A. B. ABDESSELEM AND I. ADOUANI
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LOWER BOUND OF ADMISSIBLE FUNCTIONS ON THE GRASSMANNIAN $G_{m,nm}(\mathbf{C})$

Adnène Ben Abdesselem and Ines Adouani

Abstract

We prove the existence of an "extremal" function lower bounding all admissible functions (ie plurisubharmonic functions modulo a metric) with supremum equal to zero on the complex Grassmann manifold $G_{m,nm}(\mathbf{C})$. The functions considered are invariant under a suitable automorphisms group. This gives a conceptually simple method to compute Tian's invariant in the case of a non toric manifold.

Résumé

On prouve l'existence d'une fonction "extrémale" minorant toutes les fonctions admissibles (ie plurisousharmoniques à la métrique initiale près) à sup nul sur la grassmannienne complexe $G_{m,nm}(\mathbf{C})$. Les fonctions considérées sont invariantes par un groupe d'automorphismes convenablement choisi. Cette minoration permet de calculer l'invariant de Tian sur un exemple de variétés non toriques.

1. Introduction

This article takes its origin in the problem of the existence of Kähler-Einstein metrics on a compact Kähler manifold X. This problem is one of the most fundamental problems in complex differential geometry. Let us recall that a metric g is said to be Kähler-Einstein if its Kähler form ω satisfies the following equation:

$$R = \lambda w \tag{1.1}$$

for a real number λ. R is the Ricci curvature form relative to the Kähler form ω.

In local coordinates $(z_1, \dots, z_n)$ of X, if $w = ig_{\alpha\bar{\beta}}\, dz^\alpha \wedge d\bar{z}^\beta$, then the Ricci tensor's components are given by $R_{\alpha\bar{\beta}} = -\partial_{\alpha\bar{\beta}} \log(\det(g_{\alpha\beta}))$ (where $\partial_{\alpha\bar{\beta}} = \dfrac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta}$).

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Since the cohomology class of the Ricci curvature is the first Chern class $C_1(X)$, a nessecary condition for the existence of a Kähler-Einstein metric is that $C_1(X)$ has a prescibed sign.

The problem is to find a real smooth and g-admissible function φ (ie $g_{\alpha\bar{\beta}} + \partial_{\alpha\bar{\beta}}\varphi$ is positive definite) such that:

$$g'_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}} + \partial_{\alpha\bar{\beta}}\varphi \tag{1.2}$$

is Kähler-Einstein. This leads, after a suitable normalisation, to the complex Monge-Ampère equation:

$$\log \det(g'g^{-1}) = -\varepsilon\varphi + f, \tag{1.3}$$

where f is a geometric data and where $\varepsilon = -1$ (respectively $\varepsilon = 0$, $\varepsilon = +1$) when $C_1(X)$ is negative (respectively null, positive).

The cases $C_1(X) < 0$ and $C_1(X) = 0$ (linked with Calabi conjecture) were solved independently by Aubin [A], [A1] and Yau [Y] (see also [T1]). In both cases one get a positive answer.

In the positive case, unlike the two previous ones, there exist obstructions to the existence of Kähler-Einstein metrics, given by Matsushima [M], Lichnnerowicz [L] and Futaki [F]. However, it is interesting to find conditions under which these manifolds admit or do not admit Kähler-Einstein metrics. The linearized operator of the equations cited above is no longer invertible in the positive case (since $\varepsilon = +1$ is positive). To overcome this difficulty, Thierry Aubin introduced another family of equations:

$$(*)_t : \log \det(g'g^{-1}) = -t\varphi + f, \quad \varphi \text{ is } C^\infty g\text{-admissible}, \tag{1.4}$$

and reduces the problem to the C^0-estimate of the solutions φ_t of $(*)_t$. To this end, T. Aubin in [A2] introduced an holomorphic invariant $\xi(X)$ giving a sufficient condition to the existence of a Kähler-Einstein metric on X.

A few years later, taking into account a Hörmander inequality [H] and functionals introduced by Aubin in [A2] (see also [A3]), G. Tian [T] introduced a new holomorphic invariant $\alpha_G(X)$ for C^∞ g-admissible functions, invariant under a group of automorphisms G, easier to compute than Aubin's one. Under a condition on this invariant, equation (1.1) can be solved:

THEOREM 1.1 (Tian [T]). *Let (X, g) be a compact Kähler manifold of complex dimension n with $C_1(X) > 0$. Define*:

$$\mathscr{A}_G = \{\varphi \in C^\infty(X) \mid \varphi \textit{ is } G\textit{-invariant on } X,\ g\textit{-admissible},\ \sup \varphi = 0\}$$

and

$$\alpha_G(X) = \sup\left\{\alpha \mid \exists C > 0 \textit{ such that } \int_X e^{-\alpha\varphi}\, dv \le C \textit{ for all } \varphi \in \mathscr{A}_G\right\}.$$

Then X admits a Kähler-Einstein metric whenever $\alpha_G(X) > \dfrac{n}{n+1}$.

Furthermore, the first author was interested in proving the existence of a Kähler-Einstein metric on some Fano manifolds and he gaves a "tool" to compute Tian's invariant. In fact, this method uses the underlying algebraic aspects of the manifolds under study to highlight an "extremal" function ψ lower bounding all C^∞ g-admissible functions φ with supremum equal to zero, and invariant under a suitable group of automorphisms. This fact allows to compute Tian's invariant by estimating the integral of the exponential of a single function. In [B1], the method has been initiated in the simplest case, namely the sphere $S^2 = \mathbf{P}^1(\mathbf{C})$. A similar lower bound has been proven on the complex projective space [B2], as well as on manifolds built from the projective space by blow-up ([B] [B-D] [B-C]).

