H. FUJINO
KODAI MATH. J.
38 (2015), 732–746

THE EXISTENCE OF QUASICONFORMAL HOMEOMORPHISM BETWEEN PLANES WITH COUNTABLE MARKED POINTS

Hiroki Fujino

Abstract

We consider quasiconformal deformations of $\mathbf{C}\backslash\mathbf{Z}$. We give some criteria for infinitely often punctured planes to be quasiconformally equivalent to $\mathbf{C}\backslash\mathbf{Z}$. In particular, we characterize the closed subsets of $\mathbf{R}$ whose compliments are quasiconformally equivalent to $\mathbf{C}\backslash\mathbf{Z}$.

1. Introduction

Let R be a Riemann surface. The Teichmüller space $T(R)$ is a space which describes all quasiconformal deformations of R. It is well known that $T(R)$ becomes either a finite dimensional complex manifold or a non-separable infinite dimensional Banach analytic manifold. $T(R)$ becomes finite dimensional if and only if R is of finite type. Through the investigation of quasiconformal deformations of a certain infinite type Riemann surface, a certain characteristic subspace will be found, which is separable.

The universal Teichmüller space $T(\mathbf{D})$ simultaneously describes all quasiconformal deformations of all hyperbolic type Riemann surfaces. This arises from the fact that each covering $X \to Y$ induces an embedding of $T(Y)$ into $T(X)$. On the other hand, $\mathbf{C}\backslash\mathbf{Z}$ covers a certain n-punctured Riemann sphere for each $n \geq 3$. Namely $T(\mathbf{C}\backslash\mathbf{Z})$ simultaneously describes all quasiconformal deformations of Riemann surfaces of genus 0 with at least three punctures. Needless to say, the universal Teichmüller space $T(\mathbf{D})$ also describes them. However for the reasons mentioned below, the Teichmüller space $T(\mathbf{C}\backslash\mathbf{Z})$ is more suitable to describe them than $T(\mathbf{D})$.

For each positive integer n, let $R_n = (\mathbf{C}\backslash\mathbf{Z})/\langle z+n\rangle$. R_n is an $(n+2)$-punctured Riemann sphere, and the projection $p_n : \mathbf{C}\backslash\mathbf{Z} \to R_n$ induces the embedding $p_n^* : T(R_n) \hookrightarrow T(\mathbf{C}\backslash\mathbf{Z})$. The covering transformation group of p_n is the cyclic group $\langle z+n\rangle$, so that, quasiconformal deformations of R_n correspond to periodic quasiconformal deformations of $\mathbf{C}\backslash\mathbf{Z}$ with only a period n. Then it is shown from McMullen's theorem in [4] that p_n^* is totally geodesic for the

2010 *Mathematics Subject Classification*. Primary 51M04, Secondary 51M05.
Key words and phrases. Quasiconformal mappings, extremal distance, and quasicircle.
Received January 6, 2015; revised May 29, 2015.

Teichmüller metric. By contrast, the embedding of $T(R_n)$ into $T(\mathbf{D})$ is not totally geodesic (cf. The Kra–McMullen theorem [4]). Additionally, $\mathbf{C}\backslash\mathbf{Z}$ is considered to be one of the smallest Riemann surface which has the above properties, that is, there exists no Riemann surface except $\mathbf{C}\backslash\mathbf{Z}$ which is covered by $\mathbf{C}\backslash\mathbf{Z}$ and covers R_n for all n.

Thus, in this paper, we would like to investigate quasiconformal deformations of $\mathbf{C}\backslash\mathbf{Z}$. In particular, we shall try to find all Riemann surfaces which are quasiconformally equivalent to $\mathbf{C}\backslash\mathbf{Z}$.

This attempt is reduced to the existence problem of quasiconformal homeomorphism between planes with countable marked points. In fact, if R is quasiconformally equivalent to $\mathbf{C}\backslash\mathbf{Z}$, then R is conformally equivalent to $\mathbf{C}\backslash E$ by a certain closed discrete subset $E \subset \mathbf{C}$ (cf. The removable singularity theorem, see [5, Theorem 17.3.]). Henceforth, we say that two subsets $E, E' \subset \mathbf{C}$ are quasiconformally equivalent if there exists a quasiconformal self-homeomorphism of $\mathbf{C}$ which maps E onto E'. We consider the following problem.

Problem. Let $\mathscr{P}$ be the family of all closed discrete infinite subsets $E \subset \mathbf{C}$. Find all $E \in \mathscr{P}$ which is quasiconformally equivalent to $\mathbf{Z}$.

This Problem is analogous to the problem investigated by P. MacManus. In his paper [3], he considered the *usual Cantor–middle–third set* (the *Cantor ternary set*)

$$\mathscr{C} = [0,1] \Big\backslash \bigcup_{m=1}^{\infty} \bigcup_{k=0}^{3^{m-1}-1} \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m} \right)$$

instead of $\mathbf{Z}$. Further he completely characterized subsets which is quasiconformally equivalent to $\mathscr{C}$ by several conditions of the Euclidean geometry.

First, when we take the MacManus proof into consideration, it seems significant to solve our Problem for $E \in \mathscr{P}$ contained in the real line. In this particular case, we obtain the next theorem.

Theorem A. *For a monotone increasing sequence* $E = \{a_n\}_{n \in \mathbf{Z}} \subset \mathbf{R}$ *with* $a_n \to \pm\infty$ *as* $n \to \pm\infty$, *the following conditions are equivalent.*

i. E *is quasiconformally equivalent to* $\mathbf{Z}$.
ii. *There exists a quasiconformal homeomorphism of* $\mathbf{C}$ *such that* $f(n) = a_n$ *for all* $n \in \mathbf{Z}$.
iii. *There exists* $M \geq 1$ *such that the following inequality holds for all* $n \in \mathbf{Z}$, $k \in \mathbf{N}$;

$$\frac{1}{M} \leq \frac{a_{n+k} - a_n}{a_n - a_{n-k}} \leq M.$$

The last condition derives from the concept of M-quasisymmetry. Theorem 3.1 (proved in Section 3.1) shows that if $E \in \mathscr{P}$ lying on the real line is quasiconformally equivalent to $\mathbf{Z}$, then E can not be bounded from above and below. Therefore the assumption in Theorem A is necessarily required. Further Theorem A completely characterizes the subsets of $\mathbf{R}$ which are quasiconformally equivalent to $\mathbf{Z}$.

