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THE JONES POLYNOMIAL OF RATIONAL LINKS

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Abstract

We use the Tutte polynomial to give an explicit formula for the Jones polynomial of any rational link in terms of the denominators of the canonical continued fraction of its slope.

1. Rational links and continued fractions

The class of rational links has been the core of many studies since they have been classified by Schubert [15] in terms of a rational number called the slope. Many people since then have studied different polynomial invariants of rational links. For example, the authors of [4, 12] give an explicit formula for the Conway (Alexander) polynomial invariant of rational links independently. Nakabo [13, 14] computes the HOMFLY and Jones polynomials of rational links, so that a formula for the Alexander polynomial of rational links follows from his main result in [13]. Moreover, the authors of [3, 6, 9, 10, 11, 16] have studied the Jones polynomial of rational links either directly or indirectly through studying another polynomial invariant that reduces to the Jones polynomial after some special normalization using different techniques.

In this paper, we give an explicit formula for the Jones polynomial of any rational link using a different approach than the one used in the above references. Our approach uses the Kauffman bracket state model in [7] and its relation to the Tutte polynomial of the Tait graph obtained from the diagram of the given link.

Each rational link is characterized by a rational number called the slope $\frac{p}{q}$ of a pair of relatively prime integers p, q with $\left|\frac{p}{q}\right| > 1$ and $p > 0$ by the following theorem due to Schubert [15].

Theorem 1.1. *Two rational links $L_{p/q}$ and $L_{p'/q'}$ are equivalent if and only if*

$$p = p',$$
$$\textit{and} \quad q^{\pm 1} \equiv q' \pmod{p}.$$

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A continued fraction of the rational number $\frac{p}{q}$ is a sequence of integers $b_1, b_2, \ldots, b_n$ such that

$$\frac{p}{q} = b_1 + \cfrac{1}{b_2 + \cfrac{1}{\cdots + \cfrac{1}{b_n}}}.$$

This continued fraction of the rational number $\frac{p}{q}$ will be abbreviated by $[b_1, b_2, \ldots, b_n]$ and the integers b_i are called the denominators of this continued fraction.

A diagram of a rational link can be constructed from the denominators of any continued fraction of its slope by closing the 4-braid $\sigma_2^{-b_1}\sigma_1^{b_2}\sigma_2^{-b_3}\cdots$ in the manner shown in Figure 1, where σ_1, σ_2 are shown in Figure 2 and the multiplication is defined by concatenating from left to right. It is known that this diagram is a diagram of a knot or a 2-component link according to whether the numerator p is odd or even, respectively, see for example [8, Page 21].

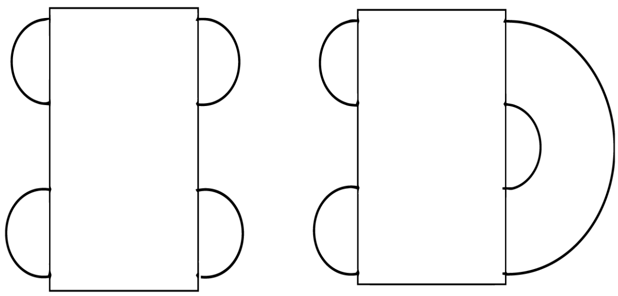

FIGURE 1. The closure of the 4-braid according to whether the last element of the product is $\sigma_2^{\pm 1}$ or $\sigma_1^{\pm 1}$, respectively.

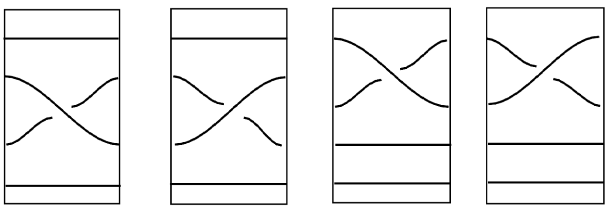

FIGURE 2. The 4-braids σ_2, σ_2^{-1}, σ_1, and σ_1^{-1}, respectively.

