A FINITE PRESENTATION OF THE LEVEL 2 PRINCIPAL CONGRUENCE SUBGROUP OF $GL(n;\mathbf{Z})$

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Abstract

It is known that the level 2 principal congruence subgroup of $GL(n;\mathbf{Z})$ has a finite generating set (see [7]). In this paper, we give a finite presentation of the level 2 principal congruence subgroup of $GL(n;\mathbf{Z})$.

1. Introduction

For $n \geq 1$, let $\Gamma_2(n) = \ker(GL(n;\mathbf{Z}) \to GL(n;\mathbf{Z}_2))$. We call $\Gamma_2(n)$ the *level 2 principal congruence subgroup* of $GL(n;\mathbf{Z})$. Note that for $A \in \Gamma_2(n)$ the diagonal entries of A are odd and the others are even.

For $1 \leq i, j \leq n$ with $i \neq j$, let E_{ij} denote the matrix whose (i,j) entry is 2, diagonal entries are 1 and others are 0, and let F_i denote the matrix whose (i,i) entry is -1, other diagonal entries are 1 and others are 0. It is known that $\Gamma_2(n)$ is generated by E_{ij} and F_i for $1 \leq i, j \leq n$ with $i \neq j$ (see [7]).

In this paper, we give a finite presentation of $\Gamma_2(n)$.

THEOREM 1.1. *For $n \geq 1$, $\Gamma_2(n)$ has a finite presentation with generators E_{ij} and F_i, for $1 \leq i, j \leq n$ with $i \neq j$, and with the following relators*
   (1) F_i^2,
   (2) $(E_{ij}F_i)^2$, $(E_{ij}F_j)^2$, $(F_iF_j)^2$ *(when $n \geq 2$)*,
   (3) (a) $[E_{ij}, E_{ik}]$, $[E_{ij}, E_{kj}]$, $[E_{ij}, F_k]$, $[E_{ij}, E_{ki}]E_{kj}^2$ *(when $n \geq 3$)*,
        (b) $[E_{ji}F_jE_{ij}F_iE_{ki}^{-1}E_{kj}, E_{ki}F_kE_{ik}F_iE_{ji}^{-1}E_{jk}]$ *for $i < j < k$ (when $n \geq 3$)*,
   (4) $[E_{ij}, E_{kl}]$ *(when $n \geq 4$)*,
where $[X, Y] = X^{-1}Y^{-1}XY$ and $1 \leq i, j, k, l \leq n$ are mutually different.

We note that a finite presentation of $\Gamma_2(n)$ has been obtained also by Fullarton [3] and Margalit-Putman.

It is clear that the above theorem is valid in the case $n = 1$. A proof of the theorem is by induction on n. In Section 3, we will prove the case $n = 2$

of Theorem 1.1, using the Reidemeister-Schreier method. In Section 4, we will prove the case $n = 3$ of Theorem 1.1, considering a simply connected simplicial complex on which $\Gamma_2(n)$ acts. In Section 5, we will introduce another simply connected simplicial complex on which $\Gamma_2(n)$ acts for $n \geq 4$. Finally, in Section 6, we will obtain the presentation of Theorem 1.1, by this action and induction on n.

We now explain about an application of Theorem 1.1. For $g \geq 1$, let N_g denote a non-orientable closed surface of genus g, that is, N_g is a connected sum of g real projective planes. Let $\cdot : H_1(N_g; R) \times H_1(N_g; R) \to \mathbf{Z}_2$ denote the mod 2 intersection form, and let $\mathrm{Aut}(H_1(N_g; R), \cdot)$ denote the group of automorphisms over $H_1(N_g; R)$ preserving the mod 2 intersection form $\cdot$, where $R = \mathbf{Z}$ or $\mathbf{Z}_2$. Consider the natural epimorphism

$$\Phi_g : \mathrm{Aut}(H_1(N_g; \mathbf{Z}), \cdot) \to \mathrm{Aut}(H_1(N_g; \mathbf{Z}_2), \cdot).$$

McCarthy and Pinkall [7] showed that $\Gamma_2(g - 1)$ is isomorphic to $\ker \Phi_g$.