In a recent paper [B-J], this method was applied in the case of a non toric manifold: the Grassmannian $G_{2,4}(\mathbf{C})$. Moreover, Tian's invariant for complex Grassmannian $G_{p,q}(\mathbf{C})$ (the space of p-planes in $\mathbf{C}^{p+q}$) has already been computed by J. Grivaux in [G] using a very clever and completely different method. In this article, our aim is to generalize the result given on $G_{2,4}(\mathbf{C})$.

2. Preliminary material

Let $G_{m,nm}(\mathbf{C})$ be the complex Grassmannian manifold of m-planes in $\mathbf{C}^{nm}$. It is a compact complex manifold of dimension $m^2(n-1)$. Let $M_{nm,m}(\mathbf{C})$ be the space of $nm \times m$ complex matrices (i-e nm rows and m columns) and let $L_{nm,m}(\mathbf{C})$ be the subset of matrices of rank m. A point of $G_{m,nm}(\mathbf{C})$ is identified by a matrix $M \in L_{nm,m}(\mathbf{C})$:

$$M = \begin{pmatrix} Z_0 \\ \vdots \\ Z_{n-1} \end{pmatrix} \tag{2.1}$$

where $Z_j = \begin{pmatrix} l_{jm+1} \\ \vdots \\ l_{(j+1)m} \end{pmatrix}$ (for $0 \le j \le n-1$) is a $m \times m$ complex matrix, and where $(l_{jm+k})_{1 \le k \le m}$ are the rows of Z_j.

Setting $I = \{i_1, \dots, i_m\}$, where $1 \le i_1 < \cdots < i_m \le nm$, Z_I denotes the $m \times m$ complex matrix given by:

$$Z_I = \begin{pmatrix} l_{i_1} \\ \vdots \\ l_{i_m} \end{pmatrix}. \tag{2.2}$$

Considering the above description, the domains of usual charts of $G_{m,nm}(\mathbf{C})$ are given by:

$$U_I = \{M \in G_{m,nm}(\mathbf{C}) \text{ such that } \det Z_I \neq 0\}. \tag{2.3}$$

For example, every point $M = \begin{pmatrix} Z_0 \\ Z_1 \\ \vdots \\ Z_{n-1} \end{pmatrix}$ in U_{I_0}, where $I_0 = \{1, 2, \dots, m\}$, can be written $M = \begin{pmatrix} Id_0 \\ Z_1' \\ \vdots \\ Z_{n-1}' \end{pmatrix}$ where Id_0 is the identity matrix of order m and, for $j \in \{1, \dots, n-1\}$, $Z_j' = Z_j Z_0^{-1}$.

We endow $G_{m,nm}(\mathbf{C})$ by the metric g obtained from the Fubini-Study on $\mathbf{P}^{r-1}\mathbf{C}$ where $r = \binom{nm}{m}$. Let $(z^\alpha)_{1 \le \alpha \le m^2(n-1)}$ be a local coordinate system in U_{I_0}, then g is given by:

$$g = a_n \partial_{\alpha\bar{\beta}} \ln\left(1 + \sum_{I \subset \{1,\dots,nm\} \setminus I_0} |\det Z_I|^2\right) dz^\alpha \otimes d\bar{z}^\beta, \tag{2.4}$$

where $a_n = nm$ and $\partial_{\alpha\bar{\beta}} = \dfrac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta}$. The integer a_n is chosen such that the metric g is in the first Chern class $C_1(G_{m,nm}(\mathbf{C}))$ of $G_{m,nm}(\mathbf{C})$. g is a Kähler metric on $G_{m,nm}(\mathbf{C})$ with fundamental form:

$$\omega = i a_n \partial_{\alpha\bar{\beta}} \ln\left(1 + \sum_{I \subset \{1,\dots,nm\} \setminus I_0} |\det Z_I|^2\right) dz^\alpha \wedge dz^{\bar{\beta}}. \tag{2.5}$$

Definition 2.1. A function $\varphi \in C^\infty(G_{m,nm}(\mathbf{C}))$ is called g-admissible if $(g + \partial_{\alpha\bar{\beta}}\varphi)$ is positive definite.

The function $\tilde{\psi}$ on $L_{nm,m}(\mathbf{C})$ of nm^2 variables given by:

$$\tilde{\psi}(M) = \ln\left(\frac{\prod_{j=0}^{j=n-1} |\det Z_j|^{2m}}{\left(\sum_{I \subset \{1,\dots,nm\}} |\det Z_I|^2\right)^{a_n}}\right) \tag{2.6}$$

induces a well defined function $\tilde{\psi}$ on $G_{m,nm}(\mathbf{C})$ outside the boundaries of the charts U_I. In fact, the right multiplication of a point M in $L_{nm,m}(\mathbf{C})$ by a matrix $A \in GL_m(\mathbf{C})$ is equal to $M' = \begin{pmatrix} Z_0 A \\ \vdots \\ Z_{n-1} A \end{pmatrix}$ and, recalling that $a_n = nm$, we have:

$$\tilde{\psi}(M') = \ln\left(\frac{|\det A|^{2nm} \prod_{j=0}^{j=n-1} |\det Z_j|^{2m}}{|\det A|^{2nm}\left(\sum_{I \subset \{1,\dots,nm\}} |\det Z_I|^2\right)^{a_n}}\right) \tag{2.7}$$

$$= \tilde{\psi}(M). \tag{2.8}$$

The expression of $\tilde{\psi}$ in the chart U_{I_0} is given by:

$$\tilde{\psi}(M) = \ln\left(\frac{\prod_{j=1}^{j=n-1} |\det Z_j|^{2m}}{(1 + \sum_{I \subset \{1,\dots,nm\}\setminus I_0} |\det Z_I|^2)^{a_n}}\right). \tag{2.9}$$