Next, we observe $E \in \mathscr{P}$ whose compliment has an automorphism of infinite order. In this case, we obtain the next theorem.

Theorem B. *Let $E \in \mathscr{P}$ which has the following form;*

$$E = \mathbf{Z} + \{a_n\}_{n=1}^{k}$$

where each a_n satisfies $\mathrm{Re}(a_n) \in [0, 1)$.

Then, E is quasiconformally equivalent to $\mathbf{Z}$ if and only if $k < +\infty$.

The assumption for E in Theorem B means that $\mathbf{C} \backslash E$ has an automorphism of infinite order $z+1$. On the other hand, if $\mathbf{C} \backslash E$ has an automorphism of infinite order, we may assume E satisfies the assumption in Theorem B by composing certain Affine transformation. Thus we immediately obtain the following application;

Let $T_0 = \bigcup_{n \in \mathbf{N}} p_n^*(T(R_n))$, namely T_0 is a subspace of $T(\mathbf{C} \backslash \mathbf{Z})$ which simultaneously describes all quasiconformal deformations of all Riemann surfaces of finite type $(0, n)$ with $n \geq 3$. Further, let T_∞ be the set of all $[S, f] \in T(\mathbf{C} \backslash \mathbf{Z})$, the Teichmüller equivalence class of the quasiconformal homeomorphism $f : \mathbf{C} \backslash \mathbf{Z} \to S$, such that there exists an automorphism of infinite order in $\mathrm{Aut}(S)$. Then Theorem B implies

Corollary C.

$$T_\infty = \bigcup_{[f] \in \mathrm{Mod}(\mathbf{C} \backslash \mathbf{Z})} [f]_*(T_0).$$

Here, $\mathrm{Mod}(\mathbf{C} \backslash \mathbf{Z})$ is the Teichmüller-Modular group of $\mathbf{C} \backslash \mathbf{Z}$. The subspace T_0 is not closed in $T(\mathbf{C} \backslash \mathbf{Z})$. However, it is easily seen that T_0 is separable by its construction. Further from the McMullen theorem, T_0 is geodesically convex with respect to the Teichmüller metric.

In addition, $\mathrm{Aut}(S)$ is isomorphic to the stabilizer $\mathrm{Stab}_{\mathrm{Mod}(\mathbf{C} \backslash \mathbf{Z})}([S, f])$ for $[S, f] \in T(\mathbf{C} \backslash \mathbf{Z})$. Therefore, the Teichmüller-Modular group $\mathrm{Mod}(\mathbf{C} \backslash \mathbf{Z})$ does not act properly discontinuously at each point of $\overline{T_\infty}$, the closure of T_∞.

Note that we can apply the above argument to the another infinite type Riemann surface $R' = \mathbf{C}^* \backslash \{2^n\}_{n \in \mathbf{Z}}$, where $\mathbf{C}^* = \mathbf{C} \backslash \{0\}$. We will discuss this case in detail in Section 4.2.

2. Preliminaries

2.1. Porous sets. We say that a subset $E \subset \mathbf{C}$ is c-porous in $\mathbf{C}$ for a constant $c \geq 1$ if any closed disk $\bar{B}_r(z')$ of radius $r > 0$ centered at $z' \in \mathbf{C}$ contains z such that $B_{r/c}(z) \subset \mathbf{C} \backslash E$.

It is easily seen that;

- $\mathbf{Z} + i\mathbf{Z}$ is not porous in $\mathbf{C}$.
- Any subset of $\mathbf{R}$ is 1-porous in $\mathbf{C}$, particularly, $\mathbf{Z}$ is 1-porous in $\mathbf{C}$.
- $E_1 = \mathbf{Z} + i\{2^n \mid n = 0, 1, 2, \dots\}$ is 8-porous.

J. Väisälä pointed out that the porosity is preserved by quasiconformal mappings in [6]. Thus it immediately follows that $\mathbf{Z} + i\mathbf{Z}$ is not quasiconformally equivalent to $\mathbf{Z}$. However, by this way, we cannot decide whether E_1 is quasiconformally equivalent to $\mathbf{Z}$ or not. (Theorem B proved in Section 4.1 shows that E_1 is not quasiconformally equivalent to $\mathbf{Z}$.)

2.2. Quasiconformal mappings and Extremal distances. Let $D \subset \mathbf{C}$ be a domain. For given continua $C_1, C_2 \subset D$,

$$\delta^D(C_1, C_2) = \operatorname{mod}(\mathscr{F}^D(C_1, C_2))$$

is called the extremal distance between C_1 and C_2 in D, where mod denotes the 2-modulus of a curve family and $\mathscr{F}^D(C_1, C_2)$ denotes the family of all rectifiable curves which join C_1 and C_2 in D. The definition of 2-modulus is given by

$$\operatorname{mod}(\mathscr{F}) := \inf_{\rho} \int_{\mathbf{C}} \rho(x + iy)^2 \, dxdy.$$

where the infimum is taken over all non-negative Borel functions with $\int_\gamma \rho |dz| \geq 1$ for all rectifiable $\gamma \in \mathscr{F}$.