It is sufficient to consider the case when the number of denominators of the continued fraction n is odd and $b_i \geq 1$ for $i = 1, 2, \dots n$ as a result of the following lemma [8, Page 24–25].

LEMMA 1.2. *There exists a unique continued fraction of* $\dfrac{p}{q} > 1$ *of positive integers with* n *odd and* $b_i \geq 1$ *for* $i = 1, 2, \dots n$.

Proof. We start with the rational number $\dfrac{p}{q} > 1$ such that $\gcd(p, q) = 1$ and $p > q > 0$. Now we have

$$\frac{p}{q} = \frac{qb_1 + q_1}{q} = b_1 + \frac{1}{\dfrac{q}{q_1}} = b_1 + \frac{1}{\dfrac{q_1 b_2 + q_2}{q_1}} = b_1 + \frac{1}{b_2 + \dfrac{q_2}{q_1}}$$

$$= \cdots = b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{\ddots + \cfrac{1}{b_n}}}}.$$

In this way, we get a continued fraction $[b_1, b_2, \dots, b_n]$ of $\dfrac{p}{q}$. Now if n is even and $b_n = 1$ then, $[b_1, b_2, \dots, b_{n-1} + 1]$ is the continued fraction with odd number of denominators. In the other case, if n is even and $b_n > 1$, then $[b_1, b_2, \dots, b_{n-1}, b_n - 1, 1]$ is the continued fraction with odd number of denominators. Finally, the uniqueness follows from applying the Euclidean algorithm at every step. □

DEFINITION 1.3. The unique continued fraction obtained using the above lemma will be called the canonical continued fraction of $\dfrac{p}{q}$ and the diagram obtained from the canonical continued fraction will be called the canonical diagram of the rational link of slope $\dfrac{p}{q}$. It is easy to see that the canonical diagram is alternating.

2. The Jones polynomial

The Jones polynomial is an invariant of links that was first defined by V. Jones in [5]. It is a Laurent polynomial in one indeterminate defined on the set of oriented links. There are many approaches to define this invariant, but we choose the approach that serves our purposes in this paper.

The Jones polynomial of a given link can be computed using the Tutte polynomial of the associated Tait graph of the given link diagram. In this paper, we restrict our work to alternating link diagrams. Therefore, the associated Tait graph will be a planar graph without signs.

The way to construct the Tait graph of a given link diagram is by using the checkerboard coloring, that is we color the regions of the link diagram in $\mathbf{R}^2$ into two colors black and white such that regions that share an arc have different colors. We then place a vertex in each black region and associate an edge to each crossing of the link that connects two vertices to obtain the graph G. By interchanging black regions with white regions, we obtain the dual graph of G. We use the convention that near each crossing, if the overcrossing arc is drawn from NW to SE then the black regions appear on the left and the right sides. For connected alternating diagrams, this rule can be followed consistently with the checkerboard coloring.

Lemma 2.1. *Using the above convention the outside region is white also the Tait graph of the canonical rational link diagram takes the form of graph G given in Figure 3. This graph consists of $2 + \sum_{i=1}^{k} b_{2i}$ vertices and $\sum_{i=1}^{2k+1} b_i$ edges with b_i denotes the number of edges that are parallel if i is odd and collinear if i is even.*

