{\mu \in \mathscr{P}(X)} \delta(\mu) \in [0, 1].$$

*Remark* 3.2. Notice that a mapping $\phi$ of $\mu \in \mathscr{P}(X)$ which satisfies (3.1) and (3.2) always exists. To see that, fix a unit vector $e \in \mathscr{H}$. Define $\phi(p) = d(p, \mathrm{bar}(\mu))e$. Then by the triangle inequality, (3.2) is satisfied.

Izeki-Naytani invariant is designed to estimate the Wang invariant in comparison with the linear spectral gap. In [7], Izeki and Nayatani proved the following.

PROPOSITION 3.3. *Let X be a complete* CAT(0) *space and G be a weighted graph. Then, we have*

$$(1 - \delta(X))\mu_1(G) \le \lambda_1(G, X) \le \mu_1(G).$$

It is known that there are complete CAT(0) spaces $X$ with $\delta(X) = 1$. Kondo [8] constructed the first examples of such spaces. On the other hand, the present author [13] proved the following criterion for a complete CAT(0) space $X$ to be $\delta(X) < 1$ (see Theorem 5.4 in [13]).

THEOREM 3.4. *Let* $0 < \theta < \frac{\pi}{2}$, $0 < \alpha < 1$ *and* $\varepsilon > 0$. *Let us say that a metric space* $(S, d_S)$ *has the property* $\mathrm{P}(\theta, \alpha, \varepsilon)$ *if there exists a finite subset* $S' \subset S$ *such that*

$$|\{s \in S' \mid \|d_S(x, s) - d_S(y, s)\| \ge \varepsilon\}| \ge \alpha|S'|$$

*holds for every* $x, y \in S$ *with* $d_S(x, y) \ge \theta$. *Let X be a complete* CAT(0) *space. If each tangent cone* $TC_pX$ *of X is isometric to a (finite or infinite) product of the Euclidean cones over metric spaces each of which has the property* $\mathrm{P}(\theta, \alpha, \varepsilon)$, *then there exists a constant* $C(\theta, \alpha, \varepsilon) < 1$ *depending only on* $\theta$, $\alpha$ *and* $\varepsilon$ *such that*

$$\delta(X) \le C(\theta, \alpha, \varepsilon).$$

The following corollary is used to prove Theorem 1.4 in Section 4.

COROLLARY 3.5. *A complete* CAT(0) *space X satisfies* $\delta(X) < 1$ *if the family* $\{S_pX\}_{p \in X}$ *consists of all spaces of directions of X is Gromov-Hausdorff precompact.*

We recall that the Gromov-Hausdorff precompactness is equivalent to the uniform total boundedness which is defined as follows.

DEFINITION 3.6. A family $\mathscr{X}$ of metric spaces is *uniformly totally bounded* if the following two conditions are satisfied:

(1) There is a constant $D > 0$ such that $\mathrm{diam}(X) \leq D$ for all $X \in \mathscr{X}$.
(2) For any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbf{N}$ such that each $X \in \mathscr{X}$ contains a subset $S_{X,\varepsilon} \subset X$ with the following property: the cardinality of $S_{X,\varepsilon}$ is at most $N(\varepsilon)$ and $X$ is covered by the union of all open $\varepsilon$-balls whose centers are in $S_{X,\varepsilon}$.

*Proof of Corollary* 3.5. It suffices to show that if $\mathscr{X}$ is a Gromov-Hausdorff precompact family of metric spaces, then there exist constants $0 < \theta < \frac{\pi}{2}$, $0 < \alpha < 1$ and $\varepsilon > 0$ such that every $X \in \mathscr{X}$ satisfies the property $\mathrm{P}(\theta, \alpha, \varepsilon)$. Since Gromov-Hausdorff precompactness is equivalent to uniform total boundedness, there exists an $N > 0$ such that each $X \in \mathscr{X}$ contains a subset $S_X \subset X$ with the following property: the cardinality of $S_X$ is no greater than $N$ and $X$ is covered by the union of all open $\frac{\pi}{12}$-balls whose centers are in $S_X$.

By the definition of the subset $S_X$, for any $x, y \in X$ with $d_X(x, y) \geq \frac{\pi}{3}$, there exist $s_0, s_1 \in S_X$ such that

$$d_X(s_0, x) \geq \frac{\pi}{4}, \quad d_X(s_0, y) \leq \frac{\pi}{12},$$
$$d_X(s_1, y) \geq \frac{\pi}{4}, \quad d_X(s_1, x) \leq \frac{\pi}{12}.$$

Hence, there exist two distinct elements $s_0, s_1 \in S$ such that

$$\|d_X(x, s_0) - d_X(y, s_0)\| \geq \frac{\pi}{6},$$
$$\|d_X(x, s_1) - d_X(y, s_1)\| \geq \frac{\pi}{6},$$

for any $x, y \in X$ with $d_X(x, y) \geq \frac{\pi}{3}$. Thus each $X \in \mathscr{X}$ has the property $\mathrm{P}\left(\frac{\pi}{3}, \frac{2}{N}, \frac{\pi}{6}\right)$. □

The following corollary is used to prove Theorem 1.6 in Section 5.

COROLLARY 3.7. *Let $X$ be a* CAT(0) *space and $p \in X$. Assume that the tangent cone $TC_pX$ is doubling with doubling constant $N \in [0, \infty)$. Then there exist $0 < \theta < \frac{\pi}{2}$, $0 < \alpha < 1$ and $\varepsilon > 0$ depending only on $N$ such that the space of directions $S_pX$ at $p$ of $X$ has the property* $\mathrm{P}(\theta, \alpha, \varepsilon)$.

