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TWO NORMALITY CRITERIA AND COUNTEREXAMPLES TO THE CONVERSE OF BLOCH'S PRINCIPLE

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Abstract

In this paper, we prove two normality criteria for a family of meromorphic functions. The first criterion extends a result of Fang and Zalcman [*Normal families and shared values of meromorphic functions II, Comput. Methods Funct. Theory, 1 (2001), 289–299*] to a bigger class of differential polynomials whereas the second one leads to some counterexamples to the converse of the Bloch's principle.

1. Introduction and main results

It is assumed that the reader is familiar with the standard notions used in the Nevanlinna value distribution theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$, $S(r,f)$ etc., one may refer to [5]. In this paper, we obtain a normality criterion for a family of meromorphic functions which involves sharing of holomorphic functions by certain differential polynomials generated by the members of the family.

In 2001, Fang and Zalcman [4, Theorem 2, p. 291] proved the following

Theorem A. *Let $\mathcal{F}$ be a family of meromorphic functions on a domain D, k be a positive integer and $a(\neq 0)$ and b be two finite values. If, for every $f \in \mathcal{F}$, all zeros of f have multiplicity at least k and $f(z)f^{(k)}(z) = a \Leftrightarrow f^{(k)}(z) = b$, then the family $\mathcal{F}$ is normal on D.*

In this paper, we extend this result as

Theorem 1.1. *Let $\mathcal{F}$ be a family of meromorphic functions on a domain D. Let $n \geq 2$, $m \geq k \geq 1$ be the positive integers and let $a(\neq 0)$ and b be two finite values. If, for each $f \in \mathcal{F}$, $f^n(z)(f^m)^{(k)}(z) = a \Leftrightarrow (f^m)^{(k)}(z) = b$, then the family $\mathcal{F}$ is normal on D.*

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Now it is natural to ask whether Theorem 1.1 still holds if a and b are holomorphic functions. In this direction, we prove the following

THEOREM 1.2. *Let $n \geq 2$, $m \geq k \geq 1$ be the positive integers. Let $a(z)(\not\equiv 0)$ and $b(z)$ be two holomorphic functions on a domain D such that multiplicity of each zero of $a(z)$ is at most p, where $p \leq \left\lceil \frac{n-1}{m} \right\rceil - 1$. Then, the family $\mathcal{F}$ of meromorphic functions on a domain D, all of whose poles are of multiplicity at least $p+1$, such that $f^n(z)(f^m)^{(k)}(z) = a(z) \Leftrightarrow (f^m)^{(k)}(z) = b(z)$, for every $f \in \mathcal{F}$, is normal on D.*

Remark 1.1. Consider the family $\mathcal{F} = \{f_l : l \in \mathbf{N}\}$, where $f_l(z) = e^{lz}$ on the unit disk $\mathbf{D}$. Then

$$(f_l^m)^{(k)}(z) = m^k l^k e^{mlz} \quad \text{and} \quad f_l^n(z)(f_l^m)^{(k)}(z) = m^k l^k e^{(n+m)lz}$$

Clearly, $f_l^n(z)(f_l^m)^{(k)}(z) = 0 \Leftrightarrow (f_l^m)^{(k)}(z) = 0$. However, $\mathcal{F}$ is not normal on $\mathbf{D}$. Thus the condition that $a \neq 0$ is essential in Theorem 1.1.

Remark 1.2. Consider the family $\mathcal{F} = \{f_l : l \in \mathbf{N}\}$, where $f_l(z) = 2lz$ on the unit disk $\mathbf{D}$. Then

$$f_l^n(z)(f_l^m)^{(k)}(z) = (2l)^{n+m} m(m-1)(m-2)\cdots(m-k)z^{n+m-k}$$

and

$$(f_l^m)^{(k)}(z) = (2l)^m m(m-1)(m-2)\cdots(m-k)z^{m-k}$$

Clearly, $f_l^n(z)(f_l^m)^{(k)}(z) = a(z) \Leftrightarrow (f_l^m)^{(k)}(z) = b(z)$, where $a(z) = z^{n+m-k}$ and $b(z) = z^{m-k}$. We can see that multiplicity of zeros of $a(z)$ is at least n. However, the family $\mathcal{F}$ is not normal on $\mathbf{D}$. Thus, the restriction on the multiplicities of the zeros of $a(z)$ is essential in Theorem 1.2.