We denote by $\mathcal{M}(N_g)$ the group of isotopy classes of diffeomorphisms over N_g. The group $\mathcal{M}(N_g)$ is called the *mapping class group* of N_g. In [7] and [4], it is shown that the natural homomorphism $\mathcal{M}(N_g) \to \mathrm{Aut}(H_1(N_g; R), \cdot)$ is surjective, where $R = \mathbf{Z}$ or $\mathbf{Z}_2$. Let $\mathcal{I}(N_g)$ denote the kernel of $\mathcal{M}(N_g) \to \mathrm{Aut}(H_1(N_g; \mathbf{Z}), \cdot)$. We say $\mathcal{I}(N_g)$ the *Torelli group* of N_g. In [5], Hirose and the author obtained a generating set of $\mathcal{I}(N_g)$ for $g \geq 4$, using Theorem 1.1.

2. Preliminaries

In this section, we explain about some facts for presentations of groups.

2.1. Basics on presentations of groups

Let G_1, G_2 and G_3 be groups with a short exact sequence

$$1 \to G_1 \xrightarrow{\phi} G_2 \xrightarrow{\pi} G_3 \to 1.$$

If G_1 and G_3 are presented then we can obtain a presentation of G_2. In particular, if G_1 and G_3 are finitely presented then G_2 can be finitely presented.

More precisely, a presentation of G_2 is obtained as follows. Let $G_1 = \langle X_1 \,|\, R_1 \rangle$ and $G_3 = \langle X_3 \,|\, R_3 \rangle$. For each $x \in X_3$, we choose $\tilde{x} \in \pi^{-1}(x)$. We put $X_2 = \{\phi(x_1), \tilde{x_3} \,|\, x_1 \in X_1, x_3 \in X_3\}$. For $r = a_1^{\varepsilon_1} a_2^{\varepsilon_2} \cdots a_k^{\varepsilon_k} \in R_3$, let $\tilde{r} = \tilde{a_1}^{\varepsilon_1} \tilde{a_2}^{\varepsilon_2} \cdots \tilde{a_k}^{\varepsilon_k}$. For $g \in \ker \pi$, let $\bar{g}$ be a word over $\phi(X_1)$ with $g = \bar{g}$. Let $A = \{\phi(r_1) \,|\, r_1 \in R_1\}$, $B = \{\tilde{r_3}\overline{\tilde{r_3}}^{-1} \,|\, r_3 \in R_3\}$ and $C = \{\tilde{x_3}\phi(x_1)\tilde{x_3}^{-1}\overline{\tilde{x_3}\phi(x_1)\tilde{x_3}^{-1}}^{-1} \,|\, x_1 \in X_1, x_3 \in X_3\}$. We put $R_2 = A \cup B \cup C$. Then we have $G_2 = \langle X_2 \,|\, R_2 \rangle$.

In addition, if there is a homomorphism $\rho : G_3 \to G_2$ such that $\pi \circ \rho = id_{G_3}$, choose $\tilde{x} = \rho(x) \in \pi(x)^{-1}$ for $x \in X_1$. Then, we have the relation $\tilde{r} = 1$ in G_2 for $r \in R_3$.

If G_2 is presented then we can examine a presentation of G_1, by the Reidemeister-Schreier method. In particular, if G_3 is a finite group, that is, the

index of $\operatorname{Im} \phi$ is finite, and G_2 can be finitely presented, then G_1 can be finitely presented.

For further information see [6].

2.2. Presentations of groups acting on a simplicial complex

Let X be a simplicial complex, and let G be a group acting on X by isomorphisms as a simplicial map. We suppose that the action of G on X is *without rotation*, that is, for a simplex $\Delta \in X$ and $g \in G$, if $g(\Delta) = \Delta$ then $g(v) = v$ for all vertices $v \in \Delta$. For a simplex $\Delta \in X$, let G_Δ be the stabilizer of Δ. For $k \geq 0$, the *k-skeleton* $X^{(k)}$ is the subcomplex of X consisting of all simplices of dimension at most k.

Consider a homomorphism $\Phi : \underset{v \in X^{(0)}}{*} G_v \to G$. For $g \in G$, if g stabilizes a vertex $w \in X^{(0)}$, we denote g by g_w as an element in $G_w < \underset{v \in X^{(0)}}{*} G_v$. For a 1-simplex $\{v, w\} \in X$ and $g \in G_v \cap G_w$, we have $g_v g_w^{-1} \in \ker \Phi$ and call $g_v g_w^{-1}$ the *edge relator*.