Let $\psi = \tilde{\psi} - \sup \tilde{\psi} = \tilde{\psi} + a_n m \ln(n)$. This new function reaches a supremum equal to zero on the point $\begin{pmatrix} Id_0 \\ \vdots \\ Id_{n-1} \end{pmatrix} \in G_{m,nm}(\mathbf{C})$ and tends to infinity in the boundaries of the charts U_I. This will be proved into Proposition 4.3.

3. An appropriate isometry group

The unitary group $U_{nm}(\mathbf{C})$ acts transitively by left multiplication on $G_{m,nm}(\mathbf{C})$ and induces an isometry group of $G_{m,nm}(\mathbf{C})$ with respect to the Fubini Study metric g. Consequently, given two points M and N of $G_{m,nm}(\mathbf{C})$, there always exist an isometry that transforms M in N.

Let us consider an isometry $i : G_{m,nm}(\mathbf{C}) \to G_{m,nm}(\mathbf{C})$ satisfying

$$i(M) = D \tag{3.1}$$

where $D = \begin{pmatrix} Id_0 \\ D_1 \\ \vdots \\ D_{n-1} \end{pmatrix}$ is the matrix obtained from M by the following description. Set

$$M = \begin{pmatrix} Id_0 \\ Z_1 \\ \vdots \\ Z_{n-1} \end{pmatrix} \in \bigcap_I \{\det Z_I \neq 0\}. \tag{3.2}$$

Since the matrices Z_j (for $1 \leq j \leq n-1$) are invertibles and according to the polar decomposition theorem, there exists a unique pair $(U_j, H_j) \in U_m(\mathbf{C}) \times H_m^+(\mathbf{C})$ such that $Z_j = U_j H_j$. However, H_j is a hermitian positive definite matrix. It follows that H_j is diagonalizable and its eigenvalues are strictly positive real numbers. In other words we have: $H_j = P_j^{-1} D_j P_j$ where $P_j \in U_m(\mathbf{C})$ and $D_j = \begin{pmatrix} \lambda_j^1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_j^m \end{pmatrix}$ is the $m \times m$ diagonal matrix whose diagonal entries are the eigenvalues of H_j.

The existence of such an automorphism is provided by the transitive action of the unitary group $U_{nm}(\mathbf{C})$ on $G_{m,nm}(\mathbf{C})$. In particular, among these automorphisms we also consider the maps P_{ij} (for $i, j \in \{0, \dots, n-1\}$) and Φ_U defined as follows:

$$P_{ij} : G_{m,nm}(\mathbf{C}) \to G_{m,nm}(\mathbf{C})$$

$$\begin{pmatrix} Z_0 \\ Z_1 \\ \vdots \\ Z_i \\ \vdots \\ Z_j \\ \vdots \\ Z_{n-1} \end{pmatrix} \mapsto \begin{pmatrix} Z_0 \\ Z_1 \\ \vdots \\ Z_j \\ \vdots \\ Z_i \\ \vdots \\ Z_{n-1} \end{pmatrix}$$

and

$$\phi_U : G_{m,nm}(\mathbf{C}) \to G_{m,nm}(\mathbf{C})$$

$$\begin{pmatrix} Z_0 \\ Z_1 \\ \vdots \\ Z_{n-1} \end{pmatrix} \mapsto \begin{pmatrix} UZ_0 \\ Z_1 \\ \vdots \\ Z_{n-1} \end{pmatrix}$$

The map P_{ij} exchanges the matrices Z_j (for $0 \le j \le n-1$) between them. Therefore ϕ_U corresponds to the left multiplication of matrices Z_j (for $0 \le j \le n-1$) by a unitary matrix $U \in U_m(\mathbf{C})$. Both Φ_U and P_{ij} are well defined. In what follows, G stands for the group generated by the isometries i. The metric g is, by definition, invariant under the action of G.

4. Main results

Here and subsequently, all calculations will be made in the chart U_{I_0} defined by

$$U_{I_0} = \{M \in G_{m,nm}(\mathbf{C}) \text{ such that } \det Z_{I_0} \neq 0\}. \tag{4.1}$$

Locally, in the chart U_{I_0}, to say a map φ defined on $G_{m,nm}(\mathbf{C})$ is G-invariant implies that it satisfies:

$$\varphi \begin{pmatrix} Id_0 \\ Z_1 \\ Z_2 \\ \vdots \\ Z_{n-1} \end{pmatrix} = \varphi \begin{pmatrix} Z_1 \\ Id_0 \\ Z_2 \\ \vdots \\ Z_{n-1} \end{pmatrix} = \varphi \begin{pmatrix} UZ_1 \\ Id_0 \\ Z_2 \\ \vdots \\ Z_{n-1} \end{pmatrix} = \varphi \begin{pmatrix} Id_0 \\ UZ_1 \\ Z_2 \\ \vdots \\ Z_{n-1} \end{pmatrix} \tag{4.2}$$

and, in the intersection of charts $(\bigcap_I \{\det Z_I \neq 0\})$, we have

$$\varphi\begin{pmatrix} Id_0 \\ Z_1 \\ Z_2 \\ \vdots \\ Z_{n-1} \end{pmatrix} = \varphi\begin{pmatrix} Z_1 \\ Id_0 \\ Z_2 \\ \vdots \\ Z_{n-1} \end{pmatrix} = \varphi\begin{pmatrix} Id_0 \\ Z_1^{-1} \\ Z_2 Z_1^{-1} \\ \vdots \\ Z_{n-1} Z_1^{-1} \end{pmatrix} \tag{4.3}$$