*Proof.* We assume that $N$ is a natural number. Since $TC_pX$ is doubling with doubling constant $N$, there exist closed balls $B_1, B_2, \ldots, B_{N^2}$ with diameter at most $\frac{1}{4}$, which cover the closed ball of radius 1 centered at the origin of the cone $TC_pX$. Hence $S_pX$ is covered by $\{\iota^{-1}(B_i)\}$, where $\iota : S_pX \to TC_pX$ is the canonical inclusion. By the inequality (2.2), each $\iota^{-1}(B_i)$ has diameter at most $\frac{\pi}{8}$. Thus the lemma follows from the similar argument as in the proof of corollary 3.5. □

Some of the known estimates of the Izeki-Nayatani invariant are:

- We have $\delta(\mathscr{H}) = 0$ for a Hilbert space $\mathscr{H}$ by definition.
- If $X$ is a finite or infinite dimensional Hadamard manifold or an $\mathbf{R}$-tree, then we have $\delta(Y) = 0$ ([7]).
- Let $X_p$ be the Euclidean building $PSL(3, \mathbf{Q}_p)/PSL(3, \mathbf{Z}_p)$ for each prime number $p$. Then, we have $\delta(X_p) \geq \dfrac{(\sqrt{p}-1)^2}{2(p-\sqrt{p}+1)}$ ([7]).
- We have $\delta(X_2) \leq 0.4122\ldots$ ([7]).
- If $X$ is a complete CAT(0) cube complex, then we have $\delta(X) \leq \frac{1}{2}$ ([3]).

## 4. CAT(0) spaces which admit proper cocompact group actions

In this section, we prove the following proposition.

PROPOSITION 4.1. *A geodesically complete* CAT(0) *space $X$ satisfies $\delta(X) < 1$ if it admits a proper cocompact isometric action of a group.*

Combining this proposition with Proposition A.1 in Appendix, we obtain the following corollary.

COROLLARY 4.2. *Let $\{X_1, X_2, \ldots, X_n\}$ be a finite set of geodesically complete* CAT(0) *spaces such that each $X_i$ admits a proper cocompact isometric action of a group. Then, there exists a constant $0 \leq c < 1$ such that any* CAT(0) *space $X$ which is isometric to a (finite or infinite) product of copies of spaces in $\{X_1, X_2, \ldots, X_n\}$ satisfies $\delta(X) \leq c$.*

Theorem 1.4 follows immediately from Corollary 4.2 and Proposition 3.3. Our proof of Proposition 4.1 consists of two lemmas.

LEMMA 4.3. *Let $X$ be a geodesically complete* CAT(0) *space. If there exists a positive real number $r > 0$ such that the family $\{B(p,r)\}_{p \in X}$ consisting of all open r-balls in $X$ is Gromov-Hausdorff precompact, then the family $\{S_pX\}_{p \in X}$ consisting of all spaces of directions is also Gromov-Hausdorff precompact.*

*Proof.* Let $p \in X$ be an arbitrary point on $X$. We denote the canonical inclusion of $S_pX$ into $TC_pX$ by $\iota$, and represent the distance functions of $S_pX$ and $TC_pX$ by $d_S$ and $d_T$ respectively.

Fix some $0 < r' < r$. By the assumption, the family $\{B(p,r)\}_{p \in X}$ is uniformly totally bounded. Hence, for any $\varepsilon > 0$, there exists a positive integer $N$ which is independent of $p$ such that each $B(p,r)$ is covered by $N$ open balls of radius $2r'\varepsilon/\pi$. Then the metric sphere

$$S(p,r') = \{q \in X \mid d_X(p,q) = r'\} \subset B(p,r)$$

is also covered by $N$ open balls of radius $2r'\varepsilon/\pi$ in $X$.

Let $F : TC_pX \to TC_pX$ be the mapping associating each element of $TC_pX$ represented by $(x,t) \in S_pX \times [0,\infty)$ to the element represented by $\left(x, \frac{1}{r'}t\right) \in S_pX \times [0,\infty)$. This mapping clearly satisfies

$$d_T(F(v),F(w)) = \frac{1}{r'} d_T(v,w) \tag{4.1}$$

for all $v, w \in TC_pX$. Then, we have $F \circ \pi_p(S(p,r')) \subset \imath(S_pX)$, where $\pi_p : X \to TC_pX$ is the 1-Lipschitz mapping defined in Definition 2.4. By (4.1), $F \circ \pi_p(S(p,r'))$ can be covered by $N$ open balls of radius $2\varepsilon/\pi$ in $TC_pX$.

Since each geodesic starting from $p$ can be extended up to $S(p,r')$ by geodesic completeness of $X$, $F \circ \pi_p(S(p,r'))$ is no other than $\imath((S_pX)^\circ)$, and $F \circ \pi_p(S(p,r'))$ is dense in $\imath(S_pX)$. Hence, $\imath(S_pX)$ is covered by $N$ open balls of radius $2\varepsilon/\pi$ in $TC_pY$. Let us denote these balls by $B_1, B_2, \ldots, B_N$. Then $\{\imath^{-1}(B_i)\}_{i=1}^N$ covers $S_pX$. By (2.2), each $\imath^{-1}(B_i)$ is covered by an open ball in $S_pX$ of radius $\varepsilon$. Hence, $S_pX$ is covered by $N$ balls of radius $\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have proved that $\{S_pX\}_{p \in X}$ is uniformly totally bounded. Thus, it is Gromov-Hausdorff precompact. □

LEMMA 4.4. *Let $X$ be a metric space. Assume that a group $\Gamma$ acts on $X$ properly and cocompactly by isometries. Then there exists some positive real number $r > 0$ such that the family $\{B(p,r)\}_{p \in X}$ consisting of all open $r$-balls in $X$ is a Gromov-Hausdorff precompact family of metric spaces.*

*Proof.* Since $\Gamma$ acts on $X$ cocompactly, there exists a compact subset $K \subset X$ such that $\bigcup_{\gamma \in \Gamma} \gamma K = X$. Since $\Gamma$ acts on $X$ properly, for every $p \in K$, there exists $r_p > 0$ such that the set $\{\gamma \in \Gamma \mid \gamma B(p,2r_p) \cap B(p,2r_p) \neq \phi\}$ is finite. Let $\{B(p_i,r_i)\}_{i=1}^N$ be one of finite subcovers of the open cover $\{B(p,r_p)\}_{p \in K}$ of $K$.