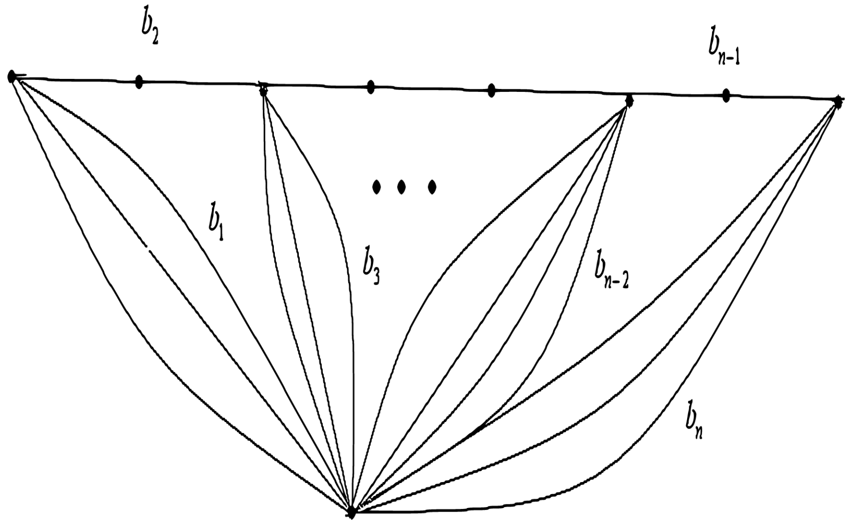

Figure 3. The canonical Tait graph associated to the sequence $\{b_1, \ldots, b_n\}$ of positive integers with an odd integer $n = 2k + 1$.

Definition 2.2. The graph G in Lemma 2.1 will be called the canonical Tait graph of the canonical diagram of the rational link of slope $\dfrac{p}{q}$. Also, we let $[b_1, b_2, \ldots, b_n]$ to be the canonical continued fraction of the corresponding rational number $\dfrac{p}{q}$.

The Jones polynomial of an oriented link can be expressed via the Tutte polynomial of the Tait graph in [17] by the following theorem:

THEOREM 2.3. *The Jones polynomial $V_L(t)$ of an alternating link L can be obtained from the Tutte polynomial $\chi(G; x, y)$ of the associated Tait graph G by the following equation*:

$$V_L(t) = (-1)^w t^{(a-b+3w)/4} \chi(G; -t, -t^{-1}) \tag{1}$$

where a is the number of white regions, b is the number of black regions, and w is the writhe of the oriented link diagram.

DEFINITION 2.4. The writhe of an oriented link is the number of positive crossings minus the number of negative crossings shown in Figure 4.

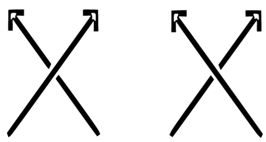

FIGURE 4. positive and negative crossings, respectively.

3. The Tutte polynomial of the canonical Tait Graph

First we recall the main properties of the Tutte polynomial of graphs and for further details and more basic reference of this polynomial see for example [1]. The following proposition lists some properties of the Tutte polynomial that we use in this paper.

PROPOSITION 3.1. *The Tutte polynomial $\chi(G; x, y) \in \mathbf{Z}[x, y]$ of a graph G satisfies*:

(1) *If the graph G consists only of one vertex v, then $\chi(v) = 1$.*
(2) *If the graph G consists only of an edge e which is not a loop, then $\chi(e) = x$.*
(3) *If the graph G consists only of one loop l, then $\chi(l) = y$.*
(4) *If $G_1 * G_2$ denotes a connected graph that consists of two graphs G_1 and G_2 having just one vertex in common, then $\chi(G_1 * G_2) = \chi(G_1)\chi(G_2)$.*
(5) *If $G_1 \sqcup G_2$ is the disjoint union of the two graphs G_1 and G_2, then $\chi(G_1 \sqcup G_2) = \chi(G_1)\chi(G_2)$.*
(6) *If e is an edge which is neither a loop nor a bridge of the graph G, then $\chi(G) = \chi(G - e) + \chi(G/e)$ where $G - e$ is the graph obtained be deleting*

the edge e in G and G/e is the graph obtained by contracting the edge e in G.

In a graph G a bridge is an edge whose removal increases the number of components of G and a loop is an edge which has the same vertex as its endpoints.

From now on, G is the canonical Tait graph of the canonical diagram D of the rational link L of slope $\frac{p}{q}$. Also, $[b_1, b_2, \dots, b_n]$ is the associated continued fraction of the rational number $\frac{p}{q}$. In this section, we give a formula for the Tutte polynomial of graph G in terms of the denominators of the associated continued fraction of the rational number $\frac{p}{q}$.