The 2-modulus coincides with the reciprocal of the extremal length introduced by L. V. Ahlfors and A. Beurling [2]. It is well known that a sense preserving homeomorphism f becomes K-quasiconformal for a constant $K \geq 1$ if and only if f satisfies the following inequality for any curve family $\mathscr{F}$ in the domain of f.

$$\frac{1}{K} \operatorname{mod}(\mathscr{F}) \leq \operatorname{mod}(f(\mathscr{F})) \leq K \operatorname{mod}(\mathscr{F}).$$

The next useful lower bound for extremal distances was presented by M. Vuorinen in [7, Lemma 4.7]; For each pair of disjoint continua $C_1, C_2 \subset \mathbf{C}$, it holds that

$$\delta^{\mathbf{C}}(C_1, C_2) \geq \frac{2}{\pi} \log\left(1 + \frac{\min_{i=1,2} \operatorname{diam}(C_i)}{\operatorname{dist}(C_1, C_2)}\right).$$

2.3. The Ahlfors three-point condition. The image of $\dot{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ under a quasiconformal self-homeomorphism of the Riemann sphere is called a quasicircle.

A characterization of quasicircles was obtained in [1]; for a Jordan curve C in the Riemann sphere which passes through ∞, C is a quasicircle if and only if there exists $A \geq 1$ such that whenever three distinct points $z_1, z_2, z_3 \in C \backslash \{\infty\}$ lie on C in this order, the following inequality holds.

$$\frac{|z_1 - z_2|}{|z_1 - z_3|} \leq A.$$

Moreover the above $A \geq 1$ can be constracted depending only on the maximal dilatation of quasiconformal homeomorphism.

This necessary and sufficient condition is called the *three-point condition* (or the *bounded turning condition*). The necessity of the three-point condition means that if a quasicircle goes far away from a certain point, it cannot return near this point above a certain rate. A similar characterization also holds for a Jordan curve which does not pass through ∞, however we will only deal with the former case.

2.4. The Ahlfors–Beurling extension theorem. An orientation preserving self-homeomorphism ϕ of $\mathbf{R}$ is called M-quasisymmetric for $M \geq 1$ if the following inequality holds for all $x \in \mathbf{R}$ and all $t > 0$.

$$\frac{1}{M} \leq \frac{\phi(x+t) - \phi(x)}{\phi(x) - \phi(x-t)} \leq M.$$

We merely say ϕ is quasisymmetric if ϕ is M-quasisymmetric for some $M \geq 1$.

The concept of M-quasisymmetry gives a characterization of orientation preserving self-homeomorphisms of real line which have a global quasiconformal extension. That is; for a given orientation preserving self-homeomorphism ϕ of $\mathbf{R}$, ϕ can be extended to a quasiconformal homeomorphism from the upper half plane onto itself if and only if ϕ is quasisymmetric (cf. The Ahlfors–Beurling extension theorem [1]). Moreover it is well known that every quasiconformal self-homeomorphism of the upper half plane is the restriction of a global quasiconformal homeomorphism. Namely, ϕ can be extended to a quasiconformal self-homeomorphism of the Riemann sphere, if and only if ϕ is quasisymmetric.

3. Proof of Theorem A

In this section, we would like to restrict ourselves to $E \in \mathscr{P}$ which lies on $\mathbf{R}$.

3.1. A criterion. We obtain the next criterion for $E \in \mathscr{P}$ contained in $\mathbf{R}$ to be quasiconformally equivalent to $\mathbf{Z}$.

THEOREM 3.1. *Let $E \in \mathscr{P}$ be contained in $\mathbf{R}$. If E is quasiconformally equivalent to $\mathbf{Z}$, then* $\sup E = +\infty$ *and* $\inf E = -\infty$.

Proof. To obtain a contradiction, assume $\inf E > -\infty$. Then since E is discrete and closed, $\sup E = +\infty$. Thus numbering E suitably we let $E = \{a_n\}_{n \in \mathbf{N}}$ be a monotone increasing sequence with $a_n \to +\infty$ as $n \to +\infty$.

Let $f : \mathbf{C} \to \mathbf{C}$ be K-quasiconformal mapping with $f(E) = \mathbf{Z}$. Composing an Affine transformation, we may assume $f(a_1) = 0$. For an arbitrary fixed constant $M \geq 1$, consider the set

$$S := \left\{ k \in \mathbf{N} \mid f(a_k) = \max_{j=1,\dots,k} f(a_j) \geq M \right\}.$$

Obviously, S consists of infinitely many elements. We number $S = \{k_j\}_{j \in \mathbf{N}}$ in ascending order. Then the sequence $\{f(a_{k_j})\}_{j \in \mathbf{N}}$ is monotone increasing. On the other hand, there exist infinitely many $n \in \mathbf{N}$ with $f(a_n) < 0$. Thus we can find $j, \ell \in \mathbf{N}$ such that $k_j < \ell < k_{j+1}$ and $f(a_\ell) < 0$. Moreover since $f(a_n) \leq f(a_{k_j})$ for all $n = 1, 2, \dots, k_{j+1} - 1$, if $f(a_m) = f(a_{k_j}) + 1$ then $m \geq k_{j+1}$. Consequently we confirmed that there exists $k \in S$ and exist $\ell, m \in \mathbf{N}$ such that

- $k < \ell < m$,
- $f(a_\ell) < 0$ and $f(a_m) = f(a_k) + 1$.