In 2004, Lahiri and Dewan [9, Theorem 1.4, p. 3] proved

THEOREM B. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D and $a(\neq 0)$, $b \in \mathbf{C}$. Suppose that $E_f = \{z \in D : f^{(k)} - af^{-n} = b\}$, where k and $n(\geq k)$ are the positive integers. If for every $f \in \mathcal{F}$*

(i) *f has no zero of multiplicity less than k*

(ii) *there exists a positive number M such that for every $f \in \mathcal{F}$, $|f(z)| \geq M$ whenever $z \in E_f$, then $\mathcal{F}$ is normal.*

In 2006, Xu and Zhang [17, Theorem 1.3, p. 5] improved Theorem B as

THEOREM C. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D and $a(\neq 0)$, $b \in \mathbf{C}$. Suppose that $E_f = \{z \in D : f^{(k)} - af^{-n} = b\}$, where k and n are the positive integers. If for every $f \in \mathcal{F}$*

(i) *f has no zero of multiplicity at least k*

(ii) *there exists a positive number M such that for every $f \in \mathcal{F}$, $|f(z)| \geq M$ whenever $z \in E_f$, then $\mathcal{F}$ is normal so long as*

(A) *$n \geq 2$ or*

(B) *$n = 1$ and $\overline{N}_k(r, 1/f) = S(r, f)$.*

In this paper, we prove the following

Theorem 1.3. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D. Let n_1, n_2, $m > k \geq 1$ be the non-negative integers such that $n_1 + n_2 \geq 1$. Suppose $\psi(z) := f^{n_1}(z)(f^m)^{(k)}(z) - af^{-n_2}(z) - b$, where $a(\neq 0)$, $b \in \mathbf{C}$. If there exists a positive constant M such that for every $f \in \mathcal{F}$, either $|f(z)| \geq M$ or $|(f^m)^{(k)}(z)| \leq M$ whenever z is a zero of $\psi(z)$, then $\mathcal{F}$ is normal in D.*

As an application of Theorem 1.3, we construct some counterexamples to the converse of Bloch's principle in the last section of this paper.

Corollary 1.4. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D. Let $n, m > k$ be the positive integers and $a(\neq 0)$ be a finite complex number. If there exists a positive constant M such that for every $f \in \mathcal{F}$, $f^n(z)(f^m)^{(k)}(z) = a \Rightarrow |(f^m)^{(k)}(z)| \leq M$, then $\mathcal{F}$ is normal in D.*

2. Some lemmas

Lemma 2.1 [21] (Zalcman's lemma). *Let $\mathcal{F}$ be a family of meromorphic functions in the unit disk $\mathbf{D}$ and α be a real number satisfying $-1 < \alpha < 1$. Then, if $\mathcal{F}$ is not normal at a point $z_0 \in \mathbf{D}$, there exist, for each $\alpha : -1 < \alpha < 1$,*

(i) *a real number $r : r < 1$,*

(ii) *points $z_n : |z_n| < r$,*

(iii) *positive numbers $\rho_n : \rho_n \to 0$,*

(iv) *functions $f_n \in F$ such that $g_n(\zeta) = \rho_n^{-\alpha} f_n(z_n + \rho_n \zeta)$ converges locally uniformly with respect to the spherical metric to $g(\zeta)$, where $g(\zeta)$ is a non constant meromorphic function on $\mathbf{C}$ and $g^{\#}(\zeta) \leq g^{\#}(0) = 1$. Moreover, the order of g is not greater than 2.*

Lemma 2.2 [22, Lemma 2.6, p. 107]. *Let $R = \dfrac{A}{B}$ be a rational function and B be non constant. Then $(R^{(k)})_\infty \leq (R)_\infty - k$, where $(R)_\infty = \deg(A) - \deg(B)$.*

Lemma 2.3. *Let $n \geq 2$, $m \geq k \geq 1$ be the positive integers. Let $a(z)(\not\equiv 0)$ be a polynomial of degree p such that $p \leq n - 2$. Then there is no function f rational on $\mathbf{C}$ which has only poles of multiplicity at least $p + 1$ such that $f^n(z)(f^m)^{(k)}(z) \neq a(z)$ and $(f^m)^{(k)}(z) \neq 0$.*

Proof. First we consider the case of a polynomial. Suppose on the contrary that there is a polynomial $f(z)$ with the given properties. Since $(f^m)^{(k)} \neq 0$ and $m \geq k$, f has zeros of multiplicity exactly one. So, we have

$$\deg(f^n(f^m)^{(k)}) \geq n \deg(f) = n > p = \deg(a(z))$$

Therefore, $f^n(z)(f^m)^k(z) - a(z)$ has a solution, which is a contradiction.

Next, suppose that f has poles. Then, we set

$$f(z) = A \frac{\prod_{i=1}^{s}(z - \alpha_i)}{\prod_{j=1}^{t}(z - \beta_j)^{n_j}}, \tag{2.1}$$

where $A \neq 0$, α_i are the distinct zeros of f with $s \geq 0$ and β_j are the distinct poles of f with $t \geq 1$.