THEOREM 4.1. *Let $\varphi \in C^\infty(G_{m,nm}(\mathbf{C}))$ be a g-admissible and G-invariant function such that* $\sup_{G_{m,nm}(\mathbf{C})} \varphi = \varphi\begin{pmatrix} Id_0 \\ \vdots \\ Id_{n-1} \end{pmatrix} = 0$. *Then*: $\varphi \geq \psi$.

This theorem asserts that all admissible and G-invariant function $\varphi \in C^\infty(G_{m,nm}(\mathbf{C}))$ with supremum equal zero are lower bounded by the function ψ defined above. The important corollary of Theorem 4.1 is established by our next theorem.

THEOREM 4.2. $\forall \alpha < \dfrac{1}{m}$, *we have the following inequality of the type Tian-Hormander* (*See* [H], [T]):

$$\int_{G_{m,nm}(\mathbf{C})} e^{-\alpha\varphi}\, dv \leqslant C \tag{4.4}$$

for every g-admissible and G-invariant function $\varphi \in C^\infty(G_{m,nm}(\mathbf{C}))$, satisfying $\sup \varphi = 0$ *on* $G_{m,nm}(\mathbf{C})$.

This theorem induces $\alpha_G(G_{m,nm}(\mathbf{C})) \geq \dfrac{1}{m}$. The remainder of this section will be devoted to the proof of these Theorems.

4.1. Proof of Theorem 4.1. The proof of Theorem 4.1 will be divided into a sequence of lemmas but let us first outline some properties of the function ψ.

PROPOSITION 4.3. *ψ is G-invariant and it reaches a supremum on $G_{m,nm}(\mathbf{C})$ equal to zero at any point in the unitry group $U_m(\mathbf{C})$. ψ and $\tilde{\psi}$ satisfy*:

$$\partial_{\lambda\bar{\mu}}\psi = \partial_{\lambda\bar{\mu}}\tilde{\psi} = -g \tag{4.5}$$

Proof. The proof falls into three steps.

STEP 1. The functions ψ and $\tilde{\psi}$ are G-invariant. Indeed, by the description $Z_j = U_j H_j = U_j P_j^{-1} D_j P_j$ (for $1 \leq j \leq n-1$) we get: $\det Z_j = \det D_j$. On the other hand, the logarithmic potential of the metric g, easily seen in the dominator of the function ψ, is invariant by the action of isometry i. It follows that $\tilde{\psi}$ is G-invariant as well as the function ψ.

STEP 2. It is easy to check that $\partial_{\lambda\bar{\mu}}\psi = \partial_{\lambda\bar{\mu}}\tilde{\psi} = -g$ since $\dfrac{\partial^2}{\partial z \partial \bar{z}}(\ln(|f(z)|^2)) = 0$ if f is a holomorphic function.

STEP 3. Under the properties of G-invariance satisfied by the function ψ, we are reduced to the case when M is equal to $\begin{pmatrix} Id_0 \\ D_1 \\ \vdots \\ D_{n-1} \end{pmatrix}$. Thus, ψ satisfy:

$$\psi(M) = \ln\left(\frac{\prod_{j=1}^{n-1} |\det D_j|^{2m}}{(1 + \sum_{I\setminus I_0} |D_I|^2)^{a_n}}\right) \tag{4.6}$$

$$= \ln\left(\frac{|(\lambda_1^1 \cdots \lambda_1^m) \cdots (\lambda_{n-1}^1 \cdots \lambda_{n-1}^m)|^{2m}}{(1 + \sum_{I\setminus I_0} |D_I|^2)^{a_n}}\right) \tag{4.7}$$

Let us show that for every point $M = \begin{pmatrix} Id_0 \\ D_1 \\ \vdots \\ D_{n-1} \end{pmatrix} \in G_{m,mn}(\mathbf{C})$, we have:

$$\psi(M) < \psi\begin{pmatrix} Id_0 \\ Id_1 \\ \vdots \\ Id_{n-1} \end{pmatrix}. \tag{4.8}$$

Let f be a function defined at every point $M = \begin{pmatrix} Id_0 \\ D_1 \\ \vdots \\ D_{n-1} \end{pmatrix}$ on the chart U_{I_0} of $G_{m,mn}(\mathbf{C})$ by:

$$f(M) = \frac{\prod_{j=1}^{n-1} |\det D_j|^{2m}}{(1 + \sum_{I\setminus I_0} |D_I|^2)^{a_n}}. \tag{4.9}$$

Observe that the function $K : (\mathbf{R}_+^n)^* \to \mathbf{R}_+^*$ defined by

$$K(x_0, \dots, x_{n-1}) = \frac{(x_0 \cdots x_{n-1})^{1/n}}{x_0 + \cdots + x_{n-1}} \tag{4.10}$$

reaches its supremum at the point $(1, \dots, 1)$. Hence, for all $M = \begin{pmatrix} Id_0 \\ D_1 \\ \vdots \\ D_{n-1} \end{pmatrix} \in G_{m,mn}(\mathbf{C})$ we have:

$$f(M) \leq \frac{\prod_{j=1}^{n-1} |\det D_j|^{2m}}{(1 + \sum_{j=1}^{n-1} |\det D_j|^2)^{a_n}} \tag{4.11}$$

The last function is nothing but the function K^{nm} (with $x_0 = 1$ and $x_j = |\det D_j|^2$ for $1 \leq j \leq n-1$). From this, we conclude that f reaches a supremum if $\det D_j = 1$ (for $1 \leq j \leq n-1$). Neverthless the point $\begin{pmatrix} Id_0 \\ Id_1 \\ \vdots \\ Id_{n-1} \end{pmatrix}$ satisfy this condition, hence the function f reaches a supremum at this point. This completes the proof.

