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ON MILNOR FIBRATIONS OF MIXED FUNCTIONS, a_f-CONDITION AND BOUNDARY STABILITY

Mutsuo Oka

Abstract

Convenient mixed functions with strongly non-degenerate Newton boundaries have a Milnor fibration ([9]), as the isolatedness of the singularity follows from the convenience. In this paper, we consider the Milnor fibration for non-convenient mixed functions. We also study geometric properties such as Thom's a_f-condition, the transversality of the nearby fibers and stable boundary property of the Milnor fibration and their relations.

1. Preliminary

Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a mixed function and write it as sum of real and imaginary part: $f = g + ih$. Writing $\mathbf{z} = (z_1, \dots, z_n)$ and $z_j = x_j + iy_j$ $(j = 1, \dots, n)$ with $x_j, y_j \in \mathbf{R}$, the mixed hypersurface $\{f = 0\}$ can be understood as the real analytic variety in $\mathbf{R}^{2n}$ defined by $\{g = h = 0\}$. The real and imaginary part g, h are also (real-valued) mixed functions and we also consider them as real analytic functions of variables $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$. By abuse of notations we use both notations $g(\mathbf{z}, \bar{\mathbf{z}})$ and $g(\mathbf{x}, \mathbf{y})$ etc. We recall some notations. The *real gradient vector* for a real-valued mixed function $k(\mathbf{x}, \mathbf{y})$ is defined as

$$\text{grad}\, k = (\text{grad}_{\mathbf{x}}\, k, \text{grad}_{\mathbf{y}}\, k) \in \mathbf{R}^{2n} \tag{1}$$

$$\text{grad}_{\mathbf{x}}\, k = (k_{x_1}, \dots, k_{x_n}), \quad \text{grad}_{\mathbf{y}}\, k = (k_{y_1}, \dots, k_{y_n}). \tag{2}$$

Here k_{x_i}, k_{y_j} are respective partial derivatives. $\mathbf{C}^n$ and $\mathbf{R}^{2n}$ are identified by $\mathbf{z} \leftrightarrow \mathbf{z}_{\mathbf{R}} = (\mathbf{x}, \mathbf{y})$. Under this identification, the Euclidean inner product in $\mathbf{R}^{2n}$ (denoted as $(*, *)_{\mathbf{R}}$) and the hermitian inner product in $\mathbf{C}^n$ (denoted as $(*, *)$) are related as $(\mathbf{z}_{\mathbf{R}}, \mathbf{z}'_{\mathbf{R}})_{\mathbf{R}} = \Re(\mathbf{z}, \mathbf{z}')$. For a mixed function k (not necessarily real-valued), we define also holomorphic and anti-holomorphic gradients as

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$$\operatorname{grad}_{\partial} k = \left(\frac{\partial k}{\partial z_1}, \dots, \frac{\partial k}{\partial z_n}\right),$$

$$\operatorname{grad}_{\bar\partial} k = \left(\frac{\partial k}{\partial \bar z_1}, \dots, \frac{\partial k}{\partial \bar z_n}\right).$$

For simplicity of notations, we use the following notations:

$$dk := \operatorname{grad} k, \quad d_{\mathbf{x}}k := \operatorname{grad}_{\mathbf{x}} k, \quad d_{\mathbf{y}}k := \operatorname{grad}_{\mathbf{y}} k,$$

$$\partial k := \operatorname{grad}_{\partial} k, \quad \bar\partial k := \operatorname{grad}_{\bar\partial} k.$$

Note that if k is real-valued,

$$\overline{\partial k} = \bar\partial k, \tag{3}$$

and real vector $dk \in \mathbf{R}^{2n}$ corresponds to the complex vector $2\overline{\partial k} \in \mathbf{C}^n$.

1.0.1. *Tangent spaces*. Let $k(\mathbf{z}, \bar{\mathbf{z}})$ is a real valued mixed function. Then the tangent space of a regular point $\mathbf{a}$ of $V_\eta := k^{-1}(\eta)$, $\eta \in \mathbf{R}$ is described as follows. For a complex vector $\mathbf{a} \in \mathbf{C}^n$, we denote the corresponding real vector as $\mathbf{a}_{\mathbf{R}} \in \mathbf{R}^{2n}$.

$$\begin{aligned} T_{\mathbf{a}} V_\eta &= \{\mathbf{v}_{\mathbf{R}} \in \mathbf{R}^{2n} \mid (\mathbf{v}_{\mathbf{R}}, dk(\mathbf{a}_{\mathbf{R}}))_{\mathbf{R}} = 0\} \\ &= \{\mathbf{v} \in \mathbf{C}^n \mid \Re(\mathbf{v}, \overline{\partial k}(\mathbf{a})) = 0\}. \end{aligned}$$

Consider the mixed hypersurface $V_\eta = f^{-1}(\eta)$, $\eta \neq 0$. We introduce two vectors in $\mathbf{C}^n$ which are more convenient to describe the Milnor fibration of the first type:

$$\mathbf{v}_1 := \overline{\partial \log f}(\mathbf{z}, \bar{\mathbf{z}}) + \bar\partial \log f(\mathbf{z}, \bar{\mathbf{z}}),$$

$$\mathbf{v}_2 := i(\overline{\partial \log f}(\mathbf{z}, \bar{\mathbf{z}}) - \bar\partial \log f(\mathbf{z}, \bar{\mathbf{z}})).$$

These vectors describe the respective tangent spaces at a regular point $\mathbf{a}$ of the real codimension 1 varieties

$$V_1 := \{\mathbf{z} \mid |f(\mathbf{z}, \bar{\mathbf{z}})| = |f(\mathbf{a}, \bar{\mathbf{a}})|\},$$

$$V_2 := \{\mathbf{z} \mid \arg f(\mathbf{z}, \bar{\mathbf{z}}) = \arg \eta\}.$$

Namely, we have shown (Lemma 30, Observation 32, [9])

$$T_{\mathbf{a}} V_1 := \{\mathbf{v} \mid \Re(\mathbf{v}, \mathbf{v}_1(\mathbf{a})) = 0\}$$

$$T_{\mathbf{a}} V_2 := \{\mathbf{v} \mid \Re(\mathbf{v}, \mathbf{v}_2(\mathbf{a})) = 0\}.$$

Note that $V_\eta = V_1 \cap V_2$. Observe that the two subspaces of dimension two

$$\langle \bar\partial g(\mathbf{a}, \bar{\mathbf{a}}), \bar\partial h(\mathbf{a}, \bar{\mathbf{a}}) \rangle_{\mathbf{R}}, \quad \langle \mathbf{v}_1(\mathbf{a}), \mathbf{v}_2(\mathbf{a}) \rangle_{\mathbf{R}}$$

are equal. In fact we have:

$$\mathbf{v}_1 = \frac{\overline{\partial f}}{\bar f} + \frac{\bar\partial f}{f} = \frac{1}{|f|^2}(f(\overline{\partial g} - i\overline{\partial h}) + \bar f(\bar\partial g + i\bar\partial h)) = \frac{1}{|f|^2}(2g\bar\partial g + 2h\bar\partial h)$$

$$\mathbf{v}_2 = i\frac{\overline{\partial f}}{\bar f} + \frac{\bar\partial f}{f} = \frac{i}{|f|^2}(f(\bar\partial g - i\bar\partial h) + \bar f(\bar\partial g + i\bar\partial h)) = \frac{1}{|f|^2}(-2h\bar\partial g - 2g\bar\partial h)$$

Proposition 1 ([8]). *Put $f = g + hi$ as before. The next conditions are equivalent.*

(1) $\mathbf{a} \in \mathbf{C}^n$ *is a critical point of the mapping* $f : \mathbf{C}^n \to \mathbf{C}$.
(2) $dg(\mathbf{a}_\mathbf{R})$, $dh(\mathbf{a}_\mathbf{R})$ *are linearly dependent over* $\mathbf{R}$.
(3) $\bar\partial g(\mathbf{a}, \bar{\mathbf{a}})$, $\bar\partial h(\mathbf{a}, \bar{\mathbf{a}})$ *are linearly dependent over* $\mathbf{R}$.
(4) *There exists a complex number α with $|\alpha| = 1$ such that* $\overline{\partial f}(\mathbf{a}, \bar{\mathbf{a}}) = \alpha\bar\partial f(\mathbf{a}, \bar{\mathbf{a}})$.

Under the above equivalent conditions, we say that $\mathbf{a}$ is a *mixed singular point* of the mixed hypersurface $f^{-1}(f(\mathbf{a}))$.

Lemma 2 (cf [3]). *Put $V_\eta = f^{-1}(\eta)$ and take $\mathbf{p} \in S_r \cap V_\eta$. Assume that $\mathbf{p}$ is a non-singular point of V_η and let $k(\mathbf{z}, \bar{\mathbf{z}})$ be a real valued mixed function. The following conditions are equivalent.*

(1) *The restriction $k|V_\eta$ has a critical point at $\mathbf{p} \in V_\eta$.*
(2) *There exists a complex number $\alpha \in \mathbf{C}^*$ such that* $\bar\partial k(\mathbf{p}) = \alpha\overline{\partial f}(\mathbf{p}, \bar{\mathbf{p}}) + \bar\alpha\bar\partial f(\mathbf{p}, \bar{\mathbf{p}})$.
(3) *There exist real numbers c, d such that*

$$\bar\partial k(\mathbf{p}) = c\bar\partial g(\mathbf{p}, \bar{\mathbf{p}}) + d\bar\partial h(\mathbf{p}, \bar{\mathbf{p}}).$$

(4) *There exist real numbers c', d' such that*

$$\bar\partial k(\mathbf{p}) = c'\bar{\mathbf{v}}_1(\mathbf{p}, \bar{\mathbf{p}}) + d'\mathbf{v}_2(\mathbf{p}, \bar{\mathbf{p}}).$$

Proof. As $\mathbf{p} \in V$ is assumed a non-singular point, (1) and (3) are equivalent. We show the implication (3) $\Rightarrow$ (2). Assume

$$\bar\partial k(\mathbf{p}) = c\bar\partial g(\mathbf{p}, \bar{\mathbf{p}}) + d\bar\partial h(\mathbf{p}, \bar{\mathbf{p}}), \quad \exists c, d \in \mathbf{R}.$$

We use the equality:

$$\bar\partial g(\mathbf{p}, \bar{\mathbf{p}}) = \frac{\bar\partial f(\mathbf{p}, \bar{\mathbf{p}}) + \overline{\partial f}(\mathbf{p}, \bar{\mathbf{p}})}{2}, \tag{4}$$

$$\bar\partial h(\mathbf{p}, \bar{\mathbf{p}}) = -\frac{i(\bar\partial f(\mathbf{p}, \bar{\mathbf{p}}) - \overline{\partial f}(\mathbf{p}, \bar{\mathbf{p}}))}{2} \tag{5}$$

to obtain the equality:

$$\bar\partial k(\mathbf{p}) = \frac{c - di}{2}\bar\partial f(\mathbf{p}, \bar{\mathbf{p}}) + \frac{c + di}{2}\overline{\partial f}(\mathbf{p}, \bar{\mathbf{p}}).$$

The implication (2) $\Rightarrow$ (3) can be shown similarly, using the equality

$$\partial f = \partial g + i\partial h, \quad \bar{\partial} f = \bar{\partial} g + i\bar{\partial} h. \tag{6}$$ □

1.0.2. *Newton boundary and strong non-degeneracy condition.* Let $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{\nu,\mu} c_{\nu\mu}\mathbf{z}^\nu \bar{\mathbf{z}}^\mu$ be a mixed polynomial. The Newton polygon $\Gamma_+(f)$ is defined by the convex hull of $\bigcup(\nu + \mu + \mathbf{R}_+^n)$ where the sum is taken for ν, μ with $c_{\nu\mu} \neq 0$. Newton boundary $\Gamma(f)$ is the union of compact faces of $\Gamma_+(f)$ as usual. f is called *convenient* if for any $i = 1, \dots, n$, $\Gamma(f)$ intersects with z_i-axis.