Although it is a well-known fact, we first confirm that $X$ is locally compact in this case. Let $q \in X$ be an arbitrary point, and let $r_0 = \min\{r_1, r_2, \ldots, r_N\}$. Observe that if there are infinitely many elements $\gamma \in \Gamma$ with $B(q,r_0) \cap \gamma K \neq \emptyset$, then there exists some $i \in \{1,\ldots,N\}$ with infinitely many elements $\gamma' \in \Gamma$ satisfying

$$B(q,r_0) \cap \gamma' B(p_i,r_i) \neq \emptyset. \tag{4.2}$$

Also, observe that if we can take $\gamma_1 \in \Gamma$ and $\gamma_2 \in \Gamma$ as $\gamma'$ in (4.2), then the element $\gamma_0 = \gamma_2^{-1}\gamma_1$ satisfies

$$B(p_i,2r_i) \cap \gamma_0 B(p_i,2r_i) \neq \emptyset \tag{4.3}$$

since both balls $B(x_i, 2r_i)$ and $\gamma_0 B(x_i, 2r_i)$ contain the point $\gamma_2^{-1}q$. Thus, if there were infinite elements $\gamma \in \Gamma$ with $B(q, r_0) \cap \gamma K \neq \emptyset$, there would be infinite $\gamma_0 \in \Gamma$ with (4.3). It contradicts the definition of $r_i$. Thus, there are only finite elements $\gamma \in \Gamma$ with $B(q, r_0) \cap \gamma K \neq \emptyset$. Let $\gamma_1, \ldots, \gamma_M$ be all such elements. Then, we have

$$B(q, r_0) \subset \bigcup_{j=1}^{M} \gamma_j K.$$

by the definition of $K$. Since the right-hand side is compact, any closed ball centered at $q$ with a radius less than $r_0$ is compact. Hence $Y$ is locally compact.

Therefore, there exists a precompact open ball $B_p \subset X$ centered at $p$ for any $p \in K$. Let $\{B_i\}_i$ be a finite subcover of the open cover $\{B_p\}_{p \in K}$ of $K$, and define $U = \bigcup_i B_i$. Then, $U$ is a precompact open subset containing $K$.

For each point $p \in K$, we define $f(p) > 0$ to be $f(p) = \sup\{\alpha > 0 \mid B(p, \alpha) \subset U\}$. Let $q \in K$ be an arbitrary point, and let $\eta > 0$ be an arbitrary positive real number. Set $\kappa = \min\{f(q), \eta\}$. Then for any $q' \in B(q, \kappa)$, we have $f(q') \geq f(q) - \eta$. Hence, $f$ is lower semi-continuous on $K$, and there exists $p_0 \in K$ on which $f$ attains the minimum value of $f$. Set $r = f(p_0)$. Then, we have $B(p, r) \subset U$ for all $p \in K$.

Now, we show that the family $\{B(p, r)\}_{p \in X}$ of all open $r$-balls in $X$ is uniformly totally bounded. Let $p \in X$ be an arbitrary point, and let $\gamma \in \Gamma$ be an element which satisfies $p \in \gamma K$. Then, since $\gamma^{-1} B(p, r) = B(\gamma^{-1} p, r)$ is covered by $U$, $B(p, r)$ is covered by precompact subset $\gamma U \subset X$ which is isometric to $U$. Uniformly total boundedness of the family $\{B(p, r)\}_{p \in X}$ follows straightforward from this, which proves the lemma. □

*Proof of Proposition* 4.1. By Lemma 4.3 and Lemma 4.4, the family $\{S_p X\}_{p \in X}$ consisting of all spaces of directions of geodesically complete CAT(0) space $X$ is Gromov-Hausdorff precompact if it admits a proper cocompact isometric action of a group. Hence, the proposition follows from Corollary 3.5. □

*Remark* 4.5. We remark that the geodesical completeness is essential in Proposition 4.1. In [8], Kondo constructed a sequence of locally compact CAT(0) cones $T_1, T_2, T_3, \ldots$ with $\lim_{i \to \infty} \delta(T_i) = 1$. For each $i = 1, 2, \ldots$, let $T_i' \subset T_i$ be a closed ball of radius $\dfrac{1}{i}$ centered at the origin. Gluing $T_1', T_2', \ldots$ by identifying the origin of every $T_i'$, then the resulting space $T'$ is a compact CAT(0) space satisfying $\delta(T') = 1$ although it is not geodesically complete.

## 5. Ultralimits and doubling CAT(0) spaces

In this section, we prove the following proposition.

PROPOSITION 5.1. *If a complete* CAT(0) *space* $X$ *is uniformly locally doubling with doubling constant* $N$, *then there exists a constant* $0 \leq C_N < 1$ *depending only on* $N$ *which satisfies* $\delta(X) < C_N$.

Combining this proposition with Proposition A.1 in Appendix, we obtain the following corollary.

COROLLARY 5.2. *If a complete* CAT(0) *space* $X$ *is isometric to a (finite or infinite) product of uniformly locally doubling* CAT(0) *spaces with a common doubling constant* $N \in [1, \infty)$, *then there exists a constant* $c < 1$ *depending only on* $N$ *such that* $\delta(X) \leq c$.

Theorem 1.6 follows immediately from Corollary 4.2 and Proposition 3.3. To prove Proposition 5.1, we show that the ultralimit of a sequence of doubling length spaces with a common doubling constant is also doubling with the same doubling constant. First, we recall the definitions of ultrafilters and ultralimits. Let $I$ be a set. A collection $\omega \subset 2^I$ of subsets of $I$ is called a *filter* on $I$ if it satisfies the following conditions:

(a) $\emptyset \notin \omega$.

(b) $A \in \omega,\ A \subset B \Rightarrow B \in \omega$.

(c) $A, B \in \omega \Rightarrow A \cap B \in \omega$.

An *ultrafilter* is a maximal filter. The maximality condition can be rephrased as the following condition:

(d) For any decomposition $I = A_1 \cup \cdots \cup A_m$ of $I$ into finitely many disjoint subsets $A_1, \ldots, A_m$, $\omega$ contains exactly one of $A_1, \ldots, A_m$.