Put

$$\sum_{j=1}^{t} n_j = N.$$

Then

$$N \geq t(p+1).$$

Now,

$$f^m(z) = A^m \frac{\prod_{i=1}^{s}(z - \alpha_i)^m}{\prod_{j=1}^{t}(z - \beta_j)^{mn_j}} \tag{2.2}$$

$$\Rightarrow (f^m)^{(k)}(z) = \frac{\prod_{i=1}^{s}(z - \alpha_i)^{m-k}}{\prod_{j=1}^{t}(z - \beta_j)^{mn_j+k}} g(z), \tag{2.3}$$

where $g(z)$ is a polynomial.

By Lemma 2.2, we have

$$(f^m)_\infty^{(k)} \leq (f^m)_\infty - k$$
$$\Rightarrow deg(g) \leq k(s + t - 1).$$

Now,

$$f^n(f^m)^{(k)} = A^n \frac{\prod_{i=1}^{s}(z-\alpha_i)^{(m+n)-k}}{\prod_{j=1}^{t}(z-\beta_j)^{(m+n)n_j+k}} g(z). \tag{2.4}$$

So,

$$(f^n(f^m)^{(k)})^{(p+1)} = \frac{\prod_{i=1}^{s}(z-\alpha_i)^{(m+n)-k-p-1}}{\prod_{j=1}^{t}(z-\beta_j)^{(m+n)n_j+k+p+1}} g_0(z), \tag{2.5}$$

where $g_0(z)$ is a polynomial.

Again, by Lemma 2.2, we have

$$(f^n(f^m)^{(k)})_{\infty}^{(p+1)} \leq (f^n(f^m)^{(k)})_{\infty} - (p+1)$$
$$\Rightarrow deg(g_0) \leq (s+t-1)(p+k+1).$$

Since $f^n(f^m)^{(k)} \neq a(z)$, we set

$$f^n(f^m)^{(k)} = a(z) + \frac{c}{\prod_{j=1}^{t}(z-\beta_j)^{(m+n)n_j+k}}, \tag{2.6}$$

where $c \neq 0$ is a constant.

So,

$$(f^n(f^m)^{(k)})^{(p+1)} = \frac{g_1(z)}{\prod_{j=1}^{t}(z-\beta_j)^{(m+n)n_j+k+p+1}}, \tag{2.7}$$

where $g_1(z)$ is a polynomial of degree at most $(p+1)(t-1)$.

On comparing (2.4) and (2.6), we have

$$s(m+n) - ks + deg(g) = N(m+n) + kt + pt$$
$$\Rightarrow N(m+n) \leq s(m+n) - k$$
$$\Rightarrow N < s,$$

for $n \geq 2$, $m \geq k \geq 1$.

Also, from (2.5) and (2.7), we have

$$deg(g_1) \geq s(m+n) - s(k+p+1).$$

Now,

$$
\begin{aligned}
(p+1)(t-1) &\geq \deg(g_1(z)) \geq s(m+n) - s(k+p+1) \\
&\Rightarrow s(m+n) \leq (p+1)(t-1) + s(k+p+1) \\
&\Rightarrow s(m+n) < (p+1)t + s(k+p+1) \\
&\Rightarrow s < \frac{p+1}{m+n}t + \frac{k+p+1}{m+n}s \\
&\Rightarrow s < \frac{1}{m+n}N + \frac{k+p+1}{m+n}s \\
&\Rightarrow s < \left(\frac{1}{m+n} + \frac{k+p+1}{m+n}\right)s \\
&\Rightarrow s < \left(\frac{k+p+2}{m+n}\right)s \\
&\Rightarrow s < s\left(\because \frac{k+p+2}{m+n} \leq 1\right),
\end{aligned}
$$

which is absurd.

Thus, if $(f^m)^{(k)}(z) \neq 0$, then $f^n(z)(f^m)^{(k)}(z) - a(z)$ has at least a solution. Hence the Lemma follows. ■

LEMMA 2.4. *Let $n \geq 2$, $m \geq k \geq 1$ be the positive integers. Then there is no transcendental meromorphic function f on $\mathbf{C}$ such that $f^n(z)(f^m)^{(k)}(z) \neq a(z)$ and $(f^m)^{(k)}(z) \neq 0$, where $a(z) \not\equiv 0$ is a small function of f.*