For any non-negative weight vector P, it defines a linear function ℓ_P on $\Gamma_+(f)$ by $\ell_P(\xi) = p_1\xi_1 + \cdots + p_n\xi_n$ where $P = {}^t(p_1, \dots, p_n)$, $\xi = (\xi_1, \dots, \xi_n) \in \Gamma_+(f)$ and the minimal value is denoted as $d(P)$ and the face where this minimal value is taken is denoted by $\Delta(P)$. In other word, $\Delta(P) := \{\xi \in \Gamma_+(f) \mid \sum_{i=1}^n p_i\xi_i = d(P)\}$. The face function associated by P is defined as $f_P := f_{\Delta(P)}$. For any coordinate subspace $\mathbf{C}^I$, we denote the restriction $f|\mathbf{C}^I$ as f^I as usual. Note that if P is strictly positive (i.e., $p_i > 0$, for any $i = 1, \dots, n$), $\Delta(P)$ is a face of $\Gamma(f)$.

To treat the case of non-convenient functions, we define *the modified Newton boundary* $\Gamma_{nc}(f)$ by adding essential non-compact faces Ξ. Here Ξ is called *an essential non-compact face* if there exists a semi-positive weight vector $P = {}^t(p_1, \dots, p_n)$ such that

(1) $\Delta(P) = \Xi$ with Ξ being a non-compact face and $f^{I(P)} \equiv 0$ where $I(P) = \{i \mid p_i = 0\}$ and
(2) for any $i \in I(P)$ and any point $\nu \in \Xi$, the half line starting from ν, $\nu + \mathbf{R}_+ E_i$ is contained in Ξ. Here E_i is the unit vector in the direction of i-th coordinate axis.

The weight vector P may not unique but $I(P)$ does not depend on P. Thus we denote it as $I(\Xi)$ and it is called *the non-compact direction* of Ξ. See

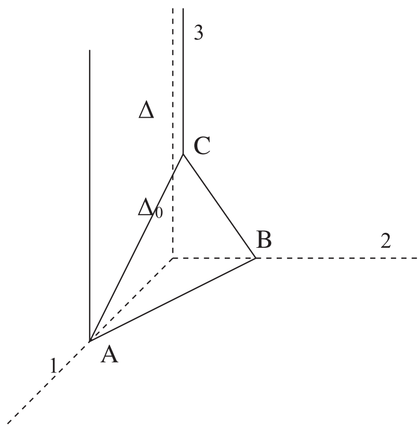

FIGURE 1. Non-compact face

Figure 1 which shows the modified Newton boundary of $f = z_1^3 + z_2^3 + z_2 z_3^2$ in Example 3.

f is called *strongly non-degenerate* if (1) for any compact face $\Delta \subset \Gamma_{nc}(f)$, the face function $f_\Delta := \sum_{\nu+\mu\in\Delta} c_{\nu\mu}\mathbf{z}^\nu\bar{\mathbf{z}}^\mu$ has no critical point as a function $f_\Delta : \mathbf{C}^{*n} \to \mathbf{C}$ and (2) for a non-comact face $\Delta \in \Gamma_{nc}(f)$, $f_{\Delta_0} : \mathbf{C}^{*n} \to \mathbf{C}$ has no critical point where $\Delta_0 = \Delta \cap \Gamma(f)$.

Example 3. Consider a holomorphic function $f = z_1^3 + z_2^3 + z_2 z_3^2$ of three variables. Note that $\Gamma_{nc}(f)$ has three vertices $A = (3,0,0)$, $B = (0,3,0)$, $C = (0,1,2)$ and the face $\Delta := \{\overline{AC} + \mathbf{R}_+ E_3\} \subset \Gamma_{nc}(f)$ where $\overline{AC}$ is the edge with endpoints A, C. The non-compact faces with edge $\overline{AB}$ and $\overline{BC}$ are not essential. They are not vanishing coordinates i.e., f does not vanish on $\{z_1 = z_3 = 0\}$ or $\{z_2 = z_3 = 0\}$. See Figure 1.

2. Milnor fibration

Asume that $f(\mathbf{z},\bar{\mathbf{z}}) = \sum_{\nu,\mu} c_{\nu\mu}\mathbf{z}^\nu\bar{\mathbf{z}}^\mu$ is a strongly non-degenerate mixed polynomial and let $V = f^{-1}(0)$. In this section, we study the Milnor fibration of f. If $f(\mathbf{z},\bar{\mathbf{z}})$ has a convenient Newton boundary, the singularity is isolated and there exists a spherical Milnor fibration (= a Milnor fibration of the first type):

$$f/|f| : S_r - K \to S^1, \quad K = V \cap S_r$$

and also a tubular Milnor fibration (= a Milnor fibration of the second type): $f : \partial E(r,\delta)^* \to S_\delta^1$ where $\partial E(r,\delta)^* = \{\mathbf{z} \in B_r \mid |f(\mathbf{z},\bar{\mathbf{z}})| = \delta\}$ for sufficiently small r, δ such that $0 < \delta \ll r$. They are C^∞-equivalent (Theorems 19, 33, 37, [9]).

For non-convenient mixed function, the singularity need not be isolated. We have proved the same assertion under an extra condition "super strongly non-degenerate" (Theorem 52, [9]). In this paper, we prove the existence of Milnor fibrations for any strongly non-degenerate functions with a weaker assumption than the assumption "super". We will study also some geometric properties behind the argument.

2.1. Smoothness of the nearby fibers. First we recall the following:

Lemma 4 (Lemma 28, [9]). *Assume that $f(\mathbf{z},\bar{\mathbf{z}})$ is a strongly non-degenerate mixed function. Then there exists a positive number r_0 and δ such that the fiber $V_\eta := f^{-1}(\eta)$ has no mixed singularity in the ball $B_{r_0}^{2n}$ for any non-zero η with $|\eta| \le \delta$.*

Proof. Though the proof is the same as that in [9], we repeat it for the beginner's convenience. We show a contradiction, assuming that the assertion does not hold. Then using the Curve Selection Lemma ([6, 4]), we can find an analytic path $\mathbf{z}(t)$, $0 \le t \le 1$ such that $\mathbf{z}(0) = O$ and $f(\mathbf{z}(t),\bar{\mathbf{z}}(t)) \ne 0$ and $\mathbf{z}(t)$

is a critical point of the function $f : \mathbf{C}^n \to \mathbf{C}$ for any $t \neq 0$. Using Proposition 1, we can find a real analytic family $\lambda(t)$ in $S^1 \subset \mathbf{C}$ such that

$$\overline{\partial f}(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = \lambda(t)\bar{\partial} f(\mathbf{z}(t), \bar{\mathbf{z}}(t)). \tag{7}$$

Put $I = \{j \mid z_j(t) \not\equiv 0\}$. We may assume for simplicity that $I = \{1, \dots, m\}$ and we consider the restriction $f^I = f \mid \mathbf{C}^I$. As $f(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = f^I(\mathbf{z}(t), \bar{\mathbf{z}}(t)) \not\equiv 0$, we see that $f^I \neq 0$. Consider the Taylor expansions of $\mathbf{z}(t)$ and $\lambda(t)$:

$$\mathbf{z}_i(t) = b_i t^{a_i} + (\text{higher terms}), \quad b_i \neq 0,\ a_i > 0,\ i = 1, \dots, m$$
$$\lambda(t) = \lambda_0 + (\text{higher terms}), \quad \lambda_0 \in S^1 \subset \mathbf{C}.$$

Consider the weight vector $A = {}^t(a_1, \dots, a_m)$ and a point in the torus $\mathbf{b} = (b_1, \dots, b_m) \in \mathbf{C}^{*I}$ and we consider the face function f_A^I of $f^I(\mathbf{z}, \bar{\mathbf{z}})$. Then we have for $j \in I$

$$\frac{\partial f}{\partial z_j}(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = \frac{\partial f_A^I}{\partial z_j}(\mathbf{b}, \bar{\mathbf{b}}) t^{d - a_j} + (\text{higher terms}),$$
$$\frac{\partial f}{\partial \bar{z}_j}(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = \frac{\partial f_A^I}{\partial z_j}(\mathbf{b}, \bar{\mathbf{b}}) t^{d - a_j} + (\text{higher terms})$$

where $d = d(A; f^I)$. The equality (7) says that

$$\overline{\frac{\partial f^I}{\partial z_j}}(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = \lambda(t) \frac{\partial f^I}{\partial \bar{z}_j}(\mathbf{z}(t), \bar{\mathbf{z}}(t)), \quad j = 1, \dots, m.$$

which implies the next equality:

$$\operatorname{ord}_t \overline{\frac{\partial f^I}{\partial z_j}}(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = \operatorname{ord}_t \frac{\partial f^I}{\partial \bar{z}_j}(\mathbf{z}(t), \bar{\mathbf{z}}(t)), \quad j = 1, \dots, m.$$

Thus we get the equality:

$$\overline{\partial f_A^I}(\mathbf{b}, \bar{\mathbf{b}}) = \lambda_0 \bar{\partial} f_A^I(\mathbf{b}, \bar{\mathbf{b}}), \quad \mathbf{b} \in \mathbf{C}^{*m}.$$

This implies that $\mathbf{b}$ is a critical point of $f_A^I : \mathbf{C}^{*I} \to \mathbf{C}$, which is a contradiction to the strong non-degeneracy of $f_A^I(\mathbf{z}, \bar{\mathbf{z}})$. □

2.2. Vanishing coordinate subspaces and essentially non-compact face functions. We assume that f is a mixed polynomial (not only mixed analytic function). We denote by $\mathscr{I}_{nv}(f)$ the set of subset $I \subset \{1, 2, \dots, n\}$ such that $f^I \not\equiv 0$ (we denoted this set as $\mathscr{N}\mathscr{V}(f)$ in [9]). We denote by $\mathscr{I}_v(f)$ the set of subset $I \subset \{1, 2, \dots, n\}$ such that $f^I \equiv 0$, and for $I \in \mathscr{I}_v(f)$ and we consider also the set of non-compact faces $\Delta \in \Gamma_{nc}(f)$ such that there exists (possibly not unique) a non-negative weight P such that $\Delta(P) = \Delta$ and $I(P) = I$. Here $I(P) = \{i \mid p_i = 0\}$. $\mathbf{C}^I$ is called *a vanishing coordinate subspace*. Note that $\mathbf{C}^I \subset V$.