An ultrafilter $\omega$ on $I$ is called *nonprincipal* if it satisfies the following condition:

(e) For any finite subset $F \subset I$, $F \notin \omega$.

Zorn's lemma guarantees the existence of nonprincipal ultrafilters on any infinite set $I$.

Fix a set $I$, and an ultrafilter $\omega$ on $I$. For a topological space $X$, a point $p \in X$, and a mapping $f : I \to X$, we write

$$\omega\text{-}\lim_i f(i) = p \tag{5.1}$$

if for every neighborhood $U$ of $p$, the preimage $f^{-1}(U)$ belongs to $\omega$. Whenever $X$ is compact and Hausdorff, for every mapping $f : I \to X$, there exists a unique $p \in X$ which satisfies (5.1).

LEMMA 5.3. *Fix a set* $I$, *an ultrafilter* $\omega$ *on* $I$, *and a subset* $J \in \omega$. *Let* $X$ *be a topological space,* $f : I \to X$ *be a mapping, and* $p \in X$. *Then, the set*

$$\omega_J = \{K \in \omega \mid K \subset J\}$$

*becomes an ultrafilter on* $J$. *Moreover, if we have* $\omega_J\text{-}\lim_j f|_J(j) = p$ *for the restriction* $f|_J$ *of* $f$ *to* $J$, *then we also have* $\omega\text{-}\lim_i f(i) = p$.

*Proof.* Since it is straightforward to see that $\omega_J$ is an ultrafilter on $J$, we only show the "moreover" part. Assume that $\omega_J\text{-}\lim_j f|_J(j) = p$ holds. Let $U \subset X$ be an arbitrary neighborhood of $p$. Then by the assumption, $f|_J^{-1}(U) \in \omega_J$. Then $f|_J^{-1}(U) \in \omega$ by the definition of $\omega_J$. Hence, we have $f^{-1}(U) \in \omega$ since we have $f|_J^{-1}(U) \subset f^{-1}(U)$, which shows that $\omega\text{-}\lim_i f(i) = p$. □

Fix a set $I$ and an ultrafilter $\omega$ on $I$. Let $\{(X_i, d_i)\}_{i \in I}$ be a sequence of metric spaces indexed by $I$, and let $\prod_{i \in I} X_i$ be the set of all sequences $\{p_i\}_{i \in I}$ with $p_i \in X_i$ for every $i \in I$. We define a relation $\sim$ on $\prod_{i \in I} X_i$ by declaring $\{p_i\} \sim \{q_i\}$ if and only if $\omega\text{-}\lim_i d_i(p_i, q_i) = 0$, which becomes an equivalence relation. We denote the set of all equivalence classes by $\omega\text{-}\lim_i (X_i, d_i)$, or simply $\omega\text{-}\lim_i X_i$. We denote the equivalence class represented by a sequence $\{p_i\} \in \prod_{i \in I} X_i$ by $\omega\text{-}\lim_i p_i$. We define the distance $d_\omega(p, q)$ between $p = \omega\text{-}\lim_i p_i$ and $q = \omega\text{-}\lim_i q_i$ by

$$d_\omega(p, q) = \omega\text{-}\lim_i d_i(p_i, q_i) \in [0, \infty].$$

Then, $(\omega\text{-}\lim_i (X_i, d_i), d_\omega)$ becomes a metric space whose distance function possibly takes the value $\infty$.

Definition 5.4. Let $\omega$ be an ultrafilter on a set $I$. Let $\{(X_i, d_i)\}_{i \in I}$ be a sequence of metric spaces indexed by $I$. We call the metric space $(\omega\text{-}\lim_i (X_i, d_i), d_\omega)$ defined above the *ultralimit* of $\{(X_i, d_i)\}_{i \in I}$ with respect to $\omega$.

An ultralimit $(\omega\text{-}\lim_i (X_i, d_i), d_\omega)$ decomposes into components consisting of points of mutually finite distance. If we are given a basepoint $p_i$ of every $X_i$, we can pick out the component consisting of points which have finite distance from $\omega\text{-}\lim_i p_i$. This component is a usual metric space where the distance between every pair of points is finite, and we denote it by $\omega\text{-}\lim_i (X_i, d_i, p_i)$.

For a sequence $\{A_i\}_{i \in I}$ of subsets $A_i \subset X_i$, we denote by $\omega\text{-}\lim_i A_i$ the subset of $\omega\text{-}\lim_i (X_i, d_i)$ consisting of all points which are represented by sequences in $\prod_{i \in I} A_i$.

Lemma 5.5. *Fix a set $I$, an ultrafilter $\omega$ on $I$, and a sequence $\{(X_i, d_i)\}_{i \in I}$ of metric spaces. Let $\{A_i^{(1)}\}_{i \in I}, \ldots, \{A_i^{(m)}\}_{i \in I}$ be sequences of subsets such that $A_i^{(k)} \subset X_i$ for every $k = 1, \ldots, m$ and every $i \in I$. Then, we have*

$$\omega\text{-}\lim_i \left( \bigcup_{k=1}^{m} A_i^{(k)} \right) = \bigcup_{k=1}^{m} \omega\text{-}\lim_i A_i^{(k)}. \tag{5.2}$$

*Proof.* The right-hand side of (5.2) is contained in the left-hand side trivially. Let $p$ be an arbitrary point in $\omega\text{-}\lim_i (\bigcup_{k=1}^m A_i^{(k)})$ represented by $\{p_i\} \in \prod_{i \in I} (\bigcup_{k=1}^m A_i^{(k)})$. For every $k \in \{1, 2, \ldots, m\}$, we set

$$I_k = \{i \in I : k = \min\{l : p_i \in A_i^{(l)}\}\}.$$

Then, $I = I_1 \cup \cdots \cup I_m$ is a decomposition of $I$ into disjoint subsets, and the ultrafilter $\omega$ contains exactly one of these subsets. Suppose that $l \in \{1, 2, \ldots, m\}$ satisfies $I_l \in \omega$. Choose a sequence $\{q_i\} \in \prod_{i \in I} A_i^{(l)}$ such that $q_i = p_i$ whenever $i \in I_l$. Then, we have

$$\omega\text{-}\lim_i \ d_i(p_i, q_i) = \omega_{I_l}\text{-}\lim_i \ d_i(p_i, q_i) = 0$$

by Lemma 5.3. Hence, such a sequence $\{q_i\} \in \prod_{i \in I} A_i^{(l)}$ also represents $p$. Thus, we have $p \in \omega\text{-}\lim_i A_i^{(l)}$, which proves the lemma. $\square$