Now we quote the following lemmas for later use whose proofs can be found in any basic reference of graph theory see for example [18].

LEMMA 3.2. *Let C_p be the cycle graph of p edges that is a graph with p vertices and p consecutive edges such that each vertex is incident to two edges, then the Tutte polynomial*

$$\chi(C_p) = \frac{x^p - 1}{x - 1} + y - 1.$$

LEMMA 3.3. *The Tutte polynomial of the dual graph of a plane graph G equals the Tutte polynomial of the original graph after interchanging x and y.*

For each integer $0 \leq l \leq k$, we associate the l-tuple $1 \leq i_1 < i_2 < \cdots < i_l \leq k$ and two finite sequences of integers as follows:

$$\begin{aligned}
c_0 &= b_1 + b_3 + \cdots + b_{2i_1 - 1}, \\
c_1 &= b_{2i_1+1} + b_{2i_1+3} + \cdots + b_{2i_2-1}, \\
&\ \vdots \\
c_l &= b_{2i_l+1} + b_{2i_l+3} + \cdots + b_{2k+1},
\end{aligned}$$

and

$$c_n^* = \begin{cases} c_n, & \text{if } 0 \leq n < l, \\ c_l - b_{2k+1}, & \text{if } n = l. \end{cases}$$

Now, we state the main theorem of this section:

THEOREM 3.4. *The Tutte polynomial of the canonical Tait graph G*

$$\chi(G) = \sum_{\substack{0 \leq l \leq k \\ 1 \leq i_1 < i_2 < \cdots < i_l \leq k}} \prod_{j=1}^{l} \left(\frac{x^{b_{2i_j}} - 1}{x - 1} \right) \prod_{j=0}^{l} \left(\frac{y^{c_j} - 1}{y - 1} + x - 1 \right).$$

Proof. We use induction on k to prove the above formula. If $k = 0$ then G is the dual graph of C_{b_1} so from Lemmas 3.3 and 3.2 we get

$$\chi(G) = \frac{y^{b_1} - 1}{y - 1} + x - 1.$$

We assume that the result holds for $k - 1$, i.e. for the number of denominators $n = 2k - 1$. Now, we can apply part 6 of Proposition 3.1 on one of the b_{2k} edges that are collinear in Figure 3 and then we use part 4 of Proposition 3.1 to get

$$\chi(G) = x^{b_{2k}-1}\left(\frac{y^{b_{2k+1}} - 1}{y - 1} + x - 1\right)\chi(G') + \chi(G''),$$

where G' is the graph obtained by deleting all the b_{2k}- and b_{2k+1}-edges, and G'' is the graph obtained by contracting one of the b_{2k}-edges. Now we repeat this process $(b_{2k} - 1)$-times on graph G'' to obtain

$$\begin{aligned}
\chi(G) &= \left(\frac{x^{b_{2k}} - 1}{x - 1}\right)\left(\frac{y^{b_{2k+1}} - 1}{y - 1} + x - 1\right)\chi(G') + \chi(G^{\alpha}) \\
&= \left(\frac{x^{b_{2k}} - 1}{x - 1}\right)\left(\frac{y^{b_{2k+1}} - 1}{y - 1} + x - 1\right) \\
&\quad \times \sum_{\substack{0 \le l \le k-1 \\ 1 \le i_1 < i_2 < \cdots < i_l \le k-1}} \prod_{j=1}^{l}\left(\frac{x^{b_{2i_j}} - 1}{x - 1}\right)\prod_{j=0}^{l}\left(\frac{y^{c_j^*} - 1}{y - 1} + x - 1\right) \\
&\quad + \sum_{\substack{0 \le l \le k-1 \\ 1 \le i_1 < i_2 < \cdots < i_l \le k-1}} \prod_{j=1}^{l}\left(\frac{x^{b_{2i_j}} - 1}{x - 1}\right)\prod_{j=0}^{l}\left(\frac{y^{c_j} - 1}{y - 1} + x - 1\right) \\
&= \sum_{\substack{0 \le l \le k \\ 1 \le i_1 < i_2 < \cdots < i_l = k}} \prod_{j=1}^{l}\left(\frac{x^{b_{2i_j}} - 1}{x - 1}\right)\prod_{j=0}^{l}\left(\frac{y^{c_j} - 1}{y - 1} + x - 1\right) \\
&\quad + \sum_{\substack{0 \le l \le k \\ 1 \le i_1 < i_2 < \cdots < i_l < k}} \prod_{j=1}^{l}\left(\frac{x^{b_{2i_j}} - 1}{x - 1}\right)\prod_{j=0}^{l}\left(\frac{y^{c_j} - 1}{y - 1} + x - 1\right) \\
&= \sum_{\substack{0 \le l \le k \\ 1 \le i_1 < i_2 < \cdots < i_l \le k}} \prod_{j=1}^{l}\left(\frac{x^{b_{2i_j}} - 1}{x - 1}\right)\prod_{j=0}^{l}\left(\frac{y^{c_j} - 1}{y - 1} + x - 1\right),
\end{aligned}$$