Definition 5. Let $\pi_I : \mathbf{C}^n \to \mathbf{C}^I$ be the projection and put $\mathbf{z}_I = \pi_I(\mathbf{z})$. Take an essential non-compact face $\Delta \in \Gamma_{nc}(f)$. Take a weight function P such that $f_P = f_\Delta$ and $I(P) = I(\Delta)$. We consider the function $\rho_\Delta(\mathbf{z}) := \|\mathbf{z}_{I(\Delta)}\|^2 = \sum_{j \in I(\Delta)} |z_j|^2$. An essential non-compact face function f_Δ is *locally tame* if there exists a positive number $r_\Delta > 0$ such that for any fixed $\mathbf{z}_{I(\Delta)} = \mathbf{a}_{I(\Delta)} \in \mathbf{C}^{*I(\Delta)}$ with $\rho_\Delta(\mathbf{z}) \le r_\Delta^2$, f_Δ has no critical points in $\mathbf{a}_{I(\Delta)} \times \mathbf{C}^{*I(\Delta)^c}$ as a mixed polynomial function of $n - |I(\Delta)|$-variables $\{z_k \mid k \notin I(\Delta)\}$. We say that f *is locally tame on the vanishing coordinate subspace* $\mathbf{C}^I$ if any face function f_Δ with $I(\Delta) = I$ is locally tame. We say that f *is locally tame along vanishing coordinate subspaces* if f is locally tame on every vanishing coordinate subspaces $\mathbf{C}^I$, $\forall I \in \mathscr{I}_v$. This is slightly weaker condition than "super strongly non-degenerate" in [9].

Put $r_I = \min\{r_\Delta \mid I(\Delta) = I\}$ for $I \in \mathscr{I}_v(f)$ and $r_{nc} = \min\{r_I \mid I \in \mathscr{I}_v(f)\}$. If f is convenient, $r_{nc} = +\infty$.

Remark 6. We say that f is "super strongly non-degenrate" if we can take $r_\Delta = \infty$ in the above definition ([9]).

2.3. Smoothness on the non-vanishing coordinate subspaces. Take $I \subset \{1, \dots, n\}$ and $\mathbf{C}^I$ is called a non-vanishing coordinate subspace if $f^I \not\equiv 0$. Put $V^\# = \bigcup_{I \in \mathscr{I}_{nv}(f)} V \cap \mathbf{C}^{*I}$. Then there exists a $r_0 > 0$ so that $V^\#$ and $V^{*I} = V \cap \mathbf{C}^{*I}$ are non-singular in the ball B_{r_0} and for any $0 < r \le r_0$, the sphere S_r and V^{*I} intersect transversely. The existence of such r_0 is shown in Theorem 16, [9].

2.4. Hamm-Lê type theorem. The following is a mixed function version of Lemma (2.1.4) (Hamm-Lê, [5]). This enables us to prove the existence of Milnor fibration with locally tame behavior assumption.

Lemma 7. *Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a strongly non-degenerate mixed polynomial which behaves locally tamely along vanishing coordinate subspaces. Put $\rho_0 = \min\{r_{nc}, r_0\}$ where r_{nc} and r_0 are described above. For any fixed positive number $r_1 \le \rho_0$, there exists positive numbers $\delta(r_1)$ (depending on r_1) such that for any non-zero $\eta, |\eta| \le \delta(r_1)$,*

(1) *the nearby fiber $V_\eta := f^{-1}(\eta)$ has no mixed singularity in the ball $B_{\rho_0}^{2n}$ and,*
(2) *for any $r, r_1 \le r \le \rho_0$, the sphere S_r and the nearby fiber $V_\eta = f^{-1}(\eta)$ intersect transversely.*

Proof. We have already proved the assertion (1) (Lemma 4). So we will prove the assertion (2). Assume that the assertion is false. By the Curve Selection Lemma, we can find a real analytic curve $\mathbf{z}(t)$ and a complex valued function $\alpha(t)$, $0 \le t \le 1$

$$z_j(t) = \alpha(t)\overline{\frac{\partial f}{\partial z_j}}(\mathbf{z}(t)) + \bar{\alpha}(t)\frac{\partial f}{\partial \bar{z}_j}(\mathbf{z}(t)),\ \forall j \tag{8}$$

where $\mathbf{z}(t)$, $\alpha(t)$ are expanded as

$$z_j(t) = b_j t^{p_j} + (\text{higher terms}), \quad b_j \neq 0 \text{ if } z_j(t) \not\equiv 0,$$
$$\alpha(t) = \alpha_0 t^m + (\text{higher terms}), \quad \alpha_0 \neq 0.$$

and $f(\mathbf{z}(t)) \neq 0$ for $t \neq 0$. Obviously $\alpha(t) \neq 0$.

Put $K = \{i \mid z_j(t) \not\equiv 0\}$ and we consider the equality in $\mathbf{C}^K$. Put $\mathbf{b} = (b_j)$ and $P = (p_j)$, $I = \{j \in K \mid p_j = 0\}$, $I_1 = K - I$ and $\Delta = \Delta(P)$. In the following, we assume $K = \{1, \dots, n\}$ as the argument is the same.

Case 1. Assume that $I \in \mathscr{I}_{nv}(f)$. Then $f^I \not\equiv 0$ and $\mathbf{b} \in V^{\#}$. We assumed that $V^{\#}$ and $S_{\|\mathbf{b}\|}$ intersect transversely for any $\mathbf{b}$, $\|\mathbf{b}\| \leq \rho_0$ and thus $S_{\|\mathbf{z}(t)\|}$ is also transverse to $V_{f(\mathbf{z}(t))}$ at $\mathbf{z}(t)$ for a small $t \ll 1$, which is a contradiction.

Case 2. Assume that $I \in \mathscr{I}_v(f)$ and so $f^I \equiv 0$. In this case, $\Delta \in \Gamma_{nc}(f)$. The above equality (8) says:

$$(9) \quad b_j t^{p_j} + (\text{higher terms}) = \left(\alpha_0 \overline{\frac{\partial f_{\Delta(P)}}{\partial z_j}}(\mathbf{b}) t^{m+d(P)-p_j} + (\text{higher terms}) \right) + \left(\bar{\alpha}_0 \frac{\partial f_{\Delta(P)}}{\partial \bar{z}_j}(\mathbf{b}) t^{m+d(P)-p_j} + (\text{higher terms}) \right), \quad j \in K.$$

We compare the order in t (= the lowest degree) of the both side. The left side has order 0 and the order of the right side is at least $d(P) + m - p_j$ for $j \notin I$ and at least $d(P) + m$ for $j \in I$. Note that $\mathbf{b} \in \mathbf{C}^{*n}$. If $d(P) + m > 0$, we get a contradiction $b_j = 0$ for $j \in I$. If $d(P) + m < 0$, we get

$$0 = \alpha_0 \overline{\frac{\partial f_{\Delta(P)}}{\partial z_j}}(\mathbf{b}) + \bar{\alpha}_0 \frac{\partial f_{\Delta(P)}}{\partial \bar{z}_j}(\mathbf{b}), \ \forall j$$

which says $\mathbf{b}$ is a mixed critical point of f_Δ, a contradiction to the strong non-degeneracy. Thus $d(P) + m = 0$ and

$$(10) \qquad b_j = \alpha_0 \overline{\frac{\partial f_{\Delta(P)}}{\partial z_j}}(\mathbf{b}) + \bar{\alpha}_0 \frac{\partial f_{\Delta(P)}}{\partial \bar{z}_j}(\mathbf{b}), \quad j \in I$$

$$(11) \qquad 0 = \alpha_0 \overline{\frac{\partial f_{\Delta(P)}}{\partial z_j}}(\mathbf{b}) + \bar{\alpha}_0 \frac{\partial f_{\Delta(P)}}{\partial \bar{z}_j}(\mathbf{b}), \quad j \in K - I.$$

The equality (11) says that the point $(b_j)_{j \in K-I}$ is a critical point of the face function f_Δ as a function of varianles $\{z_j, j \in K - I\}$, fixing $z_i = b_i$, $i \in I$ with $\rho_\Delta(\mathbf{b}) \leq \rho_0$. This is a contradiction on the assumption. □

Remark 8. The assertion (2) also follows from a_f-condition (see Proposition 11 below.)

2.5. Tubular Milnor fibration. Put

$$D(\delta_0)^* = \{\eta \in \mathbf{C} \mid 0 < |\eta| \le \delta_0\}, \quad S^1_{\delta_0} = \partial D(\delta_0)^* = \{\eta \in \mathbf{C} \mid |\eta| = \delta_0\}$$
$$E(r,\delta_0)^* = f^{-1}(D(\delta_0)^*) \cap B^{2n}_r, \quad \partial E(r,\delta_0)^* = f^{-1}(S^1_{\delta_0}) \cap B^{2n}_r.$$

By Lemma 4 and the theorem of Ehresman ([16]), we obtain the following description of the tubular Milnor fibration (i.e., the Milnor fibration of the second type) ([5]).

Theorem 9 (Tubular Milnor fibration). *Assume that $f(\mathbf{z},\bar{\mathbf{z}})$ is a strongly non-degenerate mixed function which is locally tame along the vanishing coordinate subspaces. Take positive numbers $r \le \rho_0$ and $\delta_0 \le \delta(r)$ as in Lemma 7. Then $f : E(r,\delta_0)^* \to D(\delta_0)^*$ and $f : \partial E(r,\delta_0)^* \to S^1_{\delta_0}$ are locally trivial fibrations and the topological isomorphism class does not depend on the choice of δ_0 and r.*

2.6. Spherical Milnor fibration. Consider the spherical Milnor fibration (i.e., Milnor fibration of the first kind):

$$f/|f| : S_r - K \to S^1, \quad K = V \cap S_r.$$

In the proof of the existence of the spherical fibration and the equivalence to the tubular Milnor fibration (Theorem 52, [9]), we have assumed "super strongly non-degeneracy". However this assumption is used only to prove the Hamm-Lê type assertion (Lemma 51, [9]). We have proved this Lemma with locally tameness assumption (Lemma 7). Thus we get

Theorem 10. *Assume that f is a strongly non-degenerate mixed function which is locally tame along vanishing coordinate subspaces. For a sufficiently small r, the spherical and tubular Minor fibrations exist and they are equivalent each other.*

3. Boundary stability, a_f-condition and transversality of the nearby fibers

In this section, we consider further geometric properties about mixed polynomials.