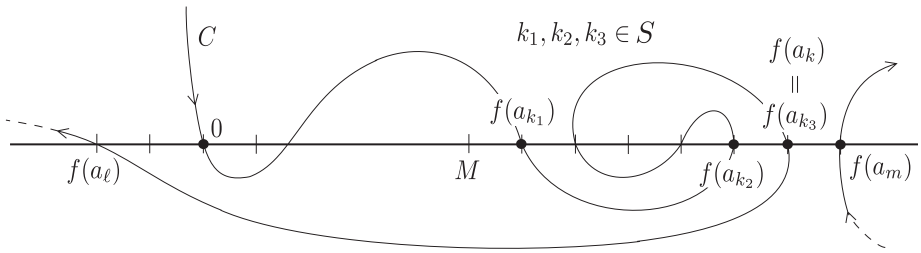

FIGURE 1

Then $f(a_k)$, $f(a_\ell)$, $f(a_m)$ lie on $C := f(\mathbf{R})$ in this order, and

$$\frac{|f(a_k) - f(a_\ell)|}{|f(a_k) - f(a_m)|} = f(a_k) - f(a_\ell) \geq M.$$

This means C can not satisfy the three-point condition for any $M \geq 1$. However this contradicts that C is a subarc of a quasicircle which passes through ∞. $\square$

This result extremely depends on the particularity of $\mathbf{Z}$.

Example. For arbitrary $r, s > 1$, $\{r^n\}_{n=0}^\infty$ is quasiconformally equivalent to $\{\pm s^n\}_{n=0}^\infty$.

3.2. A characterization. From Theorem 3.1, for our Problem it suffices to consider monotone increasing sequences $E = \{a_n\}_{n \in \mathbf{Z}} \subset \mathbf{R}$ with $a_n \to \pm\infty$ as $n \to \pm\infty$. We shall prove the next characterization theorem in this section.

Theorem A. *For a monotone increasing sequence* $E = \{a_n\}_{n \in \mathbf{Z}} \subset \mathbf{R}$ *with* $a_n \to \pm\infty$ *as* $n \to \pm\infty$, *the following conditions are equivalent.*

i. *E is quasiconformally equivalent to* $\mathbf{Z}$.
ii. *There exists a quasiconformal homeomorphism of* $\mathbf{C}$ *such that* $f(n) = a_n$ *for all* $n \in \mathbf{Z}$.
iii. *There exists* $M \geq 1$ *such that the following inequality holds for all* $n \in \mathbf{Z}$, $k \in \mathbf{N}$;

$$\frac{1}{M} \leq \frac{a_{n+k} - a_n}{a_n - a_{n-k}} \leq M.$$

That (ii) implies (i) is trivial. We prove the other implications below.

Proof ((iii) $\Rightarrow$ (ii)). Set $\phi(x) := (a_{n+1} - a_n)(x - n) + a_n$ for $x \in [n, n+1)$. Then ϕ defines an orientation preserving self-homeomorphism of $\mathbf{R}$ with $\phi(n) = a_n$, further becomes $C(M)$-quasisymmetric where $C(M) = M^4 + M^3 + M^2 + M$. Therefore we obtain a quasicnoformal extension $f : \mathbf{C} \to \mathbf{C}$ of ϕ by the Ahlfors–Beurling extension theorem.

Let $x = n + t_1$, $t = m + t_2$ $(n \in \mathbf{Z},\ m \in \mathbf{Z}_{\geq 0},\ t_1, t_2 \in [0,1))$. To prove the quasisymmetry of ϕ, we have to show the following inequality,

$$\frac{1}{C(M)} \leq I := \frac{\phi(x+t) - \phi(x)}{\phi(x) - \phi(x-t)} \leq C(M).$$

We divide the calculations into the following four cases.

i. $t_1 + t_2 \in [0,1)$ and $t_1 - t_2 \in (-1, 0)$.
ii. $t_1 + t_2 \in [0,1)$ and $t_1 - t_2 \in [0, 1)$.
iii. $t_1 + t_2 \in [1,2)$ and $t_1 - t_2 \in (-1, 0)$.
iv. $t_1 + t_2 \in [1,2)$ and $t_1 - t_2 \in [0, 1)$.

However we only check the first case here as the calculations are almost the same and easy for each case. To simplify the calculation, we use the next inequality.

Lemma 3.2. *Under the above assumptions, the following inequalities hold.*

I) *For* $n, m \in \mathbf{Z}$ $(n < m)$, *and* $k \in \mathbf{Z}_{\geq 0}$,

$$\frac{a_{m+k} - a_n}{a_m - a_n} \leq M^k + M^{k-1} + \cdots + M + 1,$$
$$\frac{a_m - a_{n-k}}{a_m - a_n} \leq M^k + M^{k-1} + \cdots + M + 1.$$

II) *For* $p, q \in \mathbf{R}$ *and* $k \in \mathbf{Z}$, *if* $k - 1 \leq p \leq k \leq q \leq k+1$ *then*

$$\frac{1}{M}(a_{k+1} - a_k)(q - p) \leq \phi(q) - \phi(p) \leq M(a_{k+1} - a_k)(q - p).$$

If we replace $(a_{k+1} - a_k)$ *by* $(a_k - a_{k-1})$, *this inequality also holds.*

Proof.

$$\text{I)}\quad \frac{a_{m+k}-a_n}{a_m-a_n} \le \frac{a_{m+k}-a_{m-1}}{a_m-a_{m-1}}$$
$$= \sum_{j=0}^{k} \frac{a_{m+j}-a_{m+j-1}}{a_m-a_{m-1}} \le M^k + M^{k-1} + \cdots + M + 1.$$

Another inequality is also proved in the same way.

$$\text{II)}\quad \phi(q)-\phi(p) = (a_{k+1}-a_k)(q-k)+a_k-(a_k-a_{k-1})(p-k+1)-a_{k-1}$$
$$= (a_{k+1}-a_k)(q-k)+(a_k-a_{k-1})(k-p)\cdots(*)$$

$$(*) \le (a_{k+1}-a_k)(q-k)+M(a_{k+1}-a_k)(k-p)$$
$$= M(a_{k+1}-a_k)(q-p)+(1-M)(a_{k+1}-a_k)(q-k) \le M(a_{k+1}-a_k)(q-p)$$
$$(*) \le M(a_k-a_{k-1})(q-k)+(a_k-a_{k-1})(k-p)$$
$$= M(a_k-a_{k-1})(q-p)+(1-M)(a_k-a_{k-1})(q-k) \le M(a_k-a_{k-1})(q-p)$$

The lower bounds are also proved in the same way. □

Continuation of Proof of Theorem A. Suppose $t_1+t_2 \in [0,1)$ and $t_1-t_2 \in (-1,0)$.