LEMMA 5.6. *Fix a set $I$, an ultrafilter $\omega$ on $I$, and a sequence $\{(X_i, d_i)\}_{i \in I}$ of length spaces. Let $p = \omega\text{-}\lim_i p_i$ be a point on the ultralimit $\omega\text{-}\lim_i(X_i, d_i)$ represented by a sequence $\{p_i\} \in \prod_{i \in I} X_i$. Then, we have*

$$\bar{B}(p, r) = \omega\text{-}\lim_i \ \bar{B}(p_i, r) \tag{5.3}$$

*for any $r > 0$, where $\bar{B}(p, r)$ denotes the closed ball in $\omega\text{-}\lim(X_i, d_i)$ of radius $r$ centered at $p$, and $\bar{B}(p_i, r)$ denotes the closed ball in $X_i$ of radius $r$ centered at $p_i$ for each $i$.*

*Proof.* The right-hand side of (5.3) is contained in the left-hand side trivially. Let $q$ be an arbitrary point in the ball $\bar{B}(p, r) \subset \omega\text{-}\lim_i(X_i, d_i)$ and let $\{q_i\}$ be a sequence representing $q$. We define a new sequence $\{q_i'\}$ as follows. If $i \in I$ satisfies $d_i(p_i, q_i) \le r$, we define $q_i' = q_i$. If $i \in I$ satisfies $d_i(p_i, q_i) > r + 1$, we define $q_i' = p_i$. If $i \in I$ satisfies

$$r + \frac{1}{m+1} < d_i(p_i, q_i) \le r + \frac{1}{m}$$

for a positive integer $m$, we take an arc-length parametrized path $\gamma : [0, L] \to X_i$ of length $L \le r + 2/m$ joining $p_i$ to $q_i$, and define $q_i'$ to be the point $\gamma(L - 2/m)$. In this case, we have $d_i(p_i, q_i') \le r$ and $d_i(q_i, q_i') \le 2/m$.

To prove that $q$ is contained in the right-hand side of (5.3), it suffices to show that the sequence $\{q_i'\}$ defined above satisfies

$$\omega\text{-}\lim_i \ d_i(q_i, q_i') = 0. \tag{5.4}$$

Let $U \subset \mathbf{R}$ be an arbitrary neighborhood of $0 \in \mathbf{R}$. Choose a positive integer $m$ large enough to satisfy $\bar{B}\left(0, \dfrac{2}{m}\right) \subset U$. Define the subset $I_m \subset I$ by

$$I_m = \left\{ i \in I \;\middle|\; d_i(p_i, q_i) \le r + \frac{1}{m} \right\}.$$

Then, we have $I_m \in \omega$ since $\omega\text{-}\lim_i d_i(p_i, q_i) \le r$. On the other hand, by the definition of $q_i'$, we have $d_i(q_i, q_i') \in \bar{B}\left(0, \dfrac{2}{m}\right)$ whenever $i \in I_m$. Thus,

$$I_m \subset \left\{ i \in I \,\middle|\, d_i(q_i, q_i') \in \bar{B}\left(0, \frac{2}{m}\right)\right\} \subset \{i \in I \mid d_i(q_i, q_i') \in U\}.$$

Hence, $\{i \in I \mid d_i(q_i, q_i') \in U\} \in \omega$, which proves (5.4). $\square$

We obtain the following proposition from Lemma 5.5 and Lemma 5.6.

PROPOSITION 5.7. *Fix a set $I$, an ultrafilter $\omega$ on $I$. Let $\{(X_i, d_i)\}_{i \in I}$ be a sequence of length spaces. If every $(X_i, d_i)$ is doubling with a common doubling constant for every $i \in I$, then the ultralimit $\omega\text{-}\lim_i(X_i, d_i)$ is also doubling with the same constant.*

*Proof.* By the assumption, there exists $N \in \mathbf{N}$ such that every $(X_i, d_i)$ is doubling with doubling constant $N$. Fix a point $p = \omega\text{-}\lim_i p_i \in \omega\text{-}\lim_i(X_i, d_i)$ and $r > 0$. Then, for each $i \in I$, there exist $N$ points $p_i^{(1)}, \dots, p_i^{(N)} \in X_i$ such that

$$\bar{B}(p_i, r) \subset \bigcup_{k=1}^{N} \bar{B}\left(p_i^{(k)}, \frac{r}{2}\right),$$

which implies that

$$\omega\text{-}\lim_i \bar{B}(p_i, r) \subset \omega\text{-}\lim_i \left\{ \bigcup_{k=1}^{N} \bar{B}\left(p_i^{(k)}, \frac{r}{2}\right)\right\}. \tag{5.5}$$

The left-hand side of (5.5) equals $\bar{B}(p, r)$ by Lemma 5.6, and the right-hand side equals $\bigcup_{k=1}^{N} \bar{B}(\omega\text{-}\lim_i p_i^{(k)}, r/2)$ by Lemma 5.5 and 5.6. Hence, we obtain

$$\bar{B}(p, r) \subset \bigcup_{k=1}^{N} \bar{B}\left(\omega\text{-}\lim_i p_i^{(k)}, \frac{r}{2}\right),$$

which proves the proposition. $\square$

PROPOSITION 5.8. *Fix a* CAT(0) *space $(X, d_X)$, $p \in X$ and a nonprincipal ultrafilter $\omega$ on $\mathbf{N}$. For every $n \in \mathbf{N}$, we define another metric $d_n$ on $X$ by*

$$d_n(p, q) = n d_X(p, q), \quad p, q \in X.$$

*Then the tangent cone $TC_pX$ isometrically embeds into $\omega\text{-}\lim_n(X, d_n, p)$.*