3.1. a_f-condition. Assume that f is a mixed polynomial and we assume that a Whitney regular stratification $\mathscr{S}$ of $\mathbf{C}^n$ is given so that $V = f^{-1}(0)$ is a union of strata $M \subset V$. We says that f *satisfies Thom's a_f-condition with respect to $\mathscr{S}$* (locally at $\mathbf{0}$) if there exist positive numbers r and $\delta \ll r$ which satisfies the following condition. *$V_\eta = f^{-1}(\eta)$ with $\eta \ne 0$, $|\eta| \le \delta$ is smooth in B_r and take any sequence $\mathbf{z}^{(\nu)}$ which converges to some $\mathbf{w} \ne \mathbf{0}$, $\mathbf{w} \in M$, where M is a stratum in $V \cap \mathscr{S}$ and suppose that the tangent space $T_{\mathbf{z}(\nu)} f^{-1}(f(\mathbf{z}^{(\nu)}))$ converges to some τ in the suitable Grassmanian space. Then $T_{\mathbf{w}} M$ is a subspace of τ.*

The following says that the nearby fiber's transversality follows from a_f-condition.

PROPOSITION 11. *Assume that f satisfies a_f-condition at* $\mathbf{0}$ *and the nearby fibers are smooth. Then there exists a $r_0 > 0$ such that for any $0 < r_1 \le r_0$, there exists a positive δ so that any nearby fiber V_η intersects transversely with the sphere S_r for $r_1 \le r \le r_0$ and $0 < |\eta| \le \delta$.*

Proof. Take r_0 so that for any $r \le r_0$, the sphere S_r intersects transversely with all strata $M \subset V$. Note that M and S_r intersect transversely if and only if for any $\mathbf{a} \in M \cap S_r$, $T_{\mathbf{a}}M$ and $T_{\mathbf{a}}S_r$ intersect transversely. That is $T_{\mathbf{a}}M \not\subset T_{\mathbf{a}}S_r$. Take a sequence of points $\mathbf{z}^{(\nu)}$ converging to $\mathbf{a} \in M \subset V$ where M is a stratum and $\mathbf{a} \neq 0$. Put $\eta_\nu = f(\mathbf{z}^{(\nu)})$ and $r_\nu = \|\mathbf{z}^{(\nu)}\|$ and $r' := \|\mathbf{a}\|$, $r_0 \ge r' \ge r_1$. Assume that V_{η_ν} intersects S_{r_ν} non-transversely at $\mathbf{z}^\nu$. Then this implies $T_{\mathbf{z}^{(\nu)}}f^{-1}(\eta_\nu) \subset T_{\mathbf{z}^{(\nu)}}S_{r_\nu}$. Assume that $T_{\mathbf{z}^{(\nu)}}f^{-1}(\eta_\nu)$ converges to τ. Then $\tau \subset T_{\mathbf{a}}S_{r'}$. On the other hand, a_f-condition says that $T_{\mathbf{a}}M \subset \tau$ and $T_{\mathbf{a}}M \not\subset T_{\mathbf{a}}S_{r'}$. This is a contradiction. □

3.2. Boundary stability condition. Assume that $r_0 > 0$ is chosen so that $\varphi = f/|f| : S_r \backslash K \to S^1$ is a fibration for any $r \le r_0$. We wish to consider the boundary condition $\bar{F}_\theta \supset K$ is satisfied or not. This property is always true for holomorphic functions but not always true for mixed functions. For the argument's simplicity, we consider as follows. Consider the Milnor fibration in a open ball:

$$\varphi_{\le r} = f/|f| : B_r - V \to S^1, \quad \varphi_{\le r}(\mathbf{z}) = f(\mathbf{z})/|f(\mathbf{z})| \tag{12}$$

and put $F_{\theta,\le r} = \varphi_{\le r}^{-1}(e^{i\theta})$. To distinguish this fibration with usual Milnor fibration on a sphere, we call this fibration *an open ball Milnor fibration*.

DEFINITION 12. We say the open Milnor fibration satisfies *the stable boundary condition* if $\overline{F_{\theta,\le r}} \supset V \cap B_r$ for any θ. Note that the Milnor fibration in a ball is homotopically equivalent to the one on a fixed sphere $f/|f| : S_r \backslash K \to S^1$.

Recall that a continuous mapping $\varphi : X \to Y$ is *an open mapping along a subset $A \subset X$* if for any point $a \in A$ and any open neighborhood U of a in X, $\varphi(U)$ is a neighborhood of $\varphi(a)$ in Y. The following is an immediate consequence of the definition.

PROPOSITION 13. *The next two conditions are equivalent.*

(1) *The boundary stability condition for the Milnor fibration of f is satisfied.*

(2) *$f : \mathbf{C}^n \to \mathbf{C}$ is an open mapping along $V \cap B_r$ for a sufficiently small $r > 0$.*

In particular, if f is a holomorphic function, it satisfies the boundary stability condition.

Lemma 14. *Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a strongly non-degenerate and locally tame along vanishing coordinate subspaces. Then the Milnor fiibration satisfies the stable boundary property.*

Proof. Take a point $\mathbf{a} = (a_1, \dots, a_n) \in V \cap \mathrm{Int}(B_r)$ and put $I = \{i \,|\, a_i \neq 0\}$.

(i) Assume that $I \in \mathscr{I}_{nv}(f)$ so that $\mathbf{a}$ is a non-singular point of V^{*I}. Then it is obvious that $\mathbf{a} \in \overline{F_\theta}$, as $\{V_\eta\}$, $|\eta| \le \delta \ll r$ is a transversal family with the spheres $S_{r'}$ for $\|\mathbf{a}\|/2 \le r' \le r$ and $V_\eta \subset F_{\theta,\le r}$ for η, $\arg \eta = \theta$.

(ii) Assume that $f^I \equiv 0$. Take an essential non-compact face $\Delta = \Delta(P)$ with $I(\Delta) = I$ and consider the face function $f_P(\mathbf{z}, \bar{\mathbf{z}})$. Put f_{P,a_I} be the restriction of f_P on $z_i = \mathbf{a}_i$, $i \in I$. Thus we consider the polynomial mapping $f_{P,a_I} : \mathbf{C}^{n-|I|} \to \mathbf{C}$. As $f_{P,\mathbf{a}_I}$ is a strongly non-degenerate function for sufficiently small $\mathbf{a}_I$, there exists $\mathbf{b} = (b_j)_{j \notin I}$ such that $f_{P,\mathbf{a}_I}(\mathbf{b}) = \rho e^{i\theta}$ for some ρ. Take an arc $\mathbf{b}(s)$, $-\varepsilon \le s \le \varepsilon$ so that $f_{P,\mathbf{a}_I}(\mathbf{b}(s)) = \rho e^{i(\theta+s)}$ and $\mathbf{b}(0) = \mathbf{b}$. This is possible as $f_{P,\mathbf{a}_I} : \mathbf{C}^{n-|I|} \to \mathbf{C}$ is a submersion. Consider the path:

$$(t,s) \mapsto \mathbf{b}(t,s) = (b_j(t,s))_{j=1}^n, \quad b_j(t,s) = \begin{cases} b_j(s)t^{p_j}, & j \notin I \\ a_j, & j \in I. \end{cases}$$

Then we have

$$\begin{aligned} f(\mathbf{b}(t,s)) &= f_{P,\mathbf{a}_I}(\mathbf{b}(s))t^{d(P)} + (\text{higher terms}) \\ &= \rho e^{i(\theta+s)} t^{d(P)} + (\text{higher terms}). \end{aligned}$$

Take a sequence $t_\nu \to 0$. As the $\arg f(\mathbf{b}(t_\nu, s)) \to \theta + s$, we can take a sequence s_ν, $-\varepsilon \le s_\nu \le \varepsilon$ such that $\arg f(\mathbf{b}(t_\nu, s_\nu)) = \theta$ for sufficiently small $|t_\nu|$. For example, assume that $\arg f(\mathbf{b}(t,0)) < \theta$. Note that $\arg f(b(t,\varepsilon)) > \theta$ as long as $t \ll 1$. Thus we use the mean value theorem to chose such a s_ν. The point $\mathbf{b}(t_\nu, s_\nu) \in F_{\theta,\le r}$ for sufficiently small $|t_\nu|$ and it converges to $\mathbf{a}$. This implies that the closure of $F_{\theta,\le r}$ contains V. □

3.3. Strongly non-degenerate polynomials which is not locally tame. (1) **Example 1**. Consider the example of M. Tibar: $f(\mathbf{z}) = z_1|z_2|^2$ ([13, 1, 2]). This is a mixed weighted homogeneous polynomial. Thus it is strongly non-degenerate. A polar weight can be $P = {}^t(1,0)$. S^1-action is defined as $\rho \circ (z_1, z_2) = (z_1\rho, z_2)$ for $\rho \in S^1$. Then for any $r > 0$, there exists a spherical Milnor fibration: $\varphi = f/|f| : S_r \backslash K \to S^1$.

First we show that the boundary stability is not satisfied. Take a fiber F_θ. K has two components, $K_1 = \{z_1 = 0\}$ and $K_2 = \{z_2 = 0\}$. The closure of F_θ is given as $\bar{F}_\theta = F_\theta \cup K_1 \cup \{(re^{i\theta}, 0)\}$. Thus the intersection $\bar{F}_\theta \cap K_2$ is a single point $(re^{i\theta}, 0)$ and this point $(re^{i\theta}, 0)$ turns along K_2 once when θ goes from 0 to 2π. Note that K_2 is a S^1-orbit of the action. We call K_2 *a rotating axis*. The function f is not locally tame along the vanishing axis $z_2 = 0$ by Lemma 14. In fact, take a point $(a, 0) \in K_2$ and put $a = \rho e^{i\theta}$. Take an open set $U = \{z_1 \,|\, |z_1 - a| < \varepsilon\} \times \{z_2 \,|\, |z_2| < \varepsilon\}$ and put α to be the small positive angle so that $\tan\alpha = \varepsilon/\rho$. Then the image of U by f is contained in the closure of the angular region $\{\eta \in \mathbf{C} \mid \theta - \alpha \le \arg\eta \le \theta + \alpha\}$ where 0 is on the boundary. Thus it is not an open mapping. More precisely we assert

ASSERTION 15. *$\overline{F_\theta}$ is homeomorphic to* Cone(K_1).