(*Upper bound*). First if $m \neq 0$, since ϕ is monotone increasing,

$$I \le \frac{\phi(n+m+1)-\phi(n)}{\phi(n)-\phi(n-m)}$$
$$= \frac{a_{n+m+1}-a_n}{a_n-a_{n-m}} \le M\frac{a_{n+m+1}-a_n}{a_{n+m}-a_n} \le M(M+1) < C(M).$$

Next if $m=0$, since $n-1 \le n+t_1+t_2 \le n \le n+t_1 \le n+1$,

$$I \le \frac{(a_{n+1}-a_n)(t_1+t_2)-(a_{n+1}-a_n)t_1}{\frac{1}{M}(a_{n+1}-a_n)t_2} = M < C(M).$$

(*Lower bound*). First if $m \neq 0,1$, by the monotonicity of ϕ

$$I > \frac{\phi(n+m+1)-\phi(n)}{\phi(n)-\phi(n-m)}$$
$$\ge \frac{1}{M}\frac{a_{n+m}-a_{n+1}}{a_{n+m+3}-a_{n+1}} \ge \frac{1}{M(M^3+M^2+M+1)} = \frac{1}{C(M)}.$$

Next if $m=0$, by the same reason of the case of upper bound,

$$I \ge \frac{(a_{n+1}-a_n)(t_1+t_2)-(a_{n+1}-a_n)t_1}{M(a_{n+1}-a_n)t_2} = \frac{1}{M} > \frac{1}{C(M)}.$$

Finally if $m = 1$, since $n \le n + t_1 \le n + 1 \le n + t_1 + 1 \le n + 2$ we have $\phi(x+t) - \phi(x) \ge \phi(x+1) - \phi(x) \ge \frac{1}{M}(a_{n+1} - a_n)$. On the other hand, $\phi(x) - \phi(x-t) \le \phi(n+1) - \phi(n-2) = a_{n+1} - a_{n-2}$. Thus we have

$$I \ge \frac{1}{M} \frac{a_{n+1} - a_n}{a_{n+1} - a_{n-2}} \ge \frac{1}{M(M^2 + M + 1)} \ge \frac{1}{C(M)}.$$

Proof ((i) $\Rightarrow$ (iii)). Let $f : \mathbf{C} \to \mathbf{C}$ be K-quasiconformal homeomorphism such that $f(E) = \mathbf{Z}$ $(K \ge 1)$.

In this proof, we shall use the following proposition.

Proposition 3.3. *For a monotone increasing sequence $E = \{a_n\}_{n \in \mathbf{Z}} \subset \mathbf{R}$ such that $a_n \to \pm\infty$ as $n \to \pm\infty$ and for any K-quasiconformal homeomorphism $f : \mathbf{C} \to \mathbf{C}$ which maps E onto $\mathbf{Z}$, there exists $L \ge 1$ depending only on K such that the following inequality holds for all $n \in \mathbf{Z}$ and $k \in \mathbf{N}$;*

$$\frac{1}{L} \le \frac{|f(a_{n+k}) - f(a_n)|}{|f(a_n) - f(a_{n-k})|} \le L.$$

Proposition 3.3 is proved in the next section.

For arbitrary fixed $n \in \mathbf{Z}$ and $k \in \mathbf{N}$, we set $r = |a_{n+k} - a_n|/|a_n - a_{n-k}|$, $r' = |f(a_{n+k}) - f(a_n)|/|f(a_n) - f(a_{n-k})|$ and $S_1 = S^1(a_n, |a_{n+k} - a_n|)$, $S_2 = S^1(a_n, |a_n - a_{n-k}|)$ where $S^1(x, R)$ denotes the circle of radius R centered at x.

If $r > 1$, then by using the Vuorinen theorem, we have

$$\begin{aligned}
\frac{2\pi}{\log r} = \delta^{\mathbf{C}}(S_1, S_2) &\ge \frac{1}{K}\delta^{\mathbf{C}}(f(S_1), f(S_2)) \\
&\ge \frac{1}{K}\frac{2}{\pi}\log\left(1 + \frac{\min_{i=1,2} \operatorname{diam} f(S_i)}{\operatorname{dist}(f(S_1), f(S_2))}\right) \\
&\ge \frac{2}{\pi K}\log\left(1 + \frac{|f(a_n) - f(a_{n-k})|}{|f(a_{n+k}) - f(a_{n-k})|}\right) \ge \frac{2}{\pi K}\log\left(1 + \frac{1}{r' + 1}\right).
\end{aligned}$$

If $r < 1$, we similarly have

$$\frac{2\pi}{\log 1/r} \ge \frac{2}{\pi K}\log\left(1 + \frac{1}{1/r' + 1}\right).$$

From Proposition 3.3, there exists $L \ge 1$ such that $1/L \le r' \le L$ where L does not depend on $n \in \mathbf{Z}$ and $k \in \mathbf{N}$. Combining the above inequalities, we obtain

$$\left(\exp\left(\frac{\pi^2 K}{\log\left(1 + \frac{1}{L+1}\right)}\right)\right)^{-1} \le \frac{|a_{n+k} - a_n|}{|a_n - a_{n-k}|} \le \exp\left(\frac{\pi^2 K}{\log\left(1 + \frac{1}{L+1}\right)}\right).$$

The proof is completed. Then remark that we can choose M depending only on K since L depends only on K. □

3.3. Proof of Proposition 3.3. We shall prove Proposition 3.3 from now on.

Under the assumptions of Proposition 3.3, we let $C = f(\mathbf{R})$. Recall that C is a subarc of a quasicircle which passes through ∞. Thus there exists a constant $A \geq 1$ depending only on K such that if arbitrary distinct three points z_1, z_2, z_3 lie on C in this order, it holds;

$$\frac{|z_1 - z_2|}{|z_1 - z_3|} \leq A.$$

Lemma 3.4. *For arbitrary $n \in \mathbf{Z}$, it holds that $|f(a_n) - f(a_{n+1})| \leq 2A$.*

Proof. Suppose $|f(a_n) - f(a_{n+1})| \geq 2$. Then we may assume $f(a_{n+1}) > f(a_n)$ since the same argument mentioned below can be applied to the case $f(a_n) > f(a_{n+1})$.