*Proof.* We construct an embedding $f : \mathrm{Cone}((S_pX)^\circ) \to \omega\text{-}\lim_n(X, d_n, p)$. For the origin $o \in \mathrm{Cone}((S_pY)^\circ)$, we define $f(o) = \omega\text{-}\lim_n p$. For $v \in$

$\mathrm{Cone}((S_pY)^\circ)\backslash\{o\}$, we define $f(v)$ as follows. Suppose that $v$ is represented by $([\gamma], r) \in (S_pY)^\circ \times (0, \infty)$, where $[\gamma]$ denotes the direction represented by a nontrivial geodesic $\gamma : [0, a] \to X$ starting from $p$. We define a sequence $\{p_n\} \in \prod_{n\in\mathbf{N}} X_n$ by

$$p_n = \begin{cases} \gamma\left(\dfrac{r}{n}\right), & \text{if } \dfrac{r}{n} \le a, \\ p, & \text{if } a < \dfrac{r}{n}, \end{cases}$$

and define $f(v) \in \omega\text{-}\lim_n(X, d_n, p)$ by

$$f(v) = \omega\text{-}\lim_n \, p_n.$$

Then, by (2.1) and the definition of the distance functions on Euclidean cones, it is easily seen that the above definition of $f$ is independent of the choices of $\gamma$, and that $f$ becomes an isometry. Since $TC_pX$ is the metric completion of $\mathrm{Cone}((S_pX)^\circ)$ and an ultralimit is always complete (see [1, Chapter I, Lemma 5.53]), $f$ extends to the isometric embedding of $TC_pX$, which completes the proof. $\square$

Combining Proposition 5.7 and Proposition 5.8, we obtain the following proposition.

Proposition 5.9. *Fix $N \in [1, \infty)$. Suppose that a* CAT(0) *space $(X, d_X)$ is uniformly locally doubling with doubling constant $N$. Then every tangent cone $TC_pX$ of $X$ is doubling with doubling constant $N$.*

*Proof.* Choose a nonprincipal ultrafilter $\omega$ on $\mathbf{N}$. Fix $p \in X$. For each $n \in \mathbf{N}$, let $d_n$ be a metric on $X$ defined by

$$d_n(p, q) = n d_X(p, q), \quad p, q \in X.$$

Since $(X, d_X)$ is locally doubling with doubling constant $N$, there exists $\varepsilon > 0$ such that the closed $\varepsilon$-ball in $(X, d_X)$ centered at $p$ is doubling with doubling constant $N$. Hence, for every $n$, the closed $n\varepsilon$-ball of $(X, d_n)$ centered at $p$ is doubling with doubling constant $N$. Fix an arbitrary $q = \omega\text{-}\lim_n q_n \in \omega\text{-}\lim_n(X, d_n, p)$ and $r > 0$. Let $s \ge 0$ be the distance between $q$ and the basepoint $\omega\text{-}\lim_n p$. We can assume that the sequence $\{q_n\}$ satisfies $d_n(p, q_n) \le 2s$ for every $n$ by replacing all points $q_n$ with $d(p, q_n) > 2s$ by $p$ if necessary. Then, the closed $r$-ball in $(X, d_n)$ centered at $q_n$ is doubling with doubling constant $N$ whenever $n \ge \dfrac{r + 2s}{\varepsilon}$ since it is contained in the closed $n\varepsilon$-ball in $(X, d_n)$ centered at $p$. Hence, by Lemma 5.6 and Proposition 5.7, the closed $r$-ball in the ultralimit $\omega\text{-}\lim_n(X, d_n, p)$ centered at the basepoint $\omega\text{-}\lim_n p$ is also doubling with doubling constant $N$. Since $TC_pY$ embeds isometrically into $\omega$-$\lim_n(Y, d_n, p)$ by Lemma 5.8, the proposition follows. $\square$

*Proof of Proposition* 5.1. By Proposition 5.7, every tangent cone $TC_pX$ of uniformly locally doubling CAT(0) space $X$ with doubling constant $N$ is doubling with constant $N$. Hence, the proposition follows from Corollary 3.5. $\square$

## 6. Applications

As we described in Section 1, Theorem 1.4 and 1.6 yield the fixed point property of a random group for a space which satisfies the hypothesis of either theorem, and coarse non-embeddability of sequences of expanders into such a space. In this section, we recall some related definitions and state these conclusions precisely.

First we recall the definition of a random group of the graph model which was introduced by Gromov [4]. A *path* on a graph $G = (V, E)$ is a finite sequence

$$(u_1, u_2), (u_2, u_3), \dots, (u_{n-1}, u_n), (u_n, u_{n+1})$$

of successive directed edges in $\vec{E}$. If all vertices $u_1, \dots, u_{n+1}$ are distinct, we call it an *embedded path*. A *cycle* is a path which starts and ends with a same vertex. The *girth* of a graph $G$ denoted by $\mathrm{girth}(G)$ is the minimal length of a cycle all of whose vertices are distinct except the starting and ending ones. If there is no such a cycle in $G$, $\mathrm{girth}(G)$ is defined to be $\infty$. The *diameter* of $G$ denoted by $\mathrm{diam}(G)$ is the supremum of the graph distance between two vertices in $G$.

Definition 6.1 (random groups of the graph model [4]). Fix a positive integer $k$, and a sequence $\{G_\ell = (V_\ell, E_\ell)\}_{\ell \in L}$ of graphs indexed by an unbounded set $L$ of positive integers. Let $\Gamma = F_k$ be the free group generated by $S = \{s_1^\pm, \dots, s_k^\pm\}$. An S-*labelling* of $G_\ell$ is a mapping $\alpha : \vec{E}_\ell \to S$ which satisfies $\alpha((u, v)) = \alpha((v, u))^{-1}$ for every $(u, v) \in \vec{E}_\ell$. We denote by $\Lambda(G_\ell, k)$ the set of all $S$-labellings of $G_\ell$. For every $\alpha \in \Lambda(G_\ell, k)$ and every path $\vec{p} = (\vec{e}_1, \dots, \vec{e}_l)$ in $G_\ell$, we denote $\alpha(\vec{p}) = \alpha(\vec{e}_1) \cdots \alpha(\vec{e}_l) \in \Gamma$. We define $R_\alpha = \{\alpha(\vec{c}) \in \Gamma \mid \vec{c}$ is a cycle in $G_\ell\}$ and $\Gamma_\alpha = \Gamma / \overline{R_\alpha}$, where $\overline{R_\alpha}$ is the normal closure of $R_\alpha$.