For example, taking $r = 1$, consider the mapping $\psi : \overline{F_\theta} \to \mathrm{Cone}(K_1)$, defined by $\psi(z_1, z_2) = (1 - |z_1|, \arg(z_2))$. Here we understand

$$\mathrm{Cone}(K_2) = [0,1] \times K_2/\{0\} \times K_2, \quad K_2 \simeq S^1.$$

M. Tibar observed that f does not have any stratification which satisfies the a_f-condition along z_1 axis ([11]). Put $f = g + ih$ with $g = x_1(x_2^2 + y_2^2)$ and $h = y_1(x_2^2 + y_2^2)$. Then the Jacobian matrix is given as

$$J(g,h) = \begin{pmatrix} x_2^2 + y_2^2 & 0 & 2x_1x_2 & 2x_1y_2 \\ 0 & x_2^2 + y_2^2 & 2y_1x_2 & 2y_1y_2 \end{pmatrix}$$

Note that the last 2×2 minor has rank one and this makes the problem at the limit. Take a point $p = (a_1 + ib_1, 0)$. Consider the rotated mixed polynomial $\tilde f := (b_1 + a_1 i)f$ and write it as $\tilde f = \tilde g + i\tilde h$. Note that $f^{-1}(f(p)) = \tilde f^{-1}(\tilde f(p))$ and $\tilde g = b_1 g - a_1 h$. Then the normalized gradient of $\tilde g$ is given by

$$\mathrm{grad}\, \tilde g = (b_1, -a_1, 0, 0).$$

Put $p = (a_1 + b_1 i, z_2)$. Thus when $z_2 \to 0$,

$$T_p f^{-1}(f(p)) \subset T_p \tilde g^{-1}(\tilde g(p)) \not\supset \mathbf{C} \times \{0\}.$$

This implies, if there is a stratification which satisfies a_f-condition, the stratum of $\mathbf{C} \times \{0\}$ which contains p can not be two dimensional at $p \in \{z_2 = 0\}$. As this is the case at any point of $\{z_2 = 0\}$, there does not exist any stratification which satisfies a_f-condition. On the other hand, we assert that

PROPOSITION 16. *f satisfies the transversality condition for the nearby fibers.*

Proof. We may assume that the sphere has radius 1, by the polar homogenuity. Assume that there is a sequence $p_\nu = (u_\nu, v_\nu) \in S_1^3$ such that $f^{-1}(f(p_\nu))$ is not transverse to S_1^3 and $f(p_\nu) \to 0$. Then either $u_\nu \to 0$ or $v_\nu \to 0$ (equivalently either $|v_\nu| \to 1$ or $|u_\nu| \to 1$). We may assume that $p_\nu = \alpha_\nu \overline{\partial f} + \bar\alpha \overline{\bar\partial f}$ by Lemma 2 which is equivalent to

$$\begin{cases} u_\nu = \alpha_\nu |v_\nu|^2 \\ v_\nu = \alpha_\nu \bar u_\nu v_\nu + \bar\alpha_\nu u_\nu v_\nu. \end{cases}$$

From the first equality, we can put $u_\nu = r_\nu e^{i\theta_\nu}$, $\alpha = \rho_\nu e^{i\theta_\nu}$. The second equality says that $1 = 2\rho_\nu r_\nu$ as $v_\nu \ne 0$. Thus $\rho_\nu \to 1/2$ if $r_\nu \to 1$ which implies $|v_\nu| \to 2$ and $|f(p_\nu)| \not\to 0$. Assume that $r_\nu \to 0$. Then $|v_\nu|^2 = r_\nu/\rho_\nu = 2r_\nu^2 \to 0$. This is also impossible, as $|p_\nu| = 1$. □

This example shows that the transversality of nearby fibers does not implies either tameness or a_f-condition. On the other hand, tameness with strong non-degeneracy implies transversality of the nearby fibers, as we will see below.

(2) Example of A. Parusinski: $f = z_1(z_2 + z_3^2)\bar{z}_2$ ([11], see also [1, 2]). Note that f is strongly non-degenerate.

PROPOSITION 17 (A. Parusinski). *Consider $I = \{1\}$ and note that $f \mid \mathbf{C}^I \equiv 0$. Then f does not satisfy a_f-condition along z_1-axis $\{z_2 = z_3 = 0\}$.*

Proof. The proof goes in the same line as that in Example 1. Consider the weight $P = {}^t(0, 1, 3)$. Then $f_P = z_1|z_2|^2$ and $d(P) = 2$. Assume that there exists a stratification $\mathscr{S}$ satisfying a_f-condition. We show a contradiction. Take a point $p = (re^{i\theta}, 0, 0)$ and assume that $p \in M$ where M is a real two dimensional stratum of $\mathbf{C}^I$. Consider the modified function $\tilde{f} = (\sin\theta + i\cos\theta)f$. Then the real part $\tilde{g}$ of $\tilde{f}$ is given as

$$\begin{aligned}\tilde{g} &= \sin\theta g - \cos\theta h\\ &= (x_1 \sin\theta - y_1 \cos\theta)|z_2|^2 + \Re(e^{i(\pi/2-\theta)} z_1 \bar{z}_2 z_3^2)\end{aligned}$$

and the gradient vector of $\tilde{g}$ at $\mathbf{z}(t) := (p, ta_2, t^3 a_3)$ for $a_2, a_3 \in \mathbf{C}^*$ fixed is given as

$$\begin{aligned}\operatorname{grad} \tilde{g}(p, ta_2, t^3 a_3) = (\sin\theta, -\cos\theta, 0, 0, 0, 0)|a_2|^2 t^2\\ + O(t^3).\end{aligned}$$

Thus the normalized gradient vector converges to

$$\mathbf{v} := (\sin\theta, -\cos\theta, 0, 0, 0, 0).$$

This implies that

$$T_{\mathbf{z}(t)} f^{-1}(f(\mathbf{z}(t)) \subset T_{\mathbf{z}(t)} \tilde{g}^{-1}(\tilde{g}(\mathbf{z}(t)) \xrightarrow{t \to 0} \mathbf{v}^{\perp} \not\supset \mathbf{C}^I.$$

This is a contradiction. □

Remark 18. We do not know (and do not care) if $f^{-1}(\eta)$, $\eta \neq 0$ is a transverse family for sufficiently small η.

(3) Example 3. Consider

$$f(\mathbf{z}, \bar{\mathbf{z}}) = z_1 k(\mathbf{z}), \quad k(\mathbf{z}) := \sum_{i=1}^{m} |z_i|^{2a_i} - \sum_{j=m+1}^{n} |z_j|^{2a_j}$$

for $2 \le m < n$. Then f is not strongly non-degenerate but polar weighted homogeneous and it has a Milnor fibration. However it is not locally tame along the vanishing coordinate subspaces and f does not satisfy the a_f-condition.

In fact the link has two components $K_1 = \{z_1 = 0\}$ and $K_2 = \{k(\mathbf{z}) = 0\}$. The component K_2 has **real codimension 1** and at any point of $K_2 \backslash K_1$, f is not open mapping and thus

$$\overline{F_\theta} = F_\theta \cup K_1 \cup \{\mathbf{z} \in S_r \mid \arg z_1 = \pm\theta\}$$

where sign is the same as that of $k(\mathbf{z})$. Thus K_2 is a rotation axis. The monodromy is the rotation arround z_1 axis:

$$h_\theta : F_0 \to F_\theta, \quad (z_1, \mathbf{z}') \mapsto (z_1 e^{i\theta}, \mathbf{z}').$$

The fiber F_θ has two components, $F_\theta^+ = \{\arg z_1 = \theta, k(\mathbf{z}) > 0\}$ and $F_\theta^- = \{\arg z_1 = -\theta, k(\mathbf{z}) < 0\}$.

Remark 19. The function $k(\mathbf{z})$ is a real valued polynomial and the fibers $k^{-1}(\eta)$ are smooth for $\eta \neq 0$ and $k^{-1}(0)$ has an isolated singularity as a real hypersurface. However as a mixed function $k : \mathbf{C}^n \to \mathbf{C}$, it has no regular points.

3.4. Thom's a_f-condition. By analyzing above examples, we notice that the limit of two independent hyperplanes $T_p g^{-1}(g(p))$ and $T_p h^{-1}(h(p))$ may not independent when p goes to some point of vanishing coordinate $\mathbf{C}^I$, and this phenomena induces a failure of a_f-condition. This problem does not occur under the tameness condition.

Theorem 20. *Assume that $f(\mathbf{z})$ is a strongly non-degenerate polynomial and assume that f is locally tame along vanishing coordinate subspaces. We consider the canonical stratification $\mathscr{S}_{can}$ which is defined by*

$$\mathscr{S}_{can} : \{V \cap \mathbf{C}^{*I}, \mathbf{C}^{*I} \backslash V \cap \mathbf{C}^I \mid I \in \mathscr{I}_{nv}(f)\} \cup \{\mathbf{C}^{*I} \mid I \in \mathscr{I}_v(f)\}.$$

Then f satisfies a_f-condition with respect to $\mathscr{S}_{can}$ in the ball $B_{\rho_0}^{2n}$ where ρ_0 is as in Lemma 7.

Proof. Take a point $\mathbf{q}^I = (q_j)_{j \in I} \in V \cap \mathbf{C}^{*I}$. Using Curve Selection Lemma, it is enough to check the a_f-condition along an arbitrary analytic path. So take any analytic path $\mathbf{z}(t)$ such that $\mathbf{z}(0) = \mathbf{q}^I$ and $\mathbf{z}(t) \in \mathbf{C}^{*J}$ for $t \neq 0$ with $I \subset J$ with $I \neq J$. As the argument is precisely the same, we assume hereafter that $J = \{1, \dots, n\}$. We will show that a_f-condition is satisfied for this curve. By non-degeneracy, we may assume that $I \in \mathscr{I}_v(f)$ so that $\mathbf{C}^I$ is a vanishing coordinate. (Otherwise, $\mathbf{q}^I$ is a smooth point of V and the a_f-condition is obviously satisfied.) Consider the Taylor expansion:

$$z_j(t) = a_j t^{p_j} + (\text{higher terms}), \quad \begin{cases} p_j = 0,\ a_j = q_j, & j \in I \\ p_j > 0, & j \notin I. \end{cases}$$

Put $P = {}^t(p_1, \dots, p_n)$, $\mathbf{a} = (a_1, \dots, a_n)$, $d = d(P)$ and $\Delta = \Delta(P)$. For notation's simplicity, we assume that $I = \{m+1, \dots, n\}$. Note that

$$\frac{\partial g}{\partial \bar{z}_j}(\mathbf{z}(t)) = \frac{\partial g_\Delta}{\partial \bar{z}_j}(\mathbf{a})t^{d-p_j} + (\text{higher terms})$$

$$\frac{\partial h}{\partial \bar{z}_j}(\mathbf{z}(t)) = \frac{\partial h_\Delta}{\partial \bar{z}_j}(\mathbf{a})t^{d-p_j} + (\text{higher terms}).$$

For simplicity, we assume that $p_1 \geq p_2 \geq \cdots \geq p_m$. For a vector $v = (v_1, \dots, v_n)$ and $1 \leq \alpha \leq \beta \leq m$, we consider the truncation

$$v_\alpha^\beta := (v_\alpha, \dots, v_\beta).$$

We choose $1 \leq \alpha \leq \beta \leq m$ as follows.