where G^{α} is the graph obtained by contracting all the b_{2k}-edges. The second equality follows from the induction hypothesis on G' and G^{α}. $\square$

COROLLARY 3.5. *The Tutte polynomial of the canonical Tait graph G of the canonical diagram of the rational link $C(b_1, b_2)$ in Conway's notation* [2]

$$\chi(G; x, y) = x\left(\frac{x^{b_2-1}-1}{x-1}\right)\left(\frac{y^{b_1}-1}{y-1}+x-1\right)+\frac{y^{b_1+1}-1}{y-1}+x-1.$$

Proof. The result follows since the canonical continued fraction of the rational link $C(b_1, b_2)$ is $[b_1, b_2-1, 1]$. □

4. Main results

We consider the case where $\frac{p}{q} > 1$ since the other case yields the mirror image of the link of slope $\left|\frac{p}{q}\right|$ and the relation between the Jones polynomial of a link and the Jones polynomial of its mirror image is given by the following theorem:

THEOREM 4.1. *Suppose K^* is the mirror image of a link K, then*

$$V_{K^*}(t) = V_K(t^{-1}).$$

We want to compute the number of white regions, the number of black regions, and the writhe of the canonical diagram D in terms of the denominators of the canonical continued fraction that will be used in the Theorem 4.5.

LEMMA 4.2. *For the Tait graph G, we have*

$$a = k+1+\sum_{i=1}^{k+1}(b_{2i-1}-1) = \sum_{i=1}^{k+1} b_{2i-1}.$$

$$b = |V_G| = 2+\sum_{i=1}^{k} b_{2i}.$$

We associate to the canonical diagram D a permutation $\sigma_D \in S_3$ on the set $\{1, 2, 3\}$. This permutation σ_D is defined in terms of the denominators of the canonical continued fraction by

$$\sigma_D = (23)^{b_1}(12)^{b_2}(23)^{b_3}\cdots(12)^{b_{2k}}(23)^{b_{2k+1}}.$$

The writhe of any link diagram depends on the orientation of each of its components. In the case of a knot diagram, then the writhe does not depend on the orientation since reversing the orientation reverses the orientation on both arcs of that crossing. In the case of link diagrams of two components, the writhe has two values according to the choice of the orientation of both components.

PROPOSITION 4.3. *The writhe of the canonical diagram of a rational knot*

$$w = \begin{cases} -b_1 - b_2 + \sum_{i=3}^{n} \varepsilon_i b_i, & \text{if } \sigma_D = (23) \text{ or } \sigma_D = (123), \\ b_1 - b_2 + \sum_{i=3}^{n} \varepsilon_i b_i, & \text{if } (\sigma_D = (13) \text{ or } \sigma_D = (132)) \text{ and } b_1 \text{ is even}, \\ b_1 + b_2 + \sum_{i=3}^{n} \varepsilon_i b_i, & \text{if } (\sigma_D = (13) \text{ or } \sigma_D = (132)) \text{ and } b_1 \text{ is odd}, \end{cases}$$

where

$$\varepsilon_i = \begin{cases} -\varepsilon_{i-2}, & \text{if } b_{i-1} \text{ is odd}, \ i-1 \text{ is even and } \varepsilon_{i-1} = -1, \\ -\varepsilon_{i-2}, & \text{if } b_{i-1} \text{ is odd}, \ i-1 \text{ is odd and } \varepsilon_{i-1} = 1, \\ \varepsilon_{i-2}, & \text{otherwise}. \end{cases}$$