Proof. Suppose on the contrary that there is a transcendental meromorphic function f on $\mathbf{C}$ satisfying the given conditions. Since $(f^m)^{(k)} \neq 0$ and $m \geq k$, f has zeros of multiplicity exactly one. Now, by second fundamental theorem of Nevanlinna for three small functions [5, Theorem 2.5, p. 47], we have

$$
\begin{aligned}
T(r, f^n(f^m)^{(k)}) &\leq \overline{N}(r, f^n(f^m)^{(k)}) + \overline{N}\left(r, \frac{1}{f^n(f^m)^{(k)}}\right) \\
&\quad + \overline{N}\left(r, \frac{1}{f^n(f^m)^{(k)} - a(z)}\right) \\
&= \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + S(r, f).
\end{aligned} \tag{2.8}
$$

Also,

$$
\begin{aligned}
T(r, f^n(f^m)^{(k)}) &\geq \frac{1}{2}\left[N(r, f^n(f^m)^{(k)}) + N\left(r, \frac{1}{f^n(f^m)^{(k)}}\right)\right] \\
&\geq \frac{n+m+k}{2}\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + S(r, f).
\end{aligned} \tag{2.9}
$$

Thus, from (2.8) and (2.9), we get

$$\frac{n+m+k}{2}\overline{N}(r,f) \le \overline{N}(r,f) + S(r,f) \tag{2.10}$$
$$\Rightarrow \overline{N}(r,f) = S(r,f).$$

Next,

$$\begin{aligned}(m+n)T(r,f) &= T(r,f^{m+n}) \\ &= T\left(r,\frac{1}{f^{m+n}}\right) + O(1) \\ &= m\left(r,\frac{1}{f^{m+n}}\right) + N\left(r,\frac{1}{f^{m+n}}\right) + O(1) \\ &= m\left(r,\frac{(f^m)^{(k)}}{f^m}\frac{1}{f^n(f^m)^{(k)}}\right) + N\left(r,\frac{1}{f^{m+n}}\right) + O(1) \\ &\le m\left(r,\frac{1}{f^n(f^m)^{(k)}}\right) + N\left(r,\frac{1}{f^{m+n}}\right) + O(1) \\ &\le T(r,f^n(f^m)^{(k)}) - N\left(r,\frac{1}{f^n(f^m)^{(k)}}\right) \\ &\quad + N\left(r,\frac{1}{f^{m+n}}\right) + S(r,f).\end{aligned} \tag{2.11}$$

Now, substituting (2.8) and (2.10) in (2.11), we get

$$\begin{aligned}(m+n)T(r,f) &\le \overline{N}\left(r,\frac{1}{f}\right) - N\left(r,\frac{1}{f^n(f^m)^{(k)}}\right) + N\left(r,\frac{1}{f^{m+n}}\right) + S(r,f) \\ &\le \overline{N}\left(r,\frac{1}{f}\right) - n\overline{N}\left(r,\frac{1}{f}\right) + (m+n)\overline{N}\left(r,\frac{1}{f}\right) + S(r,f) \\ &= (m+1)\overline{N}\left(r,\frac{1}{f}\right) + S(r,f) \\ &\le (m+1)T(r,f) + S(r,f) \\ \Rightarrow (n-1)T(r,f) &\le S(r,f),\end{aligned}$$

which is a contradiction, for $n \ge 2$.

However, if f has no zeros, then $f^n(f^m)^{(k)}$ has no zeros.

That is,

$$N\left(r,\frac{1}{f}\right)=S(r,f)\quad\text{and}\quad N\left(r,\frac{1}{f^n(f^m)^{(k)}}\right)=S(r,f).$$

Thus, by the same argument used above, we get a contradiction. ■

Lemma 2.5 [2]. *Let f be a transcendental meromorphic function and $n,m>k$ be the positive integers. Let $F=f^n(f^m)^{(k)}$. Then*

$$\left[\frac{k}{2(2k+2)}+o(1)\right]T(r,F)\le \overline{N}\left(r,\frac{1}{F-\omega}\right)+S(r,F)$$

for any small function $\omega(\not\equiv 0,\infty)$ of f.

Lemma 2.6 [2]. *Let f be a rational function and $n,m>k$ be the positive integers. Then, for $a(\neq 0)\in\mathbf{C}$, $f^n(f^m)^{(k)}-a$ has at least two distinct zeros.*

Lemma 2.7 [3]. *Let f be an entire function. If the spherical derivative $f^{\#}$ is bounded in $\mathbf{C}$, then the order of f is at most one.*

3. Proof of Theorems

Proof of Theorem 1.1. Suppose that $\mathscr{F}$ is not normal at some point $z_o\in D$. We assume $D=\mathbf{D}$. Then by Lemma 2.1, we can find a sequence $\{f_j\}$ in $\mathscr{F}$, a sequence $\{z_j\}$ of complex numbers with $z_j\to z_o$ and a sequence $\{\rho_j\}$ of positive real numbers with $\rho_j\to 0$ such that

$$g_j(\zeta)=\rho_j^{-k/(n+m)}f_j(z_j+\rho_j\zeta)$$

converges locally uniformly with respect to the spherical metric to a non-constant meromorphic function $g(\zeta)$ on $\mathbf{C}$ having bounded spherical derivative.