(A-1) For any $j < \alpha$, $\dfrac{\partial}{\partial \bar{z}_j} g_\Delta(\mathbf{a}) = 0$, $\dfrac{\partial}{\partial \bar{z}_j} h_\Delta(\mathbf{a}) = 0$ and $\left(\dfrac{\partial}{\partial \bar{z}_\alpha} g_\Delta(\mathbf{a}), \dfrac{\partial}{\partial \bar{z}_\alpha} h_\Delta(\mathbf{a})\right) \neq (0,0)$.

(A-2) Two complex vectors

$$(\bar{\partial} g_\Delta(\mathbf{a}))_\alpha^\beta = \left(\frac{\partial}{\partial \bar{z}_\alpha} g_\Delta(\mathbf{a}), \dots, \frac{\partial}{\partial \bar{z}_\beta} g_\Delta(\mathbf{a})\right)$$

$$(\bar{\partial} h_\Delta(\mathbf{a}))_\alpha^\beta = \left(\frac{\partial}{\partial \bar{z}_\alpha} h_\Delta(\mathbf{a}), \dots, \frac{\partial}{\partial \bar{z}_\beta} h_\Delta(\mathbf{a})\right)$$

are linearly independent over $\mathbf{R}$ and $(\bar{\partial} g_\Delta(\mathbf{a}))_\alpha^{\beta'}, (\bar{\partial} h_\Delta(\mathbf{a}))_\alpha^{\beta'}$ are linearly dependent over $\mathbf{R}$ for any $\beta' < \beta$. For simplicity, we use the notations:

$$v_g(t) := \bar{\partial} g(\mathbf{z}(t)) = (v_{g,1}, \dots, v_{g,m}), \quad v_h(t) := \bar{\partial} h(\mathbf{z}(t)) = (v_{h,1}, \dots, v_{g,m}).$$

We consider the order of $v_g(t) = \bar{\partial} g(\mathbf{z}(t))$ and $v_h(t) = \bar{\partial} h(\mathbf{z}(t))$. (Here the order is the lowest degree of in t.)

Suppose $\operatorname{ord} v_g = r$ and the smallest index $1 \leq i \leq m$ with $\operatorname{ord} v_{g,i} = r$ is called *leading index*. Assume that s is the leading index of $v_g(t)$. We call the coefficient of t^r in the expansion of $v_{g,s}(t)$ *the leading coefficient*. Put s' be the leading index of v_h.

For simplicity, we assume that $s \leq s'$ and if $s = s'$ we assume also $\operatorname{ord} v_g(t) \leq \operatorname{ord} v_h(t)$. This is possible by changing g and h considering $if\,(\mathbf{z}, \bar{\mathbf{z}})$, if necessary.

First we observe that

$$\operatorname{ord} v_{g,i}(t), \operatorname{ord} v_{h,i}(t) \geq d - p_i, \quad s \leq \alpha.$$

(If $s > \alpha$, this means $\dfrac{\partial}{\partial \bar{z}_\alpha} g_\Delta(\mathbf{a}) = 0$ and $\dfrac{\partial}{\partial \bar{z}_\alpha} h_\Delta(\mathbf{a}) \neq 0$ which is a contradiction to the assumption $s \leq s'$.)

STRATEGY. Put $r = \operatorname{ord} v_g(t)$, $r' = \operatorname{ord} v_h(t)$. We have three possible cases.
(1) $s' > s$ or
(2-a) $s = s'$ and the coefficients of t^r of $v_{g,s}$ and the coefficient of $t^{r'}$ of $v_{h,s}$ are linearly independent over $\mathbf{R}$ or
(2-b) $s = s'$ and the coefficients of t^r of $v_{g,s}$ and the coefficient of $t^{r'}$ of $v_{h,s}$ are linearly dependent over $\mathbf{R}$.

For (1) or (1-a), we have nothing to do. In fact, write

$$v_g(t) = v_g^\infty t^r + (\text{higher terms}), \quad v_g^\infty \in \mathbf{C}^n$$

$$v_h(t) = v_h^\infty t^{r'} + (\text{higher terms}), \quad v_h^\infty \in \mathbf{C}^n.$$

Then the normalized limit of $v_g(t)$, $v_h(t)$ are given by $v_g^\infty/\|v_g^\infty\|$, $v_h^\infty/\|v_h^\infty\|$. In this case, the limit of v_g and v_h for $t \to 0$ are complex vectors v_g^∞, v_h^∞ (up to scalar multiplications) which are in $\mathbf{C}^m \times \{0\}$. They are linearly independent over $\mathbf{R}$. Thus the limit of $T_{\mathbf{z}(t)} f^{-1}(f(\mathbf{z}(t))$ is the real orthogonal complement $\langle v_g^\infty, v_h^\infty \rangle^\perp = v_g^{\infty\perp} \cap v_h^{\infty\perp}$ which contains $\mathbf{C}^I$.

Assume $s = s'$ and the coefficients of t^r in $v_{g,s}$ and the coefficient of $t^{r'}$ in $v_{h,s}$ are linearly dependent over $\mathbf{R}$. Then we consider the following operation.

OPERATION. Put $r' = \operatorname{ord} v_h$. We have assumed $r' \geq r$. Take a unique real number λ and replace v_h by $v_h' = v_h - \lambda t^{r'-r} v_g$ with $r = \operatorname{ord} v_{g,s}$, $r' = \operatorname{ord} v_{h,s}$ to kill the coefficient of $t^{r'}$ of $v_{h,s}$. (We have assumed $r \leq r'$.)

Note that after this operation, the vector changes into

$$v_{h,j}'(t) = \left(\frac{\partial}{\partial \bar{z}_j} h_\Delta(\mathbf{a}) - \lambda\varepsilon \frac{\partial}{\partial \bar{z}_j} g_\Delta(\mathbf{a})\right) t^{d-p_j} + (\text{higher terms})$$

where $\varepsilon = 1$ or 0 according to $r' = r$ or $r' > r$ respectively. We observe that if $r' > r$, the leading term of $v_{h,j}'(t)$ does not change. If $r' = r$, the (leading) coefficient $\dfrac{\partial}{\partial \bar{z}_j} h_\Delta(\mathbf{a})$ of t^{d-p_j} in $v_{h,j}$ is changed into

$$\frac{\partial}{\partial \bar{z}_j} h_\Delta(\mathbf{a}) - \lambda \frac{\partial}{\partial \bar{z}_j} g_\Delta(\mathbf{a}),$$

the above two properties (A-1), (A-2) are unchanged.

We continue the operation as long as the leading index of v_h' is still s. Suppose that this operation stops at k-th step. Then put $s^{(k)}$ the leading index of $v_h^{(k)}$ and $r^{(k)}$ be the order of $v_h^{(k)}$. By the above two properties, $s < s^{(k)} \leq \beta$ and $r^{(k)} \leq d - p_\beta$. This implies that the limit of the normalized gradient vectors v_g and $v_h^{(k)}$, say v_g^∞, v_h^∞ are independent vectors in $\mathbf{C}^m \times \{0\} = \mathbf{C}^{I^c}$ over $\mathbf{R}$. On the other hand, by the definition of the above operations,

$$\begin{aligned} T_{\mathbf{z}(t)} f^{-1}(f(\mathbf{z}(t)) &= v_g(t)^\perp \cap v_h(t)^\perp \\ &= v_g(t)^\perp \cap v_h'(t)^\perp = \cdots = v_g(t)^\perp \cap (v_h^{(k)}(t))^\perp. \end{aligned}$$

Thus the limit of $T_{\mathbf{z}(t)}f^{-1}(f(\mathbf{z}(t))$ is nothing but $(v_g^\infty)^\perp \cap (v_h^\infty)^\perp$. Note that $(v_g^\infty)^\perp \cap (v_h^\infty)^\perp \supset \mathbf{C}^I$. This show that the a_f-property is satisfied along this curve. □

The following will be practically useful.

Lemma 21. *Let f_Δ be a face function associated with an essential non-compact face $\Delta \in \Gamma_{nc}(f)$ with $I = I(\Delta)$. Assume that $I = \{m+1, \dots, n\}$.*

(1) *For $f(\mathbf{z})$ a holomorphic function, the following is necessary and sufficient for f_Δ to be locally tame.*

$$\left(\frac{\partial}{\partial z_1} f_\Delta(\mathbf{z}), \dots, \frac{\partial}{\partial z_m} f_\Delta(\mathbf{z})\right)$$

is a non-zero vector for any $\mathbf{z}$ with $\|\mathbf{z}_I\| \le \rho_0$.

(2) *For a mixed polynomial, f_Δ is locally tame if there exists a $j \in I^c$ such that two complex numbers $\frac{\partial g_\Delta}{\partial \bar{z}_j}(\mathbf{z})$, $\frac{\partial h_\Delta}{\partial \bar{z}_j}(\mathbf{z})$ are linearly independent over $\mathbf{R}$. In other word,*

$$\Im\left(\frac{\partial g_\Delta}{\partial \bar{z}_j}(\mathbf{z}) \overline{\frac{\partial h_\Delta}{\partial \bar{z}_j}}(\mathbf{z})\right) \ne 0.$$

for any $\mathbf{z}$ with $\|\mathbf{z}_I\| \le \rho_0$.

Proof. Recall that

$$\bar{\partial} g = \frac{1}{2}(\bar{\partial} f + \overline{\partial f}), \quad \bar{\partial} h = \frac{i}{2}(\bar{\partial} f - \overline{\partial f}).$$

If f is holomorphic, $\bar{\partial} g = \frac{1}{2}\partial f$ and $\bar{\partial} h_\Delta = -i\bar{\partial} g_\Delta$ and they are perpendicular by the Euclidean inner product. Thus they are independent over $\mathbf{R}$. For the second assertion, note that the assumption is equivalent to the 2 minor

$$\det\begin{pmatrix} \frac{\partial g_\Delta}{\partial x_j}(\mathbf{a}) & \frac{\partial g_\Delta}{\partial y_j}(\mathbf{a}) \\ \frac{\partial h_\Delta}{\partial x_j}(\mathbf{a}) & \frac{\partial h_\Delta}{\partial y_j}(\mathbf{a}) \end{pmatrix} = -\Im\left(\frac{\partial g_\Delta}{\partial \bar{z}_j}(\mathbf{a}) \overline{\frac{\partial h_\Delta}{\partial \bar{z}_j}}(\mathbf{a})\right) \ne 0.$$

□

3.4.1. *Examples.* Example 1. (Modification of Tibar's example) Consider the mixed monomial $f = z_1 z_2^a \bar{z}_2$. Then we have

$$\bar{\partial} f = (0, z_1 z_2^a), \quad \overline{\partial f} = (\bar{z}_2^a z_2, a\bar{z}_1 \bar{z}_2^{a-1} z_2)$$

$$\bar{\partial} g = \frac{1}{2}(\bar{z}_2^a z_2, z_1 z_2^a + a\bar{z}_1 \bar{z}_2^{a-1} z_2)$$

$$\bar{\partial} h = \frac{i}{2}(-\bar{z}_2^a z_2, z_1 z_2^a - a\bar{z}_1 \bar{z}_2^{a-1} z_2)$$

Consider the vanishing coordinate $I = \{1\}$. Two complex numbers

$$z_1 z_2^a + a\bar{z}_1 \bar{z}_2^{a-1} z_2, \quad i(z_1 z_2^a - a\bar{z}_1 \bar{z}_2^{a-1} z_2)$$

are linearly dependent over $\mathbf{R}$ if and only if $a = 1$ as

$$\begin{aligned}(z_1 z_2^a &+ a\bar{z}_1 \bar{z}_2^{a-1} z_2)(-i)(\bar{z}_1 \bar{z}_2^a - a z_1 z_2^{a-1} \bar{z}_2) \\ &= -i(1-a^2)|z_1|^2 |z_2|^{2a} - ia(-z_1^2 z_2^{2a-1} \bar{z}_2 + \bar{z}_1^2 \bar{z}_2^{2a-1} z_2) \\ &= -i(1-a^2)|z_1|^2 |z_2|^{2a} - 2a\Im(z_1^2 z_2^{2a-1} \bar{z}_2).\end{aligned}$$

Thus the imaginary part of the above complex number is zero if and only if $a = 1$. Note that f is an open mapping along $z_2 = 0$ if and only if $a > 1$.