Lemma 4.4. *Let $\varphi \in C^\infty(G_{m,mn}(\mathbf{C}))$ be a g-admissible and G-invariant function. We get*:

$$(\varphi - \psi)\begin{pmatrix} Id_0 \\ D_1 \\ \vdots \\ D_{n-1} \end{pmatrix} \geqslant (\varphi - \psi)\begin{pmatrix} Id_0 \\ (D_1 \cdots D_{n-1})^{1/(n-1)} \\ \vdots \\ (D_1 \cdots D_{n-1})^{1/(n-1)} \end{pmatrix} \tag{4.12}$$

Proof. The proof is by induction on p. Assume that the inequality

$$(\varphi - \psi)\begin{pmatrix} Id_0 \\ D_1 \\ \vdots \\ D_{n-1} \end{pmatrix} \geqslant (\varphi - \psi)\begin{pmatrix} Id_0 \\ (D_1 \cdots D_p)^{1/p} \\ \vdots \\ (D_1 \cdots D_p)^{1/p} \\ D_{p+1} \\ \vdots \\ D_{n-1} \end{pmatrix} \tag{4.13}$$

holds for p $(1 \leqslant p < n-1)$, we will prove it for $(p+1)$, ie:

$$(\varphi-\psi)\begin{pmatrix} Id_0 \\ D_1 \\ \vdots \\ D_{n-1} \end{pmatrix} \geqslant (\varphi-\psi)\begin{pmatrix} Id_0 \\ (D_1\cdots D_{p+1})^{1/(p+1)} \\ \vdots \\ (D_1\cdots D_{p+1})^{1/(p+1)} \\ D_{p+2} \\ \vdots \\ D_{n-1} \end{pmatrix} \tag{4.14}$$

Suppose that the above inequality is not satisfied for degree $(p+1)$, then there exists a point

$$M_0 = \begin{pmatrix} Id_0 \\ D_1^0 \\ \vdots \\ D_{n-1}^0 \end{pmatrix} \in G_{m,nm}(\mathbf{C}) \tag{4.15}$$

such that

$$(\varphi-\psi)\begin{pmatrix} Id_0 \\ D_1^0 \\ \vdots \\ D_{n-1}^0 \end{pmatrix} < (\varphi-\psi)\begin{pmatrix} Id_0 \\ (D_1^0\cdots D_{p+1}^0)^{1/(p+1)} \\ \vdots \\ (D_1^0\cdots D_{p+1}^0)^{1/(p+1)} \\ D_{p+2}^0 \\ \vdots \\ D_{n-1}^0 \end{pmatrix} \tag{4.16}$$

By G-invariance of φ, we can assume that $(\lambda_j^1)_{1\leqslant j\leqslant p+1}$ (relative to $D_1^0,\dots,D_{p+1}^0$) satisfy $\lambda_1^1 \leqslant \cdots \leqslant \lambda_{p+1}^1$. Furthermore, by induction hypothesis and again by G-invariance of φ, we obtain:

$$(\varphi-\psi)\begin{pmatrix} Id_0 \\ D_1^0 \\ \vdots \\ D_{p-1}^0 \\ D_p^0 \\ D_{p+1}^0 \\ \vdots \\ D_{n-1}^0 \end{pmatrix} \geq (\varphi-\psi)\begin{pmatrix} Id_0 \\ (D_1^0\cdots D_{p-1}^0 D_p^0)^{1/p} \\ \vdots \\ (D_1^0\cdots D_{p-1}^0 D_p^0)^{1/p} \\ D_{p+1}^0 \\ \vdots \\ D_{n-1}^0 \end{pmatrix} \tag{4.17}$$

and

$$(\varphi-\psi)\begin{pmatrix} Id_0 \\ D_1^0 \\ \vdots \\ D_{p-1}^0 \\ D_{p+1}^0 \\ D_p^0 \\ \vdots \\ D_{n-1}^0 \end{pmatrix} \geq (\varphi-\psi)\begin{pmatrix} Id_0 \\ (D_1^0\cdots D_{p-1}^0 D_{p+1}^0)^{1/p} \\ \vdots \\ (D_1^0\cdots D_{p-1}^0 D_{p+1}^0)^{1/p} \\ D_p^0 \\ \vdots \\ D_{n-1}^0 \end{pmatrix} \tag{4.18}$$