At first, for any 1-simplex $\{v, w\}$, choose an orientation such that orientations are preserved by the action of G. Namely, orientations of $\{v, w\}$ and $g\{v, w\}$ are compatible for all $g \in G$. We denote the oriented 1-simplex $\{v, w\}$ by (v, w). Similarly, choose orders of 2-simplices, and denote the ordered 2-simplex $\{v_1, v_2, v_3\}$ by (v_1, v_2, v_3). For an oriented 1-simplex $e = (v, w)$, let $o(e) = v$ and $t(e) = w$. For an oriented 2-simplex $\tau = (v_1, v_2, v_3)$, we call v_1 the base point of τ.

Next, choose an oriented tree T of X such that a set of vertices of T is a set of representative elements for vertices of the orbit space $G \backslash X$. Let V denote the set of vertices of T. In addition, choose a set E of representative elements for oriented 1-simplices of $G \backslash X$ such that $o(e) \in V$ for $e \in E$ and 1-simplices of T is in E, and a set F of representative elements for ordered 2-simplices of $G \backslash X$ such that the base point of τ is in V for $\tau \in F$. For $e \in E$, let $w(e)$ denote the element in V which is equivalent to $t(e)$ by the action of G, and choose $g_e \in G$ such that $g_e(w(e)) = t(e)$ and $g_e = 1$ if $e \in T$.

For a 1-simplex $\{v, w\}$ with $v \in V$, note that $\{v, w\} = \{o(e), hg_e w(e)\}$ or $\{w(e), hg_e^{-1} o(e)\}$ for some $e \in E$ and $h \in G_v$. Then we define respectively $g_{\{v, w\}} = hg_e$ or hg_e^{-1}. Let α be a loop in X starting at a vertex of V. We denote $\alpha = \{v_i, \{v_i, v_{i+1}\} \mid 1 \leq i \leq k, v_{k+1} = v_1\}$. Note that $v_1, g_1^{-1} v_2 \in V$, where $g_1 = g_{\{v_1, v_2\}}$. For $2 \leq i \leq k$, define $g_i = g_{g_{i-1}^{-1} \cdots g_1^{-1} \{v_i, v_{i+1}\}}$, inductively. Note that for $2 \leq i \leq k$, there exists an oriented 1-simplex e_i such that $o(e_i) \in V$ and $\{v_i, v_{i+1}\} = g_1 g_2 \cdots g_{i-1} \{o(e_i), t(e_i)\}$. Let $g_\alpha = g_1 g_2 \cdots g_k$. We have $g_\alpha(v_1) = v_1$, that is, $g_\alpha \in G_{v_1}$.

For $e \in E$, put a word $\hat{g}_e$. For a 1-simplex $\{v, w\}$ with $v \in V$, let $\hat{g}_{\{v, w\}} = h\hat{g}_e$ or $h\hat{g}_e^{-1}$ if $g_{\{v, w\}} = hg_e$ or hg_e^{-1}, respectively. For a loop α in X starting at a vertex of V, let $\hat{g}_\alpha = \hat{g}_1 \hat{g}_2 \cdots \hat{g}_k$ if $g_\alpha = g_1 g_2 \cdots g_k$. Note that we can define g_τ and $\hat{g}_\tau$ for $\tau \in F$, regarding τ as a loop in X. Let $\hat{G} = \left(\underset{v \in V}{*} G_v \right) * \left(\underset{e \in E}{*} \langle \hat{g}_e \rangle \right)$.

The following theorem is a special case of the result of Brown [1].

THEOREM 2.1 ([1]). *Let X be a simply connected simplicial complex, and let G be a group acting without rotation on X by isomorphisms as a simplicial map. Then G is isomorphic to the quotient of $\hat{G}$ by the normal subgroup generated by followings*

(1) $\hat{g}_e$, *where* $e \in T$,
(2) $\hat{g}_e^{-1} A_{o(e)} \hat{g}_e (g_e^{-1} A g_e)_{w(e)}^{-1}$, *where* $e \in E$ *and* $A \in G_e$,
(3) $\hat{g}_\tau g_\tau^{-1}$, *where* $\tau \in F$.

3. Proof of the case $n = 2$ of Theorem 1.1

In this section, we prove the following proposition.

PROPOSITION 3.1. $\Gamma_2(2)$ *has a finite presentation with generators* E_{12}, E_{21}, F_1 *and* F_2, *and with relators* F_1^2, F_2^2, $(E_{12}F_1)^2$, $(E_{12}F_2)^2$, $(E_{21}F_1)^2$, $(E_{21}F_2)^2$ *and* $(F_1F_2)^2$.