Example 2. Consider the mixed polynomial

$$f(\mathbf{z}, \bar{\mathbf{z}}) = z_1^{a_1} \bar{z}_2 + z_2^{a_2} \bar{z}_3 + \cdots + z_n^{a_n} \bar{z}_1, \quad a_1, \dots, a_n \geq 2.$$

Then $\{i\} \in \mathcal{I}_v(f)$ for all $i = 1, \dots, n$. Consider for example, $I = \{n\}$. Then possible face functions are

$$f_\Delta = z_2^{a_2} \bar{z}_3 + \cdots + z_n^{a_n} \bar{z}_1 \quad \text{and} \quad f_\Xi,$$

where Ξ is a face of Δ. Now we can see that

$$\begin{aligned}\bar{\partial} f_\Delta &= (z_n^{a_n}, 0, z_2^{a_2}, \dots, z_{n-1}^{a_{n-1}}) \\ \overline{\partial f_\Delta} &= (0, a_2 \bar{z}_2^{a_2-1} z_3, \dots, a_n \bar{z}_n^{a_n-1} z_1) \\ \bar{\partial} g_\Delta &= \frac{1}{2}(z_n^{a_n}, a_2 \bar{z}_2^{a_2-1} z_3, z_2^{a_2} + a_2 \bar{z}_3^{a_3-1} z_4, \dots, z_{n-1}^{a_{n-1}} + a_n \bar{z}_n^{a_n-1} z_1) \\ \bar{\partial} h_\Delta &= \frac{i}{2}(z_n^{a_n}, -a_2 \bar{z}_2^{a_2-1} z_3, z_2^{a_2} - a_2 \bar{z}_3^{a_3-1} z_4, \dots, z_{n-1}^{a_{n-1}} - a_n \bar{z}_n^{a_n-1} z_1).\end{aligned}$$

Thus

$$(\bar{\partial} g_\Delta)_1 \cdot (\overline{\bar{\partial} h_\Delta})_1 = -\frac{i}{4}|z_n|^{2a_n}$$

and its imaginary part is non-zero, which satisfies the condition of Lemma 21. Now we consider a subset $\Xi \subset \Delta$. We consider the first monomial $z_j^{a_j} \bar{z}_{j+1}$ so that

$$z_n^{a_n} \bar{z}_1, \dots, z_{j-1}^{a_{j-1}} \bar{z}_j \notin f_\Xi, \quad z_j^{a_j} \bar{z}_{j+1} \in f_\Xi.$$

Then we have

$$\Im(\bar{\partial} g_\Xi)_j \cdot (\overline{\bar{\partial} h_\Xi})_j = -\frac{1}{4} a_k^2 |z_k|^{2a_k-2} |z_{k+1}|^2 \neq 0.$$

Thus by symmetry, we conclude that f is locally tame along each vanishing coordinate axis z_k, $k = 1, \dots, n$.

4. Some application

4.1. Mixed cyclic coverings. Consider positive integer vectors

$$\mathbf{a} := (a_1, \dots, a_n), \quad \mathbf{b} = (b_1, \dots, b_n)$$

such that $a_j > b_j \geq 0$, $j = 1, \dots, n$. We consider the mapping

$$\varphi_{\mathbf{a},\mathbf{b}} : \mathbf{C}^n \to \mathbf{C}^n, \quad (z_1, \dots, z_n) \mapsto (z_1^{a_1} \bar{z}_1^{b_1}, \dots, z_n^{a_n} \bar{z}_n^{b_n}).$$

This is a $\prod_{j=1}^n (a_j - b_j)$-fold multi-cyclic covering branched along the coordinate hyperplanes $\{z_j = 0\}$, $j = 1, \dots, n$. Consider a holomorphic function $f(\mathbf{z})$ which has a non-degenerate Newton boundary and the pull-back $\tilde{f}(\mathbf{z}, \bar{\mathbf{z}}) := f(\varphi_{\mathbf{a},\mathbf{b}}(\mathbf{z}, \bar{\mathbf{z}}))$ of f by $\varphi_{\mathbf{a},\mathbf{b}}$. This gives a strongly non-degenerate mixed function ([10]).

Proposition 22. *Assume that $f(\mathbf{z})$ is a non-degenerate holmorphic function which is locally tame along their vanishing coordinate subspaces. Then $\tilde{f}(\mathbf{w}, \overline{\mathbf{w}}) := f(\varphi_{\mathbf{a},\mathbf{b}}(\mathbf{w}, \overline{\mathbf{w}}))$ is a non-degenerate mixed function. Its vanishing coordinate subspaces are the same as that of $f(\mathbf{z})$ and it is locally tame along the vanishing coordinate subspaces.*

Proof. Take a face function $f_P(\mathbf{z})$ with weight $P = {}^t(p_1, \dots, p_n)$. Consider the weight $\tilde{P}$ which is the primitive weight vector obtained by multiplying the least common multiple of the denominators of

$${}^t\!\left(\frac{p_1}{a_1 + b_1}, \dots, \frac{p_n}{a_n + b_n}\right).$$

Then $\varphi_{\mathbf{a},\mathbf{b}}^* f(\mathbf{w}, \overline{\mathbf{w}})$ is radially weighted homogeneous with respect to the weight $\tilde{P}$. We observe also have $\tilde{f}_{\tilde{P}}(\mathbf{w}, \overline{\mathbf{w}}) = \varphi_{\mathbf{a},\mathbf{b}}^* f_P(\mathbf{w}, \overline{\mathbf{w}})$. Thus we see that the Newton boundary $\Gamma(\tilde{f})$ corresponds bijectively to that of $\Gamma(f)$ by this mapping. Suppose that $I \in \mathscr{I}_v(f) = \mathscr{I}_v(\tilde{f})$. We assume $I = \{m+1, \dots, n\}$ for simplicity. Take a non-compact face Δ with $I(\Delta) = I$ and let $\tilde{\Delta}$ be the corresponding non-compact face of $\tilde{f}$. We consider $\tilde{f}_{\tilde{\Delta}}$ as the following composition, fixing $(u_{m+1}, \dots, u_n) \in \mathbf{C}^{*I}$:

$$\tilde{f}_{\tilde{\Delta}} : \mathbf{C}^{*m} \xrightarrow{\varphi'_{\mathbf{a},\mathbf{b}}} \mathbf{C}^{*m} \xrightarrow{f_\Delta} \mathbf{C}$$

where

$$\varphi'_{\mathbf{a},\mathbf{b}}(w_1, \dots, w_m) = (w_1^{a_1} \overline{w}_1^{b_1}, \dots, w_m^{a_m} \overline{w}_m^{b_m}, u_{m+1}^{a_{m+1}} \bar{u}_{m+1}^{b_{m+1}}, \dots, u_n^{a_n} \bar{u}_n^{b_n}).$$

As $\varphi'_{\mathbf{a},\mathbf{b}}$ is an unbranched covering mapping, $\tilde{f}_{\tilde{\Delta}}$ does not have any critical points. □

4.2. Mixed functions with strongly mixed weighted homogeneous faces. We say a mixed polynomial $h(\mathbf{z}, \bar{\mathbf{z}})$ is *mixed weighted homogeneous* if it is radially weighted homogeneous and also polar weighted homogeneous. $h(\mathbf{z}, \bar{\mathbf{z}})$ is *strongly mixedr weighted homogeneous* if the polar weight and the radial weight can be the same. A mixed function $f(\mathbf{z},\bar{\mathbf{z}})$ is called of strongly mixed weighted homogeneous face type if every face function f_Δ is strongly mixed weighted homogeneous polynomial ([10]). Let $\Gamma^*(f)$ be the Newton boundary and let Σ^* be an admissible regular subdivision of $\Gamma^*(f)$ and let $\hat{\pi}: X \to \mathbf{C}^n$ be the associated toric modification. Let $\mathscr{V}$ be the vertices of Σ^* which corresponds to the exceptional divisors as in §2, [10]. Let $\mathscr{S}_I$ be the set of $|I|-1$ dimensional faces of $\Gamma(f^I)$. It is shown that $\hat{\pi}: X \to \mathbf{C}^n$ topologically resolves the mixed function $f: \mathbf{C}^n \to \mathbf{C}$ ([10]). Combining the existence of Milnor fibration and the argument in [10], we can generalize Theorem 11 ([10]) as follows. For $I \in \mathscr{I}_{nv}$, we denote by $\mathscr{S}_I$ the set of weight vectors which correspond to $|I|-1$ dimensional faces of $\Gamma(f^I)$.

The notations and definitions are the same as in Theorem 11 ([10]).