Claim.
(1) $g^n(g^m)^{(k)}\neq a$
(2) $(g^m)^{(k)}\neq 0$

Suppose that $g^n(\zeta_o)(g^m)^{(k)}(\zeta_o)=a$. Then $g(\zeta)\neq\infty$ in some small neighborhood of ζ_o. Further, $g^n(g^m)^{(k)}\not\equiv a$. Suppose $g^n(g^m)^{(k)}\equiv a$. Since g is a non-constant entire function without zeros, by Lemma 2.7, we have $g(\zeta)=e^{c\zeta+d}$, where $c\neq 0$ and d are constants. Thus

$$m^kc^ke^{(m+n)c\zeta+(m+n)d}\equiv a$$

which is impossible unless $(m+n)c=0$. Hence by Hurwitz theorem, there exist points $\zeta_j\to\zeta_o$ such that, for sufficiently large j, we have

$$a=g_j^n(\zeta_j)(g_j^m)^{(k)}(\zeta_j)=f_j^n(\zeta_j+\rho_j\zeta_j)(f_j^m)^{(k)}(\zeta_j+\rho_j\zeta_j).$$

By given condition, we have

$$(f_j^m)^{(k)}(\zeta_j + \rho_j\zeta_j) = b,$$

and hence,

$$(g_j^m)^{(k)}(\zeta_j) = \rho_j^{nk/(m+n)}(f_j^m)^{(k)}(z_j + \rho_j\zeta_j) = \rho_j^{nk/(m+n)}b$$
$$\Rightarrow (g^m)^{(k)}(\zeta_o) = \lim_{j\to\infty}(g_j^m)^{(k)}(\zeta_j) = 0$$

which contradicts that $g^n(\zeta_o)(g^m)^{(k)}(\zeta_o) = a \neq 0$. This proves claim (1).

Now, suppose $(g^m)^{(k)}(\zeta_o) = 0$ for some $\zeta_o \in \mathbf{C}$, then $g(\zeta) \neq \infty$ in some small neighborhood of ζ_o. Further, $(g^m)^{(k)} \not\equiv 0$, otherwise, g reduces to a constant since $m \geq k$. Again, by Hurwitz theorem, there exist points $\zeta_j \to \zeta_o$ such that, for sufficiently large j, we have

$$(g_j^m)^{(k)}(\zeta_j) - \rho_j^{nk/(m+n)}b = 0$$
$$\Rightarrow \rho_j^{nk/(m+n)}(f_j^m)^{(k)}(z_j + \rho_j\zeta_j) - \rho_j^{nk/(m+n)}b = 0$$
$$\Rightarrow (f_j^m)^{(k)}(z_j + \rho_j\zeta_j) = b.$$

Thus, by the given condition, we get

$$f_j^n(z_j + \rho_j\zeta_j)(f_j^m)^{(k)}(z_j + \rho_j\zeta_j) = a = g_j^n(\zeta_j)(g_j^m)^{(k)}(\zeta_j)$$
$$\Rightarrow a = \lim_{j\to\infty} g_j^n(\zeta_j)(g_j^m)^{(k)}(\zeta_j) = g^n(\zeta_o)(g^m)^{(k)}(\zeta_o) = 0$$

which is a contradiction. This proves claim (2).

Claims (1) and (2) as established contradict Lemma 2.3 and Lemma 2.4. Hence $\mathscr{F}$ is normal. ■

Proof of Theorem 1.2. Suppose that $\mathscr{F}$ is not normal at some point $z_o \in D$. We assume $D = \mathbf{D}$. We distinguish the following two cases:

Case I. $a(z_o) \neq 0$

Following the proof of Theorem 1.1, we arrive at a contradiction and hence $\mathscr{F}$ is normal in this case.

Case II. $a(z_o) = 0$

Without loss of generality, we assume that $z_o = 0$. Further, we assume $a(z) = z^p a_1(z)$, where p is a positive integer and $a_1(0) \neq 0$. We may take $a_1(0) = 1$. Now, by Lemma 2.1, we can find a sequence $\{f_j\}$ in $\mathscr{F}$, a sequence $\{z_j\}$ of complex numbers with $z_j \to 0$ and a sequence $\{\rho_j\}$ of positive real numbers with $\rho_j \to 0$ such that

$$g_j(\zeta) = \rho_j^{-(p+k)/(n+m)} f_j(z_j + \rho_j\zeta)$$

converges locally uniformly with respect to the spherical metric to a non-constant meromorphic function $g(\zeta)$ on $\mathbf{C}$ having bounded spherical derivative.