Now, for each group property $P$, we say that a *random group associated with* $\{G_\ell\}$ *has property* $P$ if we have

$$\lim_{\ell \to \infty} \frac{|\{\alpha \in \Lambda(G_\ell, k) \mid \Gamma_\alpha \text{ has the property } P\}|}{|\Lambda(G_\ell, k)|} = 1.$$

In what follows, we fix a positive integer $k$ and every random group is based on the free group $F_k$ of rank $k$. In [6], Izeki, Kondo and Nayatani proved the following fixed point theorem of a random group of the graph model. The following is a slight modification of Theorem 3.5 in [6].

THEOREM 6.2 (Izeki-Kondo-Nayatani [6]). *Fix* $C > 0$, $d > 0$ *and* $\lambda > 0$. *There exists* $\beta > 1$ *which satisfies the following statement. Let* $\mathcal{X}$ *be the set of all complete* CAT(0) *spaces* $X$ *which satisfies*

$$\lambda_1(G, X) \geq C\mu_1(G)$$

*for every graph* $G$. *Let* $\{G_\ell = (V_\ell, E_\ell)\}_{\ell \in L}$ *be a sequence of graphs indexed by an unbounded set* $L$ *of positive integers which satisfies the following conditions*:

(a) $\lim_{\ell \to \infty} |V_\ell| = \infty$,

(b) $2 \leq \deg(u) \leq d$ *for all* $\ell \in L$ *and* $u \in V_\ell$,

(c) $\mu_1(G_\ell) \geq \lambda$ *for all* $\ell \in L$,

(d) *There exists* $c > 0$ *which satisfies* $\mathrm{girth}(G_\ell) \geq \ell$ *and* $\mathrm{diam}(G_\ell) \leq c \cdot \ell$ *for every* $\ell \in L$.

(e) *There exists a constant* $c' > 0$ *such that the number of embedded paths in* $G_\ell$ *of length less than* $\dfrac{\ell}{2}$ *is less than* $c' \cdot \beta^{\ell/2}$.

*Then, the random group associated with* $\{G_\ell\}$ *is non-elementary hyperbolic and any of its isometric action on any* $X \in \mathcal{X}$ *has a global fixed point.*

Combining Theorem 1.4 and Theorem 1.6 with Theorem 6.2, we obtain the following theorem.

THEOREM 6.3. *Assume that a complete* CAT(0) *space* $X$ *satisfies the either of the following conditions.*

(i) $X$ *is a (finite or infinite) product of copies of a finite number of spaces each of which is geodesically complete and admits a cocompact proper isometric action of a group.*

(ii) $X$ *is a (finite or infinite) product of uniformly locally doubling* CAT(0) *spaces with a common doubling constant.*

*Then, the random group associated with a sequence* $\{G_\ell\}$ *of graphs which satisfies the conditions* (a), (b), (c), (d), (e) *in Theorem* 6.2 *is non-elementary hyperbolic and any of its isometric action on any* $X \in \mathcal{X}$ *has a global fixed point.*

DEFINITION 6.4 (sequences of expanders). A sequence $\{G_\ell = (V_\ell, E_\ell)\}_{\ell \in L}$ of graphs indexed by an unbounded set $L$ of positive integers is called a *sequence of expanders* if it satisfies the following conditions:

(1) $\lim_{\ell \to \infty} |V_\ell| = \infty$,

(2) There exists $d > 0$ which satisfies $\deg(u) \leq d$ for all $\ell \in L$ and $u \in V_\ell$,

(3) There exists $\lambda > 0$ which satisfies $\mu_1(G_\ell) \geq \lambda$ for all $\ell \in L$.

So the graph sequence which is used to define the random group in Theorem 6.2 and Theorem 6.3 is a sequence of expanders.

Definition 6.5. Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. A mapping $f : X \to Y$ is said to be a *coarse embedding* if there exist unbounded non-decreasing functions $\alpha, \beta : [0, \infty) \to [0, \infty)$ which satisfy

$$\alpha(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq \beta(d_X(x, x'))$$

for every $x, x' \in X$.

Let $\{G_\ell = (V_\ell, E_\ell)\}$ be a sequence of expanders, and let $d_\ell$ be the graph metric on $V_\ell$. Then, the sequence of expanders is said to be *embedded coarsely* into a metric space $(X, d_X)$ if there exist unbounded nondecreasing functions $\alpha, \beta : [0, \infty) \to [0, \infty)$ and mappings $\{f_\ell : V_\ell \to X\}_n$ which satisfy

$$\alpha(d_\ell(u, v)) \leq d_X(f_\ell(u), f_\ell(v)) \leq \beta(d_\ell(u, v))$$

for every $\ell$ and every $u, v \in V_\ell$. Coarse embeddability of a sequence of expanders into a metric space $X$ is a well-known obstruction for $X$ to be embedded coarsely into a Hilbert space (see [4] and [11]). Combining Theorem 1.4 and Theorem 1.6 with the conclusion (B) in Section 1, we see that a space satisfying the hypothesis of either theorem does not have such an obstruction.

Theorem 6.6. *If a complete* CAT(0) *space $X$ satisfies the either condition* (i) *or* (ii) *in Theorem* 6.3, *then a sequence of expanders does not embed coarsely into $X$.*

## Appendix A. Some remarks on the Izeki-Nayatani invariant

In this appendix, we collect some basic facts concerning the Izeki-Nayatani invariant, which are not mentioned in Section 3. First, the following proposition describes a basic behavior of the Izeki-Nayatani invariant under taking product, which is a slight generalization of Proposition 6.5 of [7] and quite similar to Lemma 4.3 of [13]. We include its proof for the sake of completeness.