Letting $m \in \mathbf{Z}_{\leq n}$ satisfy

$$f(a_m) = \max\{f(a_j) \mid j \in \mathbf{Z}_{\leq n} \text{ s.t. } f(a_n) \leq f(a_j) < f(a_{n+1})\}$$

and $\ell \in \mathbf{Z}$ satisfy $f(a_\ell) = f(a_m) + 1$ (then $\ell \geq n+1$ by the constraction), we can constract $m, \ell \in \mathbf{Z}$ which satisfy the following conditions;

i. $m \leq n$ and $n+1 \leq \ell$,

ii. $f(a_n) \leq f(a_m) < f(a_{n+1})$ and $f(a_n) < f(a_\ell) \leq f(a_{n+1})$,

iii. $|f(a_m) - f(a_\ell)| = 1$. (See, Figure 2.)

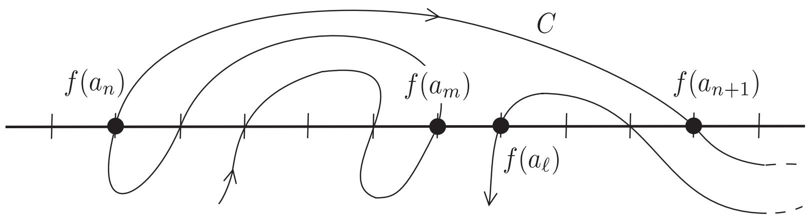

Figure 2

First, suppose $f(a_m) - f(a_n) \geq (f(a_{n+1}) - f(a_n))/2$. Then $f(a_m)$, $f(a_n)$, $f(a_\ell)$ are distinct since $f(a_m) - f(a_n) \geq 1$, and lie on C in this order. From the three-point condition,

$$\begin{aligned} A &\geq \frac{|f(a_m) - f(a_n)|}{|f(a_m) - f(a_\ell)|} \\ &= f(a_m) - f(a_n) \geq \frac{f(a_{n+1}) - f(a_n)}{2}. \end{aligned}$$

Thus we have $f(a_{n+1}) - f(a_n) \leq 2A$.

Next, suppose $f(a_m) - f(a_n) < (f(a_{n+1}) - f(a_n))/2$. Then $f(a_{n+1}) - f(a_m) > (f(a_{n+1}) - f(a_n))/2$ holds. Since $f(a_{n+1}) - f(a_n) > 2$,

$$f(a_{n+1}) - f(a_\ell) > \frac{f(a_{n+1}) - f(a_n)}{2} - 1 \geq 0,$$

that is, $\ell \neq n+1$. Therefore $f(a_m)$, $f(a_{n+1})$, $f(a_\ell)$ are distinct and are in this order on C. Similarly we have $f(a_{n+1}) - f(a_n) \leq 2A$. □

Lemma 3.5. *For arbitrary* $n \in \mathbf{Z}$ *and* $k \in \mathbf{N}$ $(k \neq 1)$, *it follows*;

$$\frac{k-1}{2A} \leq |f(a_n) - f(a_{n+k})| \leq 2Ak.$$

Proof. (*Upper bound*). By using the triangle inequality, it immediately follows from Lemma 3.4 that $|f(a_n) - f(a_{n+k})| \leq 2Ak$.

(*Lower bound*). Suppose $k \neq 1$. The open interval $\left(f(a_n) - \frac{k-1}{2}, f(a_n) + \frac{k-1}{2}\right)$ contains at most $(k-1)$ integer points. Thus there exists $m \in \mathbf{Z}$ $(n < m < n+k)$ such that

$$|f(a_n) - f(a_m)| \geq \frac{k-1}{2}.$$

From the three-point condition, we obtain

$$A \geq \frac{|f(a_n) - f(a_m)|}{|f(a_n) - f(a_{n+k})|} \geq \frac{k-1}{2|f(a_n) - f(a_{n+k})|},$$

that is, $|f(a_n) - f(a_{n+k})| \geq (k-1)/2A$. □

Proof of Proposition 3.3. If $k \neq 1$, it immediately follows from Lemma 3.5 that

$$\frac{1}{L} \leq \frac{|f(a_{n+k}) - f(a_n)|}{|f(a_n) - f(a_{n-k})|} \leq L$$

for $L = 8A^2$. Moreover, even if $k = 1$, it follows from Lemma 3.4

$$8A^2 > 2A \geq \frac{|f(a_{n+1}) - f(a_n)|}{|f(a_n) - f(a_{n-1})|} \geq \frac{1}{2A} > \frac{1}{8A^2}.$$ □

4. Proof of Theorem B

In this section, first, we shall prove Theorem B. Next, we introduce an another example for which almost the same result holds. Finally, we would like to suggest a natural question arising from the above observations.

4.1. Proof of Theorem B.

THEOREM B. *Let $E \in \mathscr{P}$ which has the following form;*

$$E = \mathbf{Z} + \{a_n\}_{n=1}^{k}$$

where each a_n satisfies $\mathrm{Re}(a_n) \in [0, 1)$.