Subcase I. Suppose there exist a subsequence of $\frac{z_j}{\rho_j}$, we may take $\frac{z_j}{\rho_j}$ itself, such that $\frac{z_j}{\rho_j} \to \infty$ as $j \to \infty$.

Let

$$G_j(\zeta) = z_j^{-(p+k)/(n+m)} f_j(z_j + z_j\zeta).$$

Then, by the given condition $f^n(z)(f^m)^{(k)}(z) = a(z) \Leftrightarrow (f^m)^{(k)}(z) = b(z)$, we have

$$G_j^n(\zeta)(G_j^m)^{(k)}(\zeta) = (1+\zeta)^p a_1(z_j + z_j\zeta) \Leftrightarrow (G_j^m)^{(k)}(\zeta) = z_j^l b(z_j + z_j\zeta),$$

where

$$l = -\frac{m(p+k)}{n+m} + k > 0.$$

Thus, by Case I, $\{G_j\}$ is normal on $\mathbf{D}$ and $G_j \to G$ (say) on $\mathbf{D}$. Hence, by Marty's theorem, there exist a compact subset E of $\mathbf{D}$ and a constant $M > 0$ such that

$$G_j^{\#}(\xi) \le M \quad \text{for } \xi \in E.$$

Claim. $G^{\#}(0) = 0$. Suppose $G^{\#}(0) \neq 0$. Then for $\zeta \in \mathbf{C}$, we have

$$\begin{aligned} g^{\#}(\zeta) &= \lim_{j\to\infty} g_j^{\#}(\zeta) \\ &= \lim_{j\to\infty} \rho_j^{-(p+k)/(n+m)} f_j^{\#}(z_j + \rho_j\zeta) \\ &= \lim_{j\to\infty} \left(\frac{z_j}{\rho_j}\right)^{(p+k)/(n+m)} G_j^{\#}\left(\frac{\rho_j}{z_j}\zeta\right) \\ &= \infty \end{aligned}$$

which is a contradiction to the fact that g has bounded spherical derivative.

Now, $G^{\#}(0) = 0 \Rightarrow G'(0) = 0$. For any $\zeta \in \mathbf{C}$, we have

$$\begin{aligned} g_j'(\zeta) &= \rho_j^{-(p+k)/(n+m)+1} f_j'(z_j + \rho_j\zeta) \\ &= \left(\frac{\rho_j}{z_j}\right)^{-(p+k)/(n+m)+1} G_j'\left(\frac{\rho_j}{z_j}\zeta\right) \xrightarrow{\chi} 0 \end{aligned}$$

on $\mathbf{C}$ as $\frac{p+k}{n+m} < 1$. Thus $g'(\zeta) \equiv 0$ implies that g is constant and this is a contradiction.

Subcase II. Suppose there exist a subsequence of $\frac{z_j}{\rho_j}$, we may take $\frac{z_j}{\rho_j}$ itself, such that $\frac{z_j}{\rho_j} \to c$ as $j \to \infty$, where c is a finite number.

Then, we have

$$H_j(\zeta) = \rho_j^{-(p+k)/(n+m)} f_j(\rho_j \zeta) = g_j\left(\zeta - \frac{z_j}{\rho_j}\right) \xrightarrow{\chi} g(\zeta - c) := H(\zeta).$$

Thus, by the given condition, we have

$$H_j^n(\zeta)(H_j^m)^{(k)}(\zeta) = \zeta^p a_1(\rho_j\zeta) \Leftrightarrow (H_j^m)^{(k)}(\zeta) = \rho_j^l b(\rho_j\zeta),$$

where

$$l = -\frac{m(p+k)}{n+m} + k > 0.$$

Claim.