For $t\in[(\lambda_p^1)^{1/p},(\lambda_{p+1}^1)^{1/p}]$, consider the curve $c(t)$ defined by:

$$C(t)=\begin{pmatrix} \begin{pmatrix} \zeta_{[1]}^1(t) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \zeta_{[1]}^m(t) \end{pmatrix} \\ \vdots \\ \vdots \\ \begin{pmatrix} \zeta_{[p]}^1(t) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \zeta_{[p]}^m(t) \end{pmatrix} \\ \begin{pmatrix} \zeta_{[p+1]}^1(t) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \zeta_{[p+1]}^m(t) \end{pmatrix} \\ D_{p+2} \\ \vdots \\ D_{n-1} \end{pmatrix}$$

where

$$\begin{cases} (\zeta^1_{[j]}(t))_{1\leqslant j\leqslant p} = (\lambda^1_1\cdots\lambda^1_{p-1})^{1/p}t \\ (\zeta^i_{[j]}(t))_{1\leqslant j\leqslant p} = (\lambda^i_1\cdots\lambda^i_p)^{1/p}\left(\dfrac{t}{(\lambda^1_p)^{1/p}}\right)^{\{\ln((\lambda^i_{p+1})^{1/p}/(\lambda^i_p)^{1/p})/\ln((\lambda^1_{p+1})^{1/p}/(\lambda^1_p)^{1/p})\}}, \\ \qquad \forall 2\leqslant i\leqslant m \end{cases}$$

and

$$\begin{cases} \zeta^1_{[p+1]}(t) = \dfrac{\lambda^1_p\lambda^1_{p+1}}{t^p} \\ \zeta^i_{[p+1]}(t) = \lambda^i_{p+1}\left(\dfrac{t^p}{\lambda^1_p}\right)^{\{\ln((\lambda^i_p)^{1/p}/(\lambda^i_{p+1})^{1/p})/\ln((\lambda^1_{p+1})^{1/p}/(\lambda^1_p)^{1/p})\}}, \quad \forall 2\leqslant i\leqslant m \end{cases}$$

By the inequality (4.16), the matrices D^0_j (for $1\leqslant j\leqslant p+1$) can not be all equal. Since we have chosen λ^i_j to satisfy $\lambda^1_1\leqslant\cdots\leqslant\lambda^1_{p+1}$, it follows that the curve $c(t)$ passes at $t=(\lambda^1_p)^{1/p}$ through the point

$$P_1 = \begin{pmatrix} Id_0 \\ (D^0_1\cdots D^0_{p-1}D^0_p)^{1/p} \\ \vdots \\ (D^0_1\cdots D^0_{p-1}D^0_p)^{1/p} \\ D^0_{p+1} \\ \vdots \\ D^0_{n-1} \end{pmatrix}, \tag{4.19}$$

and at $t=(\lambda^1_{p+1})^{1/p}$ through the point

$$P_2 = \begin{pmatrix} Id_0 \\ (D^0_1\cdots D^0_{p-1}D^0_{p+1})^{1/(p+1)} \\ \vdots \\ (D^0_1\cdots D^0_{p-1}D^0_{p+1})^{1/(p+1)} \\ D^0_p \\ \vdots \\ D^0_{n-1} \end{pmatrix}. \tag{4.20}$$

It is easy to check that at $t_0 = ((\lambda_p^1 \lambda_{p+1}^1)/(\lambda_1^1 \cdots \lambda_{p-1}^1)^{1/p}) \in [(\lambda_p^1)^{1/p}, (\lambda_{p+1}^1)^{1/p}]$, the curve $c(t)$ passes through the point

$$P_3 = \begin{pmatrix} Id_0 \\ (D_1^0 \cdots D_{p+1}^0)^{1/(p+1)} \\ \vdots \\ (D_1^0 \cdots D_{p+1}^0)^{1/(p+1)} \\ D_{p+2}^0 \\ \vdots \\ D_{n-1}^0 \end{pmatrix}. \tag{4.21}$$

Consequently, from the inequalities (4.16), (4.17) and (4.18), we conclude that:

$$(\varphi - \psi)(P_3) > (\varphi - \psi)(P_1) \quad \text{and} \quad (\varphi - \psi)(P_3) > (\varphi - \psi)(P_2).$$

This shows that the function $(\varphi - \psi)$ has a local maximum on the curve $c(t)$. Applying the G-invariance argument of $(\varphi - \psi)$ again, we get:

$$(\varphi - \psi)(c(t)) = (\varphi - \psi)\begin{pmatrix} \begin{pmatrix} \zeta_{[1]}^1(t) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \zeta_{[1]}^m(t) \end{pmatrix} \\ \vdots \\ \vdots \\ \begin{pmatrix} \zeta_{[p]}^1(t) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \zeta_{[p]}^m(t) \end{pmatrix} \\ \begin{pmatrix} \zeta_{[p+1]}^1(t) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \zeta_{[p+1]}^m(t) \end{pmatrix} \\ D_{p+2} \\ \vdots \\ D_{n-1} \end{pmatrix}$$

$$= (\varphi - \psi)\begin{pmatrix} \begin{pmatrix} \zeta^1_{[1]}(te^{i\theta}) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \zeta^m_{[1]}(te^{i\theta}) \end{pmatrix} \\ \vdots \\ \vdots \\ \begin{pmatrix} \zeta^1_{[p]}(te^{i\theta}) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \zeta^m_{[p]}(te^{i\theta}) \end{pmatrix} \\ \begin{pmatrix} \zeta^1_{[p+1]}(te^{i\theta}) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \zeta^m_{[p+1]}(te^{i\theta}) \end{pmatrix} \\ D_{p+2} \\ \vdots \\ D_{n-1} \end{pmatrix}$$

$$= (\varphi - \psi)\begin{pmatrix} \begin{pmatrix} \zeta^1_{[1]}(z) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \zeta^m_{[1]}(z) \end{pmatrix} \\ \vdots \\ \vdots \\ \begin{pmatrix} \zeta^1_{[p]}(z) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \zeta^m_{[p]}(z) \end{pmatrix} \\ \begin{pmatrix} \zeta^1_{[p+1]}(z) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \zeta^m_{[p+1]}(z) \end{pmatrix} \\ D_{p+2} \\ \vdots \\ D_{n-1} \end{pmatrix}$$

$$= (\varphi - \psi)(c(z))$$

where $c(z)$ is the curve defined on the annulus $\{(\lambda^1_p)^{1/p} \leqslant |z| \leqslant (\lambda^1_{p+1})^{1/p}\}$.