Then, E is quasiconformally equivalent to $\mathbf{Z}$ *if and only if $k < +\infty$.*

Proof. (*Necessity*). To obtain a contradiction, assume $k = +\infty$. Let $f : \mathbf{C} \to \mathbf{C}$ be a K-quasiconformal homeomorphism with $f(\mathbf{Z}) = E$, and by composing an Affine transformation, we may assume $0 \in E$, and $\sup\{\mathrm{Im}\, a_n \mid n \in \mathbf{N}\} = \infty$.

Under the above assumptions, we prove the following lemma.

LEMMA 4.1.

$$\sup_{m \in \mathbf{Z}} |\mathrm{Im}\, f(m) - \mathrm{Im}\, f(m+1)| = \infty.$$

Proof. Since $\mathbf{Z}$ is porous, by Väisälä's theorem, E is c-porous for some $c \geq 1$. For any $r > 1$ let $x = i\{(\sqrt{2}c + 1)r + 1\}$. Then by porousity of E, there exist $z \in \overline{B}_{\sqrt{2}cr}(x)$ such that $B_{\sqrt{2}r}(z) \subset \mathbf{C} \backslash E$. Then the square domain $\{w = u + iv \mid |u - \mathrm{Re}\, z| < r, |v - \mathrm{Im}\, z| < r\}$ does not intersect with E.

It is easily confirmed that

- $E \cap \{w \mid \mathrm{Im}\, z - r < \mathrm{Im}\, w < \mathrm{Im}\, z + r\} = \emptyset$, since $z + 1 \in \mathrm{Aut}(\mathbf{C} \backslash E)$.
- $E \cap \{w \mid \mathrm{Im}\, w \geq \mathrm{Im}\, z + r\} \neq \emptyset$, since $\sup\{\mathrm{Im}\, a \mid a \in E\} = \infty$.
- $E \cap \{w \mid \mathrm{Im}\, z - r \geq \mathrm{Im}\, w\} \neq \emptyset$, since $0 \in E$ and $\mathrm{Im}\, z - r \geq 1$.

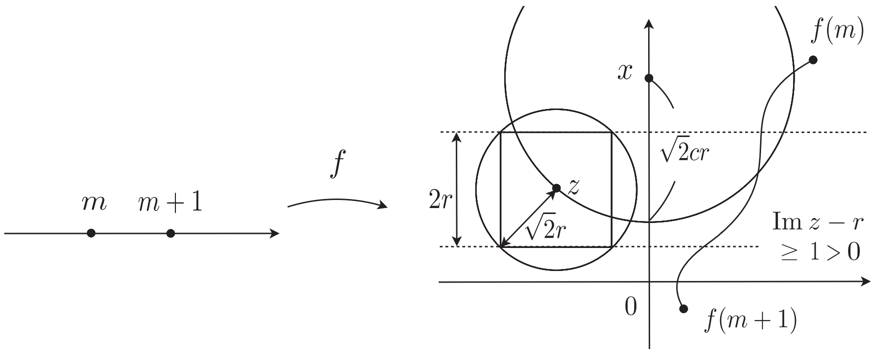

FIGURE 3

Therefore when we consider the image of real line under f, it immediately follows there exists $m \in \mathbf{Z}$ such that $|\mathrm{Im}\, f(m) - \mathrm{Im}\, f(m+1)| \geq 2r$. □

Continuation of Proof of Theorem B. By Lemma 4.1, there exists $m \in \mathbf{Z}$ such that

$$\ell := |\mathrm{Im}\, f(m) - \mathrm{Im}\, f(m+1)| > \exp\left(\frac{K\pi^2}{\log 2}\right).$$

Let

$$C_1' := \{f(m) + t \mid t \in [0,1]\}, \qquad C_1 := f^{-1}(C_1')$$
$$C_2' := \{f(m+1) + t \mid t \in [0,1]\}, \quad C_2 := f^{-1}(C_2').$$

Then we have,

i. by quasiconformality of f

$$\delta^{\mathbf{C}}(C_1, C_2) \le K\delta^{\mathbf{C}}(C_1', C_2'),$$

ii. since C_1' and C_2' are separeted by the annulus $\{w \mid 1 < |w - f(m)| < \ell\}$

$$\delta^{\mathbf{C}}(C_1', C_2') \le \frac{2\pi}{\log \ell} < \frac{2}{K\pi} \log 2,$$

iii. from Vuorinen's theorem,

$$\delta^{\mathbf{C}}(C_1, C_2) \ge \frac{2}{\pi} \log\left(1 + \frac{\min_{i=1,2} \mathrm{diam}(C_i)}{\mathrm{dist}(C_1, C_2)}\right) \ge \frac{2}{\pi} \log\left(1 + \min_{i=1,2} \mathrm{diam}\, C_i\right).$$

Combining the above inequalities, we obtain

$$\min_{i=1,2} \mathrm{diam}\, C_i < 1.$$

On the other hand, since each endpoints of C_i are in the integer set, $\mathrm{diam}\, C_i \ge 1$ $(i = 1, 2)$. This is a contradiction.