3.1. The Reidemeister Schreier method
Let x, y and z be

$$x = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

At first, we prove the next lemma.

LEMMA 3.2. $GL(2; \mathbf{Z})$ *has a presentation with*

$$GL(2; \mathbf{Z}) = \langle x, y, z \mid xyxy^{-1}x^{-1}y^{-1}, (xy)^6, z^2, xzyz \rangle.$$

Proof. In [8], it is known that $SL(2; \mathbf{Z})$ has a presentation with

$$SL(2; \mathbf{Z}) = \langle x, y \mid xyxy^{-1}x^{-1}y^{-1}, (xy)^6 \rangle.$$

Consider the short exact sequence

$$1 \to SL(2; \mathbf{Z}) \to GL(2; \mathbf{Z}) \to \{\pm 1\} \to 1.$$

Note that $\{\pm 1\} = \langle \det z \mid (\det z)^2 \rangle$. Then we have that $GL(2; \mathbf{Z})$ has a presentation with generators x, y and z, and with the following relations

- $xyxy^{-1}x^{-1}y^{-1} = 1$, $(xy)^6 = 1$,
- $z^2 = 1$,
- $zxz^{-1} = y^{-1}$, $zyz^{-1} = x^{-1}$.

Since $z^2 = 1$, we have $zxzy = 1$ and $zyzx = 1$. Moreover the equation $zxzy = zyzx = 1$ is obtained from $xzyz = 1$. Therefore, we obtain the claim. $\qquad\square$

Next we consider the short exact sequence

$$1 \to \Gamma_2(2) \to GL(2; \mathbf{Z}) \xrightarrow{\pi} GL(2; \mathbf{Z}_2) \to 1.$$

For $0 \le i \le 5$, let $a_i \in GL(2; \mathbf{Z})$ be

$$a_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

$$a_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad a_4 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad a_5 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

and let $U = \{a_0, a_1, a_2, a_3, a_4, a_5\}$. Since each of a_i is denoted by $a_0 = 1$, $a_1 = x^{-1}$, $a_2 = y$, $a_3 = z$, $a_4 = x^{-1}z$ and $a_5 = yz$, as a word over $\{x, y, z\}$, we have that U is a Schreier transversal for $\Gamma_2(2)$ in $GL(2; \mathbf{Z})$ (see [6]). For $A \in GL(2; \mathbf{Z})$, we define $\bar{A} = a_i$ if $\pi(A) = \pi(a_i)$. Let B be the set of matrices $\overline{wa_i}^{-1}wa_i$ with $wa_i \notin U$, where $0 \le i \le 5$ and $w = x^{\pm 1}$, $y^{\pm 1}$ and z. Then we have

$$B = \left\{ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} \right\}$$

(see Table 1). Note that B is a generating set of $\Gamma_2(2)$ (see [6]). It is clear that

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1}, \quad \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^{-1}.$$

Thus, by Tietze transformations, we obtain the generating set $B' = \{g_1, g_2, g_3, g_4\}$ of $\Gamma_2(2)$, where

$$g_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad g_4 = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}.$$

$\overline{wa_i}^{-1}wa_i$	$w = x$	$w = x^{-1}$	$w = y$	$w = y^{-1}$	$w = z$
$i = 0$	$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$i = 1$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$i = 2$	$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$i = 3$	$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$i = 4$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$i = 5$	$\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Table 1. The matrix $\overline{wa_i}^{-1}wa_i$.

We now prove the next lemma.

LEMMA 3.3. *Let $r = r_1 r_2 \cdots r_n \in GL(2; \mathbf{Z})$. Then for $0 \leq i \leq 5$ and $1 \leq j \leq n - 1$, we have*

$$\overline{r_j \overline{(r_{j+1} \cdots r_n) a_i}} = \overline{(r_j r_{j+1} \cdots r_n) a_i}.$$

Proof. Note that $\bar{A} = \bar{B}$ if and only if $\pi(A) = \pi(B)$. We calculate

$$\pi(r_j \overline{(r_{j+1} \cdots r_n) a_i}) = \pi(r_j)\pi(\overline{(r_{j+1} \cdots r_n) a_i})$$
$$= \pi(r_j)\pi((r_{j+1} \cdots r_n) a_i)$$
$$= \pi((r_j r_{j+1} \cdots r_n) a_n).$$