Thus, the function $(\varphi - \psi)$ has a local maximum inside the annulus, it follows that its Hessian is negative in this points. Consequently,

$$\frac{\partial^2[(\varphi-\psi)(c(z))]}{\partial z \partial \bar{z}}(z_0) = \frac{\partial^2(\varphi-\psi)}{\partial z^i \partial \bar{z}^j}(c(z_0))\dot{c}^i(z_0)\dot{\bar{c}}^j(z_0) < 0. \tag{4.22}$$

This contradicts the admissibility of φ.

Lemma 4.5. *Let $\varphi \in C^\infty(G_{m,nm})$ be a g-admissible and G-invariant function. Then*:

$$(\varphi-\psi)\begin{pmatrix} Id_0 \\ (D_1\cdots D_{n-1})^{1/(n-1)} \\ \vdots \\ (D_1\cdots D_{n-1})^{1/(n-1)} \end{pmatrix} \geqslant (\varphi-\psi)\begin{pmatrix} Id_0 \\ Id_1 \\ \vdots \\ Id_{n-1} \end{pmatrix} \tag{4.23}$$

Proof. Set

$$A = (D_1\cdots D_{n-1})^{1/(n-1)} \tag{4.24}$$

$$= \begin{pmatrix} (\lambda_1^1\cdots\lambda_{n-1}^1)^{1/(n-1)} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & (\lambda_1^m\cdots\lambda_{n-1}^m)^{1/(n-1)} \end{pmatrix}$$

By G-invariance of the function $(\varphi - \psi)$, we can write:

$$(\varphi-\psi)\begin{pmatrix} Id_0 \\ A \\ A \\ \vdots \\ A \end{pmatrix} = (\varphi-\psi)\begin{pmatrix} A^{-1} \\ Id_1 \\ \vdots \\ Id_{n-1} \end{pmatrix} = (\varphi-\psi)\begin{pmatrix} Id_0 \\ A^{-1} \\ Id_2 \\ \vdots \\ Id_{n-1} \end{pmatrix} \tag{4.25}$$

By Lemma 4.4, we obtain:

$$(\varphi-\psi)\begin{pmatrix} Id_0 \\ A^{-1} \\ Id_2 \\ \vdots \\ Id_{n-1} \end{pmatrix} \geq (\varphi-\psi)\begin{pmatrix} Id_0 \\ (A^{-1}Id_2\cdots Id_{n-1})^{1/(n-1)} \\ \vdots \\ (A^{-1}Id_2\cdots Id_{n-1})^{1/(n-1)} \end{pmatrix} \tag{4.26}$$

$$\geq (\varphi-\psi)\begin{pmatrix} Id_0 \\ (A^{-1})^{1/(n-1)} \\ \vdots \\ (A^{-1})^{1/(n-1)} \end{pmatrix} \tag{4.27}$$

Repeating the previous process q more times ($q \in \mathbf{N}^*$), we deduce that:

$$(\varphi-\psi)\begin{pmatrix} Id_0 \\ A \\ \vdots \\ A \end{pmatrix} \geqslant (\varphi-\psi)\begin{pmatrix} Id_0 \\ A^{(-1)^q/(n-1)^q} \\ \vdots \\ A^{(-1)^q/(n-1)^q} \end{pmatrix} \tag{4.28}$$

$$= (\varphi-\psi)\begin{pmatrix} Id_0 \\ (D_1\cdots D_{n-1})^{(-1)^q/(n-1)^{q+1}} \\ \vdots \\ (D_1\cdots D_{n-1})^{(-1)^q/(n-1)^{q+1}} \end{pmatrix} \tag{4.29}$$

where

$$(D_1\cdots D_{n-1})^{(-1)^q/(n-1)^{q+1}} \tag{4.30}$$
$$= \begin{pmatrix} (\lambda_1^1\cdots\lambda_{n-1}^1)^{(-1)^q/(n-1)^{q+1}} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & (\lambda_1^m\cdots\lambda_{n-1}^m)^{(-1)^q/(n-1)^{q+1}} \end{pmatrix}$$

Set $x_1 = (\lambda_1^1\cdots\lambda_{n-1}^1)^{(-1)^q/(n-1)^{q+1}}, \ldots, x_m = (\lambda_1^m\cdots\lambda_{n-1}^m)^{(-1)^q/(n-1)^{q+1}}$. These sequences $(x_i)_{1\le i\le m}$ go to 1 when q goes to infinity. This proves the lemma.