(1) $H^n(\zeta)(H^m)^{(k)}(\zeta) \neq \zeta^p$ on $\mathbf{C} - \{0\}$

(2) $(H^m)^{(k)}(\zeta) \neq 0$ on $\mathbf{C} - \{0\}$

Suppose that $H^n(\zeta_o)(H^m)^{(k)}(\zeta_o) = \zeta_o^p$, $\zeta_o \neq 0$. Then, $H(\zeta) \neq \infty$ on some small neighborhood of ζ_o. Further, $H^n(\zeta)(H^m)^{(k)}(\zeta) \not\equiv \zeta^p$. If $H^n(\zeta)(H^m)^{(k)}(\zeta) \equiv \zeta^p$, then $\zeta = 0$ is the only possible zero of H. If H is a transcendental function, then, clearly $H^n(H^m)^{(k)}$ is also a transcendental function, which is not true. If H is a rational function and $\zeta = 0$ is a zero of H, then H is a polynomial. Thus, $\deg(H^n(H^m)^{(k)}) \geq n \deg(H) \geq n$, which is a contradiction to the fact that $H^n(\zeta)(H^m)^{(k)}(\zeta) \equiv \zeta^p$, $p \leq n-2$. By Hurwitz's theorem, there exist points $\zeta_j \to \zeta_o$ such that, for sufficiently large j, we have

$$H_j^n(\zeta_j)(H_j^m)^{(k)}(\zeta_j) - \zeta_j^p a_1(\rho_j\zeta_j) = 0$$
$$\Rightarrow (H_j^m)^{(k)}(\zeta_j) - \rho_j^l b(\rho_j\zeta_j) = 0.$$

Thus,

$$\begin{aligned}(H^m)^{(k)}(\zeta_o) &= \lim_{j\to\infty} (H_j^m)^{(k)}(\zeta_j) \\ &= \lim_{j\to\infty} \rho_j^l b(\rho_j\zeta_j) \\ &= 0\end{aligned}$$

which contradicts that $H^n(\zeta_o)(H^m)^{(k)}(\zeta_o) = \zeta_o^p \neq 0$. This proves claim (1).

Next, suppose $(H^m)^{(k)}(\zeta_o) = 0$ for some $\zeta_o \in \mathbf{C} - \{0\}$. Then $H(\zeta) \neq \infty$ on some small neighborhood of ζ_o. Further, $(H^m)^{(k)} \not\equiv 0$, otherwise, H reduces to a constant since $m \geq k$. Thus, by Hurwitz theorem, there exist points $\zeta_j \to \zeta_o$ such that, for sufficiently large j, we have

$$(H_j^m)^{(k)}(\zeta_j) - \rho_j^l b(\rho_j \zeta_j) = 0$$

$$\Rightarrow H_j^n(\zeta_j)(H_j^m)^{(k)}(\zeta_j) - \zeta_j^p a_1(\rho_j \zeta_j) = 0$$

and so

$$\begin{aligned} H^n(\zeta_o)(H^m)^{(k)}(\zeta_o) &= \lim_{j\to\infty} H_j^n(\zeta_j)(H_j^m)^{(k)}(\zeta_j) \\ &= \lim_{j\to\infty} \zeta_j^p a_1(\rho_j\zeta_j) \\ &= \zeta_o^p \end{aligned}$$

which is a contradiction. This proves claim (2).

Claims (1) and (2) as established contradict Lemma 2.3 and Lemma 2.4. Hence $\mathscr{F}$ is normal. ■

Proof of Theorem 1.3. Suppose that $\mathscr{F}$ is not normal at some point $z_0 \in D$. Then by Lemma 2.1, we can find a sequence $\{f_j\}$ in $\mathscr{F}$, a sequence $\{z_j\}$ of complex numbers with $z_j \to z_o$ and a sequence $\{\rho_j\}$ of positive real numbers with $\rho_j \to 0$ such that

$$g_j(\zeta) = \rho_j^{-k/(n_1+n_2+m)} f_j(z_j + \rho_j \zeta)$$

converges locally uniformly with respect to the spherical metric to a non-constant meromorphic function $g(\zeta)$ on $\mathbf{C}$ having bounded spherical derivative. Now, by Lemma 2.5 and Lemma 2.6, $g^n(\zeta)(g^m)^{(k)}(\zeta) - a$ has at least one zero for $n \geq 1$, $m > k \geq 1$. Suppose that $g^n(\zeta_0)(g^m)^{(k)}(\zeta_0) - a = 0$ for some $\zeta_0 \in \mathbf{C}$. Clearly, $g(\zeta) \neq 0, \infty$ in some neighborhood of ζ_0. Thus, we have

$$g^{n_1}(\zeta_0)(g^m)^{(k)}(\zeta_0) - ag^{-n_2}(\zeta_0) = 0,$$

where $n = n_1 + n_2 \geq 1$.