Proof. We prove the case for which $\sigma_D = (23)$. In this case, the canonical rational knot diagram appears as in Figure 5.

The set of all crossings in the canonical diagram D forms a partition of n elements such that the i-th element of this partition contains all the crossings that form $\sigma_2^{-b_i}$ if i is odd and $\sigma_1^{b_i}$ if i is even in the braid form. It is clear that crossings of the same element of the partition have the same sign. Therefore, we have $w = \sum_{i=1}^{2k+1} \varepsilon_i b_i$.

Now after we choose an orientation as given in Figure 5, we obtain that left semicircular arcs (bridges) are oriented clockwise and counterclockwise for the lower one and for the upper one, respectively. Therefore, the crossings in the first and the second elements of the partition are negative, i.e. $\varepsilon_1 = \varepsilon_2 = -1$. Assume that we determine the value of ε_i for $1 \leq i \leq m-1$ and we want to determine the value of ε_m. We note that the value of ε_m depends on the parity of b_{m-1} and the value of ε_{m-2}. Therefore, we can consider the value of b_{m-1} of being 1 or 2 in the case that b_{m-1} is odd or even respectively. Now we show one case as in Figure 5 and the other cases will be treated similarly. □

PROPOSITION 4.4. *The writhe of the canonical diagram of a rational link of two components has two values according to the choice of orientation*

(1) *If the left semicircular arcs (bridges) are oriented clockwise, then the writhe*

$$w = \begin{cases} b_1 + b_2 + \sum_{i=3}^{n} \varepsilon_i b_i, & \text{if } (\sigma_D = (1) \text{ or } \sigma_D = (12)) \text{ and } b_1 \text{ is odd}, \\ b_1 - b_2 + \sum_{i=3}^{n} \varepsilon_i b_i, & \text{if } (\sigma_D = (1) \text{ or } \sigma_D = (12)) \text{ and } b_1 \text{ is even}, \end{cases}$$

(2) *If the lower left semicircular arc (bridge) is oriented clockwise and the upper left semicircular arc (bridge) is oriented counterclockwise, then*

the writhe

$$w = -b_1 - b_2 + \sum_{i=3}^{n} \varepsilon_i b_i.$$

where

$$\varepsilon_i = \begin{cases} -\varepsilon_{i-2}, & \text{if } b_{i-1} \text{ is odd, } i-1 \text{ is even and } \varepsilon_{i-1} = -1, \\ -\varepsilon_{i-2}, & \text{if } b_{i-1} \text{ is odd, } i-1 \text{ is odd and } \varepsilon_{i-1} = 1, \\ \varepsilon_{i-2}, & \text{otherwise.} \end{cases}$$

Proof. The proof is similar to the proof of Proposition 4.3 by considering all cases. □

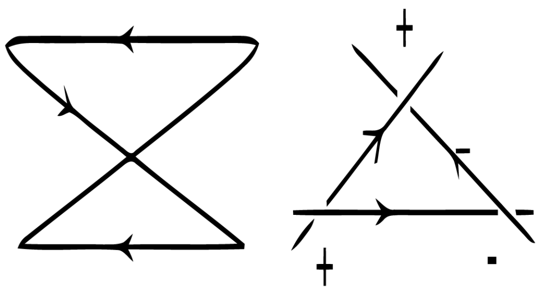

FIGURE 5. The case where $\sigma_D = (23)$ and one of the cases of ε_i.