Therefore, we obtain the claim. $\qquad\square$

Let R be the set of relators of $GL(2; \mathbf{Z})$ in Lemma 3.2. For any $r = r_1 r_2 \cdots r_n \in R$ and $0 \leq i \leq 5$, we define a word s_{ri} over B' as follows.

$$s_{ri} = (a_i^{-1} r_1 \overline{(r_2 \cdots r_n) a_i})(\overline{(r_2 \cdots r_n) a_i}^{-1} r_2 \overline{(r_3 \cdots r_n) a_i}) \cdots (\overline{r_n a_i}^{-1} r_n a_i).$$

Let $\hat{S} = \{s_{ri} \mid r \in R, 0 \leq i \leq 5\}$. Then $\hat{S}$ is a set of relators of $\Gamma_2(2)$ (see [6]). Hence we have $\Gamma_2(2) = \langle B' \mid \hat{S} \rangle$.

3.2. Proof of Proposition 3.1

We now write all elements in $\hat{S}$ as a product of elements in B'. Let $[w] = \bar{w}^{-1} w$.

For $r = xyxy^{-1}x^{-1}y^{-1}$, we have

$$s_{r0} = [xa_1][ya_4][xa_3][y^{-1}a_5][x^{-1}a_2][y^{-1}a_0]$$
$$= (g_4 g_3^{-1})^2,$$

$$s_{r1} = [xa_0][ya_2][xa_5][y^{-1}a_3][x^{-1}a_4][y^{-1}a_1]$$
$$= (g_1^{-1} g_3 g_4)^2,$$

$$s_{r2} = [xa_5][ya_3][xa_4][y^{-1}a_1][x^{-1}a_0][y^{-1}a_2]$$
$$= g_4^2,$$

$$s_{r3} = [xa_4][ya_1][xa_0][y^{-1}a_2][x^{-1}a_5][y^{-1}a_3]$$
$$= (g_2 g_1^{-1})^2,$$

$$s_{r4} = [xa_3][ya_5][xa_2][y^{-1}a_0][x^{-1}a_1][y^{-1}a_4]$$
$$= (g_3^{-1} g_1 g_2)^2,$$

$$s_{r5} = [xa_2][ya_0][xa_1][y^{-1}a_4][x^{-1}a_3][y^{-1}a_5]$$
$$= g_2^2.$$

For $r = (xy)^6$, we have

$$s_{r0} = [xa_1][ya_4][xa_3][ya_5][xa_2][ya_0][xa_1][ya_4][xa_3][ya_5][xa_2][ya_0]$$
$$= (g_4 g_3^{-1} g_1 g_2)^2,$$
$$s_{r1} = [xa_0][ya_2][xa_5][ya_3][xa_4][ya_1][xa_0][ya_2][xa_5][ya_3][xa_4][ya_1]$$
$$= (g_1^{-1} g_3 g_4 g_2)^2,$$
$$s_{r2} = [xa_5][ya_3][xa_4][ya_1][xa_0][ya_2][xa_5][ya_3][xa_4][ya_1][xa_0][ya_2]$$
$$= (g_4 g_2 g_1^{-1} g_3)^2,$$
$$s_{r3} = [xa_4][ya_1][xa_0][ya_2][xa_5][ya_3][xa_4][ya_1][xa_0][ya_2][xa_5][ya_3]$$
$$= (g_2 g_1^{-1} g_3 g_4)^2,$$
$$s_{r4} = [xa_3][ya_5][xa_2][ya_0][xa_1][ya_4][xa_3][ya_5][xa_2][ya_0][xa_1][ya_4]$$
$$= (g_3^{-1} g_1 g_2 g_4)^2,$$
$$s_{r5} = [xa_2][ya_0][xa_1][ya_4][xa_3][ya_5][xa_2][ya_0][xa_1][ya_4][xa_3][ya_5]$$
$$= (g_2 g_4 g_3^{-1} g_1)^2.$$

For $r = z^2$ and $0 \le i \le 5$, since $\overline{z a_i}^{-1} z a_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we have $s_{ri} = 1$. For $r = xzyz$, we have

$$s_{r0} = [xa_1][za_5][ya_3][za_0] = 1,$$
$$s_{r1} = [xa_0][za_3][ya_5][za_1] = g_1^{-1} g_1 = 1,$$
$$s_{r2} = [xa_5][za_1][ya_4][za_2] = g_4^2,$$
$$s_{r3} = [xa_4][za_2][ya_0][za_3] = 1,$$
$$s_{r4} = [xa_3][za_0][ya_2][za_4] = g_3^{-1} g_3 = 1,$$
$$s_{r5} = [xa_2][za_4][ya_1][za_5] = g_2^2.$$