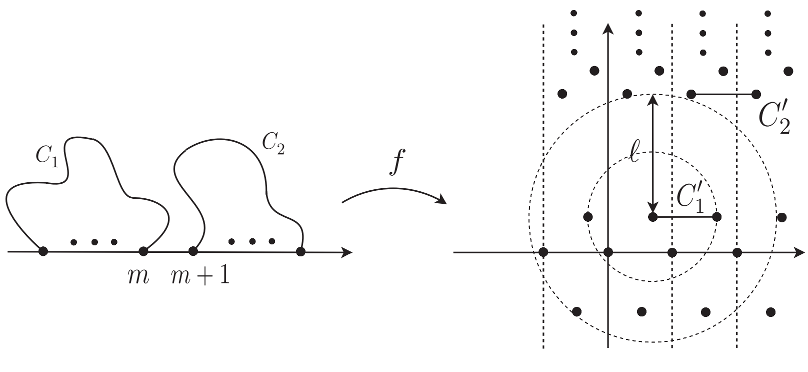

Figure 4

(*Sufficiency*). Since $(\mathbf{C}\backslash\mathbf{Z})/\langle z+k\rangle$ and $(\mathbf{C}\backslash E)/\langle z+1\rangle$ are $(k+2)$-punctured Riemann sphere, there exists a quasiconformal homeomorphism between them which fixes 0 and ∞. Then it can be lifted to a quasiconfromal homeomorphism between $\mathbf{C}\backslash\mathbf{Z}$ and $\mathbf{C}\backslash E$. $\square$

Remark. The necessity and the proof of the sufficiency part shows that if a Riemann surface R which has an automorphism of infinite order is quasiconformally equivalent to $\mathbf{C}\backslash\mathbf{Z}$, then there exists a periodic quasiconformal deformation from $\mathbf{C}\backslash\mathbf{Z}$ to R coming from the deformation of finitely punctured Riemann sphere.

Corollary C.

$$T_\infty = \bigcup_{[f]\in \mathrm{Mod}(\mathbf{C}\backslash\mathbf{Z})} [f]_*(T_0).$$

Here, symbols used in Corollary C are defined in Introduction.

4.2. Another example. For a Riemann surface R, let $\mathrm{Aut}_\infty(R) \subset \mathrm{Aut}(R)$ be the set of all automorphisms of infinite order. The following Theorem 4.2 is a mere rephrasing of Theorem B.

Theorem 4.2. *For $E \in \mathscr{P}$ with $\mathrm{Aut}_\infty(\mathbf{C}\backslash E) \neq \emptyset$, the following are equivalent.*
i. *$\mathbf{C}\backslash E$ is quasiconformally equivalent to $\mathbf{C}\backslash\mathbf{Z}$.*
ii. *For any $h \in \mathrm{Aut}_\infty(\mathbf{C}\backslash E)$, the quotient space $(\mathbf{C}\backslash E)/\langle h\rangle$ is a finitely punctured Riemann sphere.*
iii. *There exists $h \in \mathrm{Aut}_\infty(\mathbf{C}\backslash E)$ such that the quotient space $(\mathbf{C}\backslash E)/\langle h\rangle$ is a finitely punctured Riemann sphere.*

Now, we would like to consider the another infinite type Riemann surface $R' = \mathbf{C}^*\backslash\{2^n\}_{n\in\mathbf{Z}}$, where $\mathbf{C}^* = \mathbf{C}\backslash\{0\}$. In this case, a similar theorem is proved far more easily than the case of $\mathbf{C}\backslash\mathbf{Z}$, because of the relative compactness of the fundamental domain of $\langle 2^n z\rangle$ (contrary to this, the fundamental domain of $\langle z+n\rangle$ is not relatively compact in $\mathbf{C}$).

Theorem 4.3. *For a closed discrete infinite subset $E \subset \mathbf{C}^*$ with $\mathrm{Aut}_\infty(\mathbf{C}^*\backslash E) \neq \emptyset$, the following are equivalent.*
i. *$\mathbf{C}^*\backslash E$ is quasiconformally equivalent to $\mathbf{C}^*\backslash\{2^n\}_{n\in\mathbf{Z}}$.*
ii. *For any $h \in \mathrm{Aut}_\infty(\mathbf{C}^*\backslash E)$, the quotient space $(\mathbf{C}^*\backslash E)/\langle h\rangle$ is a finitely punctured torus.*
iii. *There exists $h \in \mathrm{Aut}_\infty(\mathbf{C}^*\backslash E)$ such that the quotient space $(\mathbf{C}^*\backslash E)/\langle h\rangle$ is a finitely punctured torus.*

Moreover, a theorem similar to Corollary C also holds. In this case, the space corresponding to T_0 simultaneously describes all quasiconformal deformations of all Riemann surfaces of finite type $(1,n)$ with $n \geq 1$, and has the same properties of T_0, separability and geodesic convexity.

4.3. Natural question. With the observations mentioned above, a natural question arises; does an analogous theorem hold for Riemann surfaces which have the following property?

- It has an automorphism of infinite order.
- For any automorphism of infinite order, the quotient space by the action of its cyclic group is of finite type.

In other words, is the above property preserved by quasiconformal deformations?

For example, the Riemann surface defined by $w^2 = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$ has the above property.

Acknowledgements. The author would like to thank the referee for many specific suggestions to improve the manuscript.

References

[1] L. V. Ahlfors, Quasiconformal reflections, Acta Math. **109** (1963), 291–301.
[2] A. Beurling and L. V. Ahlfors, Conformal invariants and function-theoretic null-sets, Acta Math. **83** (1950), 101–129.
[3] P. MacManus, Catching sets with quasicircles, Rev. Mat. Iberoamericana **15** (1999), 267–277.
[4] C. McMullen, Amenability, Poincaré series and quasiconformal maps, Invent. Math. **97** (1989), 95–127.
[5] J. Väisälä, Lectures on n-dimensional quasiconformal mappings, Lecture notes in mathematics **229**, Springer-Verlag, Berlin-New York, 1971.
[6] J. Väisälä, Porous sets and quasisymmetric maps, Trans. Amer. Math. Soc. **299** (1987), 525–533.
[7] M. Vuorinen, On Teichmüller's modulus problem in $\mathbf{R}^n$, Math. Scand. **63** (1988), 315–333.

Hiroki Fujino
Graduate School of Mathematics
Nagoya University
Furo-cho Chikusa-ku Nagoya 464-8602
Japan
E-mail: m12040w@math.nagoya-u.ac.jp