From Theorem 3.4, Equation 1 and Lemma 4.2, we get a formula of the Jones polynomial of rational links.

THEOREM 4.5. *The Jones polynomial of the rational link L*

$$V_L(t) = (-1)^w t^{(\sum_{i=1}^{k+1} b_{2i-1} - 2 - \sum_{i=1}^{k} b_{2i} + 3w)/4} \chi(G; -t, -t^{-1}), \tag{2}$$

where $\chi(G; -t, -t^{-1})$ is the Tutte polynomial of the canonical Tait graph G computed in Theorem 3.4 and w is the writhe computed in Propositions 4.3 and Proposition 4.4.

COROLLARY 4.6. *The determinant of the rational link L*

$$\det(L) = \sum_{\substack{0 \le l \le k \\ 1 \le i_1 < i_2 < \cdots < i_l \le k}} \prod_{j=1}^{l} b_{2i_j} \prod_{j=0}^{l} c_j. \tag{3}$$

COROLLARY 4.7. *The Jones polynomial of the rational link $C(b_1)$ in Conway's notation* [2]

$$V_L(t) = (-1)^{w+1} t^{(b_1+2+3w)/4} + \sum_{i=1}^{b_1-1} (-1)^{w+i} t^{(b_1-2+3w-4i)/4}.$$

Proof. According to Lemmas 3.2 and 3.3

$$\chi(G; -t, -t^{-1}) = \frac{(-t^{-1})^{b_1} - 1}{-t^{-1} - 1} - t - 1.$$

Equation 2 implies

$$\begin{aligned} V_L(t) &= (-1)^w t^{(b_1-2+3w)/4} \left(\frac{(-t^{-1})^{b_1} - 1}{-t^{-1} - 1} - t - 1 \right) \\ &= (-1)^w t^{(b_1+2+3w)/4} + \sum_{i=1}^{b_1-1} (-1)^{w+i} t^{(b_1-2+3w-4i)/4}. \end{aligned}$$ □

COROLLARY 4.8. *The Jones polynomial of the rational link $C(b_1, b_2)$ in Conway's notation* [2]

$$\begin{aligned} V_L(t) = (-1)^w t^{(b_1-b_2+3w)/4} \Bigg(&\left(\frac{(-t)^{b_2-1} - 1}{-t - 1} \right) \left(\frac{(-t^{-1})^{b_1} - 1}{-t^{-1} - 1} - t - 1 \right) (-t) \\ &+ \left(\frac{(-t^{-1})^{b_1+1} - 1}{-t^{-1} - 1} - t - 1 \right) \Bigg). \end{aligned}$$

Proof. The canonical continued fraction of the rational link $C(b_1, b_2)$ is $[b_1, b_2 - 1, 1]$. Now corollary 3.5 implies

$$\begin{aligned} \chi(G; -t, -t^{-1}) = -t &\left(\frac{(-t)^{b_2-1} - 1}{-t - 1} \right) \left(\frac{(-t^{-1})^{b_1} - 1}{-t^{-1} - 1} - t - 1 \right) \\ &+ \frac{(-t^{-1})^{b_1+1} - 1}{-t^{-1} - 1} - t - 1. \end{aligned}$$

Also equation 2 implies

$$\begin{aligned} V_L(t) = (-1)^w t^{(b_2-b_1+3w)/4} \Bigg(&\left(\frac{(-t)^{b_2-1} - 1}{-t - 1} \right) \left(\frac{(-t^{-1})^{b_1} - 1}{-t^{-1} - 1} - t - 1 \right) (-t) \\ &+ \left(\frac{(-t^{-1})^{b_1+1} - 1}{-t^{-1} - 1} - t - 1 \right) \Bigg). \end{aligned}$$ □

Remark 4.9. The writhe in the above two corollaries can be computed using Proposition 4.3 and Proposition 4.4.

Remark 4.10. We thank the anonymous referee for his/her valuable comments and suggestions that improved this paper especially in suggesting a new notation that simplified the formulas in this paper.

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