Note that $s_{(xy)^6 0} = s_{(xy)^6 4} = s_{(xy)^6 5}$, $s_{(xy)^6 1} = s_{(xy)^6 2} = s_{(xy)^6 3}$, up to conjugation, and $s_{xzyz2} = s_{xyxy^{-1}x^{-1}y^{-1}2}$, $s_{xzyz5} = s_{xyxy^{-1}x^{-1}y^{-1}5}$. Therefore, $\Gamma_2(2)$ has a presentation with generators g_1, g_2, g_3, g_4 and with relators $(g_4 g_3^{-1})^2$, $(g_1^{-1} g_3 g_4)^2$, g_4^2, $(g_2 g_1^{-1})^2$, $(g_3^{-1} g_1 g_2)^2$, g_2^2, $(g_4 g_3^{-1} g_1 g_2)^2$ and $(g_1^{-1} g_3 g_4 g_2)^2$.

Finally, we put $E_{12} = g_1$, $E_{21} = g_3$, $F_1 = g_4 g_3^{-1}$ and $F_2 = g_2 g_1^{-1}$. Note that $g_1 = E_{12}$, $g_2 = F_2 E_{12}$, $g_3 = E_{21}$ and $g_4 = F_1 E_{21}$. By Tietze transformations, we conclude that $\Gamma_2(2)$ has a finite presentation with generators E_{12}, E_{21}, F_1 and F_2, and with relators F_1^2, F_2^2, $(E_{12} F_1)^2$, $(E_{12} F_2)^2$, $(E_{21} F_1)^2$, $(E_{21} F_2)^2$ and $(F_1 F_2)^2$.

Thus, the proof of Proposition 3.1 is completed. Therefore, Theorem 1.1 is valid when $n = 2$.

4. Proof of the case $n = 3$ of Theorem 1.1

In this section, we prove the following proposition.

PROPOSITION 4.1. $\Gamma_2(3)$ *has a finite presentation with generators* E_{12}, E_{13}, E_{21}, E_{23}, E_{31}, E_{32}, F_1, F_2 *and* F_2, *and with the following relators*

(1) F_1^2, F_2^2, F_3^2,

(2) $(E_{12}F_1)^2$, $(E_{12}F_2)^2$, $(E_{13}F_1)^2$, $(E_{13}F_3)^2$, $(E_{21}F_2)^2$, $(E_{21}F_1)^2$, $(E_{23}F_2)^2$, $(E_{23}F_3)^2$, $(E_{31}F_3)^2$, $(E_{31}F_1)^2$, $(E_{32}F_3)^2$, $(E_{32}F_2)^2$, $(F_1F_2)^2$, $(F_1F_3)^2$, $(F_2F_3)^2$,

(3) (a) $[E_{12}, E_{13}]$, $[E_{21}, E_{23}]$, $[E_{31}, E_{32}]$, $[E_{21}, E_{31}]$, $[E_{12}, E_{32}]$, $[E_{13}, E_{23}]$, $[E_{12}, F_3]$, $[E_{21}, F_3], [E_{13}, F_2], [E_{31}, F_2], [E_{23}, F_1], [E_{32}, F_1], [E_{32}, E_{13}]E_{12}^2, [E_{23}, E_{12}]E_{13}^2$, $[E_{31}, E_{23}]E_{21}^2$, $[E_{13}, E_{21}]E_{23}^2$, $[E_{21}, E_{32}]E_{31}^2$, $[E_{12}, E_{31}]E_{32}^2$,

(b) $[E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32}, E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23}]$.

4.1. Preparation

For $R = \mathbf{Z}$ or $\mathbf{Z}_2$, let $\mathscr{B}_n(R)$ denote the simplicial complex whose $(k-1)$-simplex $\{x_1, x_2, \ldots, x_k\}$ is the set of k-vectors $x_i \in R^n$ such that $x_1, x_2, \ldots, x_k$ are mutually different column vectors of a matrix $A \in GL(n; R)$. In [2], Day and Putman proved that $\mathscr{B}_n(\mathbf{Z})$ is $(n-2)$-connected. Here, a simplicial complex X is m-connected if its geometric realization $|X|$ is m-connected. In addition, X is -1-connected if X is nonempty. Note that there is the natural left action $\Gamma_2(n) \times \mathscr{B}_n(\mathbf{Z}) \to \mathscr{B}_n(\mathbf{Z})$ defined by $A\{x_1, x_2, \ldots, x_k\} = \{Ax_1, Ax_2, \ldots, Ax_k\}$ for $A \in \Gamma_2(n)$ and $\{x_1, x_2, \ldots, x_k\} \in \mathscr{B}_n(\mathbf{Z})$, and that the action is without rotation.