4.2. Proof of Theorem 4.1. Let φ be a function satisfying assumptions of Theorem 4.1. In the chart

$$U_{I_0} = \{M \in G_{m,nm}(\mathbf{C}) \text{ such that } \det Z_{I_0} \neq 0\} \tag{4.31}$$

and at the point $M = \begin{pmatrix} Id_0 \\ Z_1 \\ \vdots \\ Z_{n-1} \end{pmatrix}$ satisfying $\det Z_j \neq 0$, for $1 \leqslant j \leqslant n-1$, Lemma 4.4 gives

$$(\varphi-\psi)\begin{pmatrix} Id_0 \\ Z_1 \\ \vdots \\ Z_{n-1} \end{pmatrix} \geqslant (\varphi-\psi)\begin{pmatrix} Id_0 \\ (D_1\cdots D_{n-1})^{1/(n-1)} \\ \vdots \\ (D_1\cdots D_{n-1})^{1/(n-1)} \end{pmatrix} \tag{4.32}$$

where D_j is the diagonal matrix constituted by the eigenvalues of the Hermitian matrix H_j given by the polar decomposition of the matrix Z_j (for $1 \leq j \leqslant n-1$). Applying Lemma 4.5, it follows that:

$$(\varphi - \psi)\begin{pmatrix} Id_0 \\ Z_1 \\ \vdots \\ Z_{n-1} \end{pmatrix} \geqslant (\varphi - \psi)\begin{pmatrix} Id_0 \\ Id_1 \\ \vdots \\ Id_{n-1} \end{pmatrix} \tag{4.33}$$

Consequently, $\varphi \geq \psi$ at every point M in the chart U_{I_0} satisfying $\det Z_j \neq 0$ (for $1 \leqslant j \leqslant n-1$). Concerning the point $M = \begin{pmatrix} Id_0 \\ Z_1 \\ \vdots \\ Z_{n-1} \end{pmatrix}$ with at least one matrix Z_j such that $\det Z_j = 0$, we get $\varphi \geqslant \psi$ since $\psi = -\infty$. This proves the Theorem 4.1.

4.3. Proof of Theorem 4.2: Tian's Invariant of the Grassmann $G_{m,nm}(\mathbf{C})$. Let $\varphi \in C^\infty(G_{m,nm}(\mathbf{C}))$ be a g-admissible and G-invariant function satisfying $\sup \varphi = 0$ on $G_{m,nm}(\mathbf{C})$. Theorem 4.1 yields $\varphi \geqslant \psi$ at every point $M \in G_{m,mn}(\mathbf{C})$. Thus, for all $\alpha \geqslant 0$ it follows that:

$$\int_{G_{m,nm}(\mathbf{C})} e^{-\alpha\varphi}\, dv \leqslant \int_{G_{m,nm}(\mathbf{C})} e^{-\alpha\psi}\, dv \tag{4.34}$$

Let us evaluate the last integral in the map U_{I_0} defined by $U_{I_0} = \{\det Z_0 \neq 0\}$. In this map, the volume element related to the metric g defined on $G_{m,nm}(\mathbf{C})$ is given by (See [G]):

$$dv = b_n \left(1 + \sum_{I \subset \{1,\ldots,nm\}\setminus I_0} |\det Z_I|^2\right)^{-a_n} dz_J \wedge d\bar{z}_J \tag{4.35}$$

where $a_n = nm$, $b_n = \left(\frac{i}{2}\right)^{m^2(n-1)}$ and $J = \{1, \ldots, (n-1)m^2\}$. Hence, we obtain:

$$\int_{G_{m,nm}(\mathbf{C})} e^{-\alpha\psi}\, dv \tag{4.36}$$

$$= b_n \int_{\mathbf{C}^{(n-1)m^2}} \left(\frac{\prod_{j=1}^{j=n-1} |\det Z_j|^{2m}}{(1 + \sum_{I \subset \{1,\ldots,nm\}\setminus I_0} |\det Z_I|^2)^{a_n}}\right)^{-\alpha} \times \left(1 + \sum_{I \subset \{1,\ldots,nm\}\setminus I_0} |\det Z_I|^2\right)^{-a_n} dz_J \wedge d\bar{z}_J \tag{4.37}$$

$$= b_n \int_{\mathbf{C}^{(n-1)m^2}} \frac{\prod_{j=1}^{j=n-1} |\det Z_j|^{-2m\alpha}}{(1+\sum_{I\subset\{1,\dots,nm\}\setminus I_0} |\det Z_I|^2)^{a_n(1-\alpha)}} dz_J \wedge d\bar{z}_J \tag{4.38}$$

$$= b_n \int_0^{+\infty} \cdots \int_0^{+\infty} \frac{(1+\sum_{I\subset\{1,\dots,nm\}\setminus I_0} \det D'_I)^{a_n(\alpha-1)}}{\prod_{j=1}^{j=n-1} (\det D'_j)^{m\alpha}} \tag{4.39}$$

$$\times (du_1^1 \cdots du_1^m) \cdots (du_{n-1}^1 \cdots du_{n-1}^m)$$

where $(D'_j)_{1\le j\le n-1} = \begin{pmatrix} u_j^1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & u_j^m \end{pmatrix}$ and $(u_j^i)_{1\le i\le m} = ((\lambda_j^i)^2)_{1\le i\le m}$.

This integral converges for $\alpha < \frac{1}{m}$, and the proof is complete.

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Adnène Ben Abdesselem
University of Paris 6
U.F.R.929, 4, place Jussieu
75005 Paris
France

Current Address
Institute of Mathematics of Jussieu
E-mail: adnene.ben-abdesselem@imj-prg.fr

Ines Adouani
University of Sousse
School of Science and Technology Hamam Sousse
Rue Lamine Abassi 4011 H. Sousse

Current Address
Laboratory of Physical-Mathematical
Special Functions and Applications
E-mail: ines.adouani@imj-prg.fr