Proposition A.1. *Let $X_1, X_2, X_3, \ldots$ be complete* CAT(0) *spaces. Let $X$ be a product of $X_1, X_2, X_3, \ldots$ (with respect to some basepoints). Then we have*

$$\delta(X) = \sup\{\delta(X_i) \mid i = 1, 2, 3, \ldots\}.$$

*Proof.* The inequality $\delta(X) \geq \sup\{\delta(X_i) \mid i = 1, 2, 3, \ldots\}$ is obvious since every $X_i$ is isometrically embedded into $X$. Let $\mu = \sum_{i=1}^m t_i \operatorname{Dirac}_{p_i}$ be a finitely supported probability measure on $X$ whose support contains at least two points. We write $p_i = (p_i^{(1)}, p_i^{(2)}, p_i^{(3)}, \ldots) \in \prod_n X_n = X$ for each $i = 1, \ldots, n$. For each $n$, we define a probability measure $\mu_n$ on $X_n$ to be

$$\mu_n = \sum_{i=1}^m t_i \operatorname{Dirac}_{p_i^{(n)}}.$$

Let $\mathrm{bar}(\mu) = (b_1, b_2, b_3, \dots)$ be the barycenter of $\mu$. Then we have $\mathrm{bar}(\mu_n) = b_n$ for every $n$ since if we had $\mathrm{bar}(\mu_n) \neq b_n$ for some $n$, then it would follow that

$$\int_X d_X(p, b')^2 \mu(dp) < \int_X d_X(p, \mathrm{bar}(\mu))^2 \mu(dp),$$

where $b'$ denotes the point on $X$ such that the $n$-th component is $b_n$ and the $i$-th component is $\mathrm{bar}(\mu_i)$ for every $i \neq n$.

For each $n$, let $\phi_n : \mathrm{Supp}(\mu_n) \to \mathscr{H}_n$ be a realization of $\mu_n$ which satisfies

$$\delta(\mu_n) = \frac{\|\int_{X_n} \phi_n(p)\mu_n(dp)\|^2}{\int_{X_n} \|\phi_n(p)\|^2 \mu_n(dp)}.$$

Such a realization $\phi_n$ exists for every $n$ since the space of all realizations of $\mu_n$ is compact. Define a mapping $\phi : \mathrm{Supp}(\mu) \to \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \mathscr{H}_3 \oplus \cdots$ as

$$\phi(p_i) = (\phi_1(p_i^{(1)}), \phi_2(p_i^{(2)}), \phi_3(p_i^{(3)}), \dots), \quad i = 1, \dots, m.$$

Then, it is easily seen that $\phi$ is a realization of $\mu$. And, we have

$$\begin{aligned}
\delta(\mu) &\leq \frac{\|\int_X \phi(p)\mu(dp)\|^2}{\int_X \|\phi(p)\|^2 \mu(dp)} = \frac{\sum_{n=1}^{\infty} \|\sum_{i=1}^m t_i \phi_n(p_i^{(n)})\|^2}{\sum_{n=1}^{\infty} \sum_{i=1}^m t_i \|\phi_n(p_i^{(n)})\|^2} \\
&\leq \sup_n \frac{\|\sum_{i=1}^m t_i \phi_n(p_i^{(n)})\|^2}{\sum_{i=1}^m t_i \|\phi_n(p_i^{(n)})\|^2} \leq \sup_n \delta(\mu_n),
\end{aligned}$$

which proves the desired inequality $\delta(X) \leq \sup\{\delta(X_i) \mid i = 1, 2, 3, \dots\}$. □

Although the Izeki-Nayatani invariant is defined as a global invariant of the space, it can be estimated by the local property of the space. To see this, we define the following notation, which is introduced in [7].

Definition A.2 (Izeki-Nayatani [7]). Let $X$ be a complete CAT(0) space, and $p \in X$. We define $\delta(X, p) \in [0, 1]$ to be

$$\delta(X, p) = \sup\{\delta(\nu) \mid \nu \in \mathscr{P}(X), \mathrm{bar}(\nu) = p\},$$

where $\mathscr{P}(X)$ is the space of all finitely supported probability measures on $Y$ whose supports contain at least two points. If no such $\nu$ exists, we define $\delta(X, p) = -\infty$.

Although, the following proposition is basic and important, it seems that there is no reference containing its complete proof.

Proposition A.3. *Suppose that $X$ is a complete* CAT(0) *space. Then we have*

$$\delta(X) = \sup\{\delta(TC_pX, o) \mid p \in X\} = \sup\{\delta(TC_pX) \mid p \in X\}, \tag{A.1}$$

*where $o$ denotes the origin of the tangent cone $TC_pX$.*

*Proof.* The inequality

$$\delta(X) \leq \sup\{\delta(TC_pX, o) \mid p \in X\}$$

was proved in [7, Lemma 6.2], and the inequality

$$\sup\{\delta(TC_pX, o) \mid p \in X\} \leq \sup\{\delta(TC_pX) \mid p \in X\}$$

is trivial from the definition. So we need only to prove the inequality

$$\sup\{\delta(TC_pX) \mid p \in X\} \leq \delta(X). \tag{A.2}$$

To this end, it suffices to show that $\delta(TC_pX) \leq \delta(X)$ for any $p \in X$. By Proposition 4.2 of [5], if $\{X_n\}_{n \in \mathbf{N}}$ is a sequence of complete CAT(0) spaces, $\omega$ is a nonprincipal ultrafilter on $\mathbf{N}$, and $X_\omega$ is the ultralimit of $\{X_n\}$ with respect to $\omega$, then $\delta(X_\omega) \leq \omega\text{-}\lim_n \delta(X_n)$ holds. Combining this with Proposition 5.8 in Section 5, the inequality (A.2) follows immediately. □

*Remark* A.4. Although the previous version of this paper [14] also includes the proof of Proposition A.3 without using the notion of ultralimit, we omit it here to avoid redundancy.

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