Theorem 23. *Let $f(\mathbf{z},\bar{\mathbf{z}})$ a non-degenerate mixed polynomial of strongly mixed positive weighted homogeneous face type which is locally tame along vanishing coordinate subspaces. Let $V = f^{-1}(V)$ be a germ of hypersurface at the origin and let $\tilde{V}$ be the strict transform of V to X. Then*

(1) *$\tilde{V}$ is topologically smooth and real analytic smooth variety outside of the union of the exceptional divisors $\bigcup_{P\in\mathscr{V}} \hat{E}(P)$.*

(2) *The zeta function of the Milnor fibration of $f(\mathbf{z},\bar{\mathbf{z}})$ is given by the formula*

$$\zeta(t) = \prod_I \zeta_I(t), \quad \zeta_I(t) = \prod_{P\in\mathscr{S}_I} (1 - t^{\mathrm{pdeg}(P, f_P^I)})^{-\chi(P)/\mathrm{pdeg}(P, f_P^I)}$$

where $\chi(P)$ is the Euler characteristic of the torus Milnor fiber of f_P^I

$$F_P^* = \{\mathbf{z}_I \in \mathbf{C}^{*I} \mid f_P^I(\mathbf{z}_I) = 1\}, \quad P \in \mathscr{S}_I.$$

4.2.1. *Example*: 1. *Curves with mixed Brieskorn faces.* Consider a mixed polynomial

$$f(\mathbf{z},\bar{\mathbf{z}}) = z_1^2 z_2^2 (z_1^6 \bar{z}_1^3 + z_2^4 \bar{z}_2^2)(z_1^4 \bar{z}_1^2 + z_2^6 \bar{z}_2^3)$$

f is strongly non-degenerate and has two faces which are strongly polar weighted homogeneous: Put $P = {}^t(2,3)$, $Q := {}^t(3,2)$. There are two faces corresponding to P and Q.

$$f_P(\mathbf{z},\bar{\mathbf{z}}) = z_1^6 \bar{z}_1^2 z_2^2 (z_1^6 \bar{z}_1^3 + z_2^4 \bar{z}_2^2), \quad f_Q(\mathbf{z},\bar{\mathbf{z}}) = z_1^2 z_2^6 \bar{z}_2^2 (z_1^4 \bar{z}_1^2 + z_2^6 \bar{z}_2^3)$$

and f_P, f_Q are strongly polar weighted with $\mathrm{pdeg}\, f_P = \mathrm{pdeg}\, f_Q = 20$. Thus the contribution of f_P to the zeta-function is $(1 - t^{20})^{-\chi(P)/20}$ where $\chi(P)$ is the Euler characteristic of

$$F_P^* := \{\mathbf{z} \in \mathbf{C}^{*2} \mid f_P(\mathbf{z},\bar{\mathbf{z}}) = 1\}.$$

F_P^* is diffeomorphic to

$$F_P' := \{\mathbf{z} \in \mathbf{C}^{*2} \mid z_1^4 z_2^2 (z_1^3 + z_2^2) = 1\}$$

by Theorem 10 ([8]). Thus $\chi(P) = \chi(F_P^*) = -20$. Thus using the symmetry of f_P and f_Q, we get $\zeta(t) = (1 - t^{20})^2$.

In general, for a non-degenerate non-convenient mixed polynomial of two variables $f(\mathbf{z},\bar{\mathbf{z}})$, consider the right end monomial $z_1^m\bar{z}_1^n z_2^a\bar{z}_2^b$. Right end means that $\Gamma(f)$ is in the space $\{(\nu,\mu)\,|\,\nu\le m+n,\mu\ge a+b\}$. If $a+b\ge 1$, z_1 axis is a vanishing coordinate. It is locally tame along z_1-axis if and only if $a-b\ne 0$.

Example 2. Consider D_n singularity:

$$D_n : f(z_1,z_2,z_3) = z_1^2 + z_2^2 z_3 + z_3^{n-1}.$$

Then the Milnor number $\mu(f)$ of f is n and the zeta function is given as $\zeta(t) = (t^{n-1}+1)(t^2-1)$. f has a vanishing axis z_2 but V is non-singular except at the origin. Consider

$$\begin{aligned}\tilde{f}(\mathbf{w},\bar{\mathbf{w}}) &= \varphi_{2,1}^* f(\mathbf{w},\bar{\mathbf{w}})\\ &= w_1^4\bar{w}_1^2 + w_2^4\bar{w}_2^2 w_3^2\bar{w}_3 + w_3^{2(n-1)}\bar{w}_3^{n-1}\end{aligned}$$

$\tilde{f}$ has a vanishing coordinate axis w_2 but the data for the zeta function is exactly same as f. As $\varphi_{2,1}$ is a homeomorhism, $\mu(\tilde{f}) = n$ and it has the same zeta functions as f. See also Corollary 15, [10].

4.3. Join type polynomials. We consider the join type polynomial

$$f(\mathbf{z},\bar{\mathbf{z}},\mathbf{w},\bar{\mathbf{w}}) = f_1(\mathbf{z},\bar{\mathbf{z}}) + f_2(\mathbf{w},\bar{\mathbf{w}}),\quad (\mathbf{z},\mathbf{w})\in\mathbf{C}^n\times\mathbf{C}^m.$$

PROPOSITION 24. *Assume that f_1 and f_2 are strongly non-degenerate mixed polynomial. Then f is also strongly non-degenerate. We assume that f_1, f_2 do not have any linear term so that they have a critical point at the respective origin. Then we have*

(1) *If f_1 and f_2 are locally tame along vanishing coordinate subspaces, f is also locally tame along vanishing coordinate subspaces. In particular, f satisfies a_f-condition.*

(2) *If f_1 or f_2 does not satisfy a_f-condition, f does not satisfy a_f-condition.*

Proof. Assume that $I_1\in I_v(f_1)$ and $I_2\in I_v(f_2)$. Then $f\,|\,\mathbf{C}^{I_1}\times\mathbf{C}^{I_2}\equiv 0$. Take $\Delta_1\in\Gamma_{nc}(f_1)$ with $I(\Delta_1)=I_1$ and $\Delta_2\in\Gamma_{nc}(f_2)$ with $I(\Delta_2)=I_2$. Then $\Delta := \Delta_1 * \Delta_2 \in \Gamma_{nc}(f)$ and $f_\Delta(\mathbf{z},\bar{\mathbf{z}},\mathbf{w},\bar{\mathbf{w}}) = f_{1\Delta_1}(\mathbf{z},\bar{\mathbf{z}}) + f_{2\Delta_2}(\mathbf{w},\bar{\mathbf{w}})$ satisfies certainly the local tameness condition. (Here $\Delta_1 * \Delta_2$ is the convex polyhedron spanned by Δ_1 and Δ_2.) Conversely suppose that $\Delta\subset\Gamma_v(f)$ with $I = I(\Delta)$. Then $f\,|\,\mathbf{C}^I\equiv 0$. Put $I_1 = I\cap\{1,\dots,n\}$ and $I_2 = I\backslash I_1$. Then $I_1\in\mathscr{I}_v(f_1)$ and $I_2\in\mathscr{I}_v(f_2)$. Take P so that $I(P) = I$ and put $\Delta_1 = \Delta\cap\mathbf{C}^n$ and $\Delta_2 = \Delta\cap\mathbf{C}^m$. Then $\Delta = \Delta_1 * \Delta_2$. Let P_1, P_2 be the projection to $\mathbf{C}^n$ or $\mathbf{C}^m$ respectively. Then $\Delta(P_1) = \Delta_1$ and $\Delta(P_2) = \Delta_2$ and $f_P = f_{1,P_1}(\mathbf{z},\bar{\mathbf{z}}) + f_{2,P_2}(\mathbf{w},\bar{\mathbf{w}})$ and it is certainly locally tame. This proves (1).

To prove the assertion (2), assume for example f_1 does not satisfies a_f-condition. Take a stratification $\mathscr{S}$ such that its restriction to $\mathbf{C}^n$ and $\mathbf{C}^m$ are stratification $\mathscr{S}_1$ and $\mathscr{S}_2$ for f_1 and f_2 respectively. By the assumption, there exists $\mathbf{p} \in V_1 = V(f_1)$ and a stratum M of $\mathscr{S}_1$ with $\mathbf{p} \in M$ and an analytic curve $\mathbf{z}(t)$ in $\mathbf{C}^n \setminus \{\mathbf{0}\}$ such that $\mathbf{z}(0) = \mathbf{p}$ and a_f-condition is not satisfied along this curve. Write $f_1 = g_1 + ih_1$, $f_2 = g_2 + ih_2$ and $f = g + ih$. We may assume that

$$\bar\partial g_1(\mathbf{z}(t)) = v_{g_1}^\infty t^{s_1} + (\text{higher terms})$$

and it converges to $v_{g_1}^\infty$. We assume ord $\bar\partial h_1(\mathbf{z}(t)) \geq$ ord $\bar\partial g_1(\mathbf{z}(t))$. By the same technique as in the proof of Theorem 20, we take a new vector $\bar\partial h_1' := \bar\partial h_1(t) - k(t)\bar\partial g_1(\mathbf{z}(t))$ so that

$$\bar\partial h_1'(t) = v_{h_1}^\infty t^d + (\text{higher terms})$$

so that the leading coefficient vectors v_{g_1}, v_{h_1} are linearly independent over $\mathbf{R}$. Thus the limit of the tangent space $T_{\mathbf{z}(t)} f_1^{-1}(f_1(\mathbf{z}(t))$ is given by $v_{g_1}^{\infty\perp} \cap v_{h_1}^{\infty\perp}$. By the assumption, we have that $v_{g_1}^{\infty\perp} \cap v_{h_1}^{\infty\perp} \not\supset T_{\mathbf{z}(0)} M$. Note that d is the order of $\bar\partial h_1'(\mathbf{z}(t))$. Consider the analytic path $(\mathbf{z}(t), \mathbf{w}(t))$ where $\mathbf{w}(t) = (t^{3d}, \ldots, t^{3d})$. Let us consider

$$\bar\partial h_2'(\mathbf{w}(t)) := \bar\partial h_2(\mathbf{w}(t)) - k(t)\bar\partial g_2(\mathbf{w}(t)).$$

Then it is easy to see that ord $\bar\partial g_2(\mathbf{w}(t)), \bar\partial h_2'(\mathbf{w}(t)) \geq 2d$. Put

$$\bar\partial h'(\mathbf{z}(t), \mathbf{w}(t)) = \partial h(\mathbf{z}(t), \mathbf{w}(t)) - k(t)\bar\partial g(\mathbf{z}(t), \mathbf{w}(t)).$$

Thus this implies that ord $\bar\partial g(\mathbf{z}(t), \mathbf{w}(t)) =$ ord $\bar\partial g_1(\mathbf{z}(t))$ and ord $\bar\partial h'(\mathbf{z}(t), \mathbf{w}(t)) =$ ord $\bar\partial h_1'(\mathbf{z}(t)) \geq 2d$ and the normalized limits are given as

$$\bar\partial g(\mathbf{z}(t), \mathbf{w}(t)) \to (v_{g_1}, \mathbf{0}), \quad \bar\partial h'(\mathbf{z}(t), \mathbf{w}(t)) \to (v_{h_1}, \mathbf{0})$$

which implies the limit of $T_{(\mathbf{z}(t), \mathbf{w}(t))} f^{-1}(f(\mathbf{z}(t), \mathbf{w}(t))$ is $(v_{g_1}^{\infty\perp} \cap v_{h_1}^{\infty\perp}) \times \mathbf{C}^m$. By the assumption, $(v_{g_1}^{\infty\perp} \cap v_{h_1}^{\infty\perp}) \times \mathbf{C}^m \not\supset T_{\mathbf{z}(0)} M$. □

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Mutsuo Oka
DEPARTMENT OF MATHEMATICS
TOKYO UNIVERSITY OF SCIENCE
1-3, KAGURAZAKA, SHINJUKU-KU
TOKYO 162-8601
JAPAN
E-mail: oka@rs.kagu.tus.ac.jp