Now, in some neighborhood of ζ_0, we have

$$\begin{aligned} &g_j^{n_1}(\zeta_0)(g_j^m)^{(k)}(\zeta_0) - ag_j^{-n_2}(\zeta_0) - \rho_j^{kn_2/(n+m)} b \\ &\quad = \rho_j^{kn_2/(n+m)} \{f_j^{n_1}(z_j + \rho_j\zeta_0)(f_j^m)^{(k)}(z_j + \rho_j\zeta_0) - af_j^{-n_2}(z_j + \rho_j\zeta_0) - b\} \end{aligned}$$

By Hurwitz's theorem, there exists a sequence $\zeta_j \to \zeta_0$ such that for all large values of j,

$$f_j^{n_1}(z_j + \rho_j\zeta_j)(f_j^m)^{(k)}(z_j + \rho_j\zeta_j) - af_j^{-n_2}(z_j + \rho_j\zeta_j) - b = 0$$

Thus, by the assumption, if $|f_j(z_j + \rho_j\zeta_j)| \geq M$, then we have

$$|g_j(\zeta_j)| = \rho_j^{-k/(n+m)} |f_j(z_j + \rho_j\zeta_j)| \geq \rho_j^{-k/(n+m)} M.$$

Since $g_j(\zeta)$ converges uniformly to $g(\zeta)$ in some neighborhood of ζ_0, for all large values of j and for every $\varepsilon > 0$, we have

$$|g_j(\zeta) - g(\zeta)| < \varepsilon \quad \text{for all } \zeta \text{ in that neighborhood of } \zeta_o.$$

Thus, in a neighborhood of ζ_o, for all large values of j, we have

$$|g(\zeta_j)| \geq |g_j(\zeta_j)| - |g(\zeta_j) - g_j(\zeta_j)| > \rho_j^{-k/(n+m)} M - \varepsilon$$

which is a contradiction to the fact that ζ_0 is not a pole of $g(\zeta)$.

Again, by the assumption, if $|(f_j^m)^{(k)}(z_j + \rho_j\zeta_j)| \leq M$, then we have

$$|(g_j^m)^{(k)}(\zeta_j)| = \rho_j^{k-mk/(n_1+n_2+m)} |(f_j^m)^{(k)}(z_j + \rho_j\zeta_j)| \leq \rho_j^{k-mk/(n_1+n_2+m)} M$$

so that

$$(g^m)^{(k)}(\zeta_o) = \lim_{j\to\infty} (g_j^m)^{(k)}(\zeta_j) = 0$$

which contradicts $g^n(\zeta_o)(g^m)^{(k)}(\zeta_o) = a \neq 0$. Hence $\mathscr{F}$ is normal. ■

4. Counterexamples to the converse of the Bloch's principle

The Bloch's principle as noted by Robinson [14] is one of the twelve mathematical problems requiring further consideration; it is a heuristic principle in function theory. The Bloch's principle states that a family of holomorphic (meromorphic) functions satisfying a property $\mathscr{P}$ in a domain D is likely to be a normal family if the property $\mathscr{P}$ reduces every holomorphic (meromorphic) function on **C** to a constant. The Bloch's principle is not universally true, for example one can see [15].

The converse of the Bloch's Principle states that if a family of meromorphic functions satisfying a property $\mathscr{P}$ on an arbitrary domain D is necessarily a normal family, then every meromorphic function on **C** with property $\mathscr{P}$ reduces to a constant. Like Bloch's principle, its converse is not true. For counterexamples one can see [1], [8], [10], [16], [18], [20]. In order to construct counterexamples to the converse, one needs to prove a suitable normality criterion. Here Theorem 1.3 is such a criterion. Infact, following is a direct consequence of Theorem 1.3:

THEOREM 4.1. *Let $\mathscr{F}$ be a family of meromorphic functions in a domain D. Let n_1, n_2, $m > k \geq 1$ be the non-negative integers such that $n_1 + n_2 \geq 1$. Suppose $\psi(z) := f^{n_1}(z)(f^m(z))^{(k)} - af^{-n_2}(z) - b$, where $a(\neq 0)$, $b \in \mathbf{C}$, has no zeros in D. Then $\mathscr{F}$ is normal in D.*

Now by Theorem 4.1, we have the following four counterexamples to the converse of the Bloch's principle:

Consider $f(z) = e^z$. Then for $n_1 = 1$, $n_2 = 0$, $m = 2$, $k = 1$, $a = -1$, and $b = 1$, $\psi(z) := f(z)(f^2)'(z) + 1 - 1 = 2e^{3z}$ has no zeros in **C**. Thus there is a non constant entire function with property $\mathscr{P} : \psi(z)$ has no zeros in $\mathbf{C}$. Hence in view of Theorem 4.1, this is a counterexample to the converse of Bloch's principle.

Similarly, for the same values of the constants n_1, n_2, m, k, a, and b, the meromorphic functions

$$\frac{1}{z}, \quad \frac{1}{e^z+1}, \quad \tan z \pm i,$$

provide three more counterexamples to the converse of the Bloch's principle.

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