In this section, we consider the case $n = 3$. Since $GL(3; \mathbf{Z}_2)$ is the quotient of $GL(3; \mathbf{Z})$ by $\Gamma_2(3)$, it follows that the orbit space $\Gamma_2(3)\backslash\mathscr{B}_3(\mathbf{Z})$ is isomorphic to $\mathscr{B}_3(\mathbf{Z}_2)$. Let $\varphi : \mathscr{B}_3(\mathbf{Z}) \to \mathscr{B}_3(\mathbf{Z}_2)$ be a natural surjection induced by the surjection $GL(3; \mathbf{Z}) \twoheadrightarrow GL(3; \mathbf{Z}_2)$.

For $1 \leq i \leq 7$, let v_i be $v_1 = e_1$, $v_2 = e_2$, $v_3 = e_3$, $v_4 = e_1 + e_2$, $v_5 = e_1 + e_3$, $v_6 = e_2 + e_3$ and $v_7 = e_1 + e_2 + e_3$, where e_1, e_2 and e_3 are canonical normal vectors in $\mathbf{Z}^3$. Then, the vertices of $\mathscr{B}_3(\mathbf{Z}_2)$ are $\varphi(v_i)$, the 1-simplices are $\varphi(\{v_i, v_j\})$, and the 2-simplices are $\varphi(\{v_i, v_j, v_k\})$, where $\{i, j, k\}$ is not $\{1, 2, 4\}$, $\{1, 3, 5\}$, $\{1, 6, 7\}$, $\{2, 3, 6\}$, $\{2, 5, 7\}$, $\{3, 4, 7\}$ and $\{4, 5, 6\}$. (Note that $\{v_1, v_2, v_4\}$, $\{v_1, v_3, v_5\}$, $\{v_1, v_6, v_7\}$, $\{v_2, v_3, v_6\}$, $\{v_2, v_5, v_7\}$, $\{v_3, v_4, v_7\}$ and $\{v_4, v_5, v_6\}$ are not 2-simplices of $\mathscr{B}_3(\mathbf{Z})$.)

We prove the next lemma.

LEMMA 4.2. $\Gamma_2(3)$ *is isomorphic to the quotient of* $\underset{1 \leq i \leq 7}{*} \Gamma_2(3)_{v_i}$ *by the normal subgroup generated by edge relators.*

For the definition of the edge relator, see Subsection 2.2.

Proof. We set followings

- $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$,
- $T = \{(v_1, v_i) \mid 2 \leq i \leq 7\} \cup V$,
- $E = \{(v_i, v_j) \mid 1 \leq i < j \leq 7\}$,
- $F = \{(v_i, v_j, v_k) \mid 1 \leq i < j < k \leq 7, \varphi(\{v_i, v_j, v_k\}) \in \mathscr{B}_3(\mathbf{Z}_2)\}$.

For $e = (v_i, v_j) \in E$, since $w(e) = t(e)$, we choose $g_e = 1$, and write $g_{ij} = g_e$. By Theorem 2.1, $\Gamma_2(3)$ is isomorphic to the quotient of $\left(\underset{1 \leq i \leq 7}{*} \Gamma_2(3)_{v_i} \right) *$ $\left(\underset{1 \leq i < j \leq 7}{*} \langle \hat{g}_{ij} \rangle \right)$ by the normal subgroup generated by followings

(1) $\hat{g}_{1i}$, where $2 \leq i \leq 7$,
(2) $\hat{g}_{ij}^{-1} X_{v_i} \hat{g}_{ij} X_{v_j}^{-1}$, where $1 \leq i < j \leq 7$ and $X \in \Gamma_2(3)_{(v_i, v_j)}$,
(3) $\hat{g}_\tau g_\tau^{-1}$, where $\tau \in F$.

Note that $g_\tau = g_{ij} g_{jk} g_{ik}^{-1}$ for $\tau = (v_i, v_j, v_k)$. Hence, the relation $\hat{g}_\tau g_\tau^{-1} = 1$ is equivalent to the relation $\hat{g}_{ij} \hat{g}_{jk} = \hat{g}_{ik}$. Since $\hat{g}_{1i} = 1$ for $2 \leq i \leq 7$, we have the relation $\hat{g}_{ij} = 1$ for $2 \leq i < j \leq 7$ except for $(i, j) = (2, 4)$, $(3, 5)$ and $(6, 7)$. For example, the relation $\hat{g}_{23} = 1$ is obtained from the relation $\hat{g}_{12} \hat{g}_{23} = \hat{g}_{13}$. In addition, relations $\hat{g}_{24} = 1$, $\hat{g}_{35} = 1$ and $\hat{g}_{67} = 1$ are obtained from relations $\hat{g}_{23} \hat{g}_{34} = \hat{g}_{24}$, $\hat{g}_{23} \hat{g}_{35} = \hat{g}_{25}$ and $\hat{g}_{26} \hat{g}_{67} = \hat{g}_{27}$, respectively. Hence, we have the relation $\hat{g}_{ij} = 1$ for $1 \leq i < j \leq 7$. Therefore, $\Gamma_2(3)$ is isomorphic to the quotient of $\underset{1 \leq i \leq 7}{*} \Gamma_2(3)_{v_i}$ by the normal subgroup generated by $A = \{ X_{v_i} X_{v_j}^{-1} \mid 1 \leq i < j \leq 7,$ $X \in \Gamma_2(3)_{(v_i, v_j)} \}$. Since A is the set of edge relators, we obtain the claim. □

We next consider presentations of $\Gamma_2(3)_{v_i}$ for all $1 \leq i \leq 7$ and edge relators.

4.2. Presentations of $\Gamma_2(3)_{v_i}$

LEMMA 4.3. *For $1 \leq t \leq n$ there is a short exact sequence*

$$0 \to \mathbf{Z}^{n-1} \to \Gamma_2(n)_{e_t} \to \Gamma_2(n-1) \to 1.$$

Proof. We first note that $A \in \Gamma_2(n)_{e_t}$ is a matrix whose t-column vector is e_t. For $\mathbf{Z}^{n-1}$ we give the presentation $\mathbf{Z}^{n-1} = \langle x_1, x_2, \ldots, x_{n-1} \mid x_i x_j x_i^{-1} x_j^{-1}$ $(1 \leq i < j \leq n-1) \rangle$. Let $\mathbf{Z}^{n-1} \to \Gamma_2(n)_{e_t}$ be the homomorphism which sends x_i to E_{ti} when $i < t$ and to E_{ti+1} when $i \geq t$. Let $\Gamma_2(n)_{e_t} \to \Gamma_2(n-1)$ be the homomorphism which sends A to A_{tt}, where A_{ij} is the $(n-1)$-submatrix of A obtained by removing the i-row vector and the j-column vector of A. Then, it follows that the sequence $0 \to \mathbf{Z}^{n-1} \to \Gamma_2(n)_{e_t} \to \Gamma_2(n-1) \to 1$ is exact. □

Remark 4.4. Let $\rho_t : \Gamma_2(n-1) \to \Gamma_2(n)_{e_t}$ be the homomorphism defined by

$$\rho_t(E_{ij}) = \begin{cases} (E_{ij})_{e_t} & \text{(when } i, j \leq t-1), \\ (E_{ij+1})_{e_t} & \text{(when } i \leq t-1, j \geq t), \\ (E_{i+1j})_{e_t} & \text{(when } j \leq t-1, i \geq t), \\ (E_{i+1j+1})_{e_t} & \text{(when } i, j \geq t), \end{cases}$$

$$\rho_t(F_i) = \begin{cases} (F_i)_{e_t} & \text{(when } i \leq t-1), \\ (F_{i+1})_{e_t} & \text{(when } i \geq t), \end{cases}$$

where subscripts e_t are added in order to indicate that these are the elements of $\Gamma_2(n)_{e_t}$, that is, we write A_{e_t} for $A \in \Gamma_2(n)_{e_t}$. Put $\Gamma_2(n-1) = \langle X | Y \rangle$. Then, from Lemma 4.3, $\Gamma_2(n)_{e_t}$ is generated by

- $(E_{ti})_{e_t}$ for $1 \le i \le n$ with $i \ne t$,
- $(E_{ij})_{e_t}$, $(F_i)_{e_t}$ for $1 \le i, j \le n$ with $i \ne j$ and $i, j \ne t$,

and has relators

(1) $[(E_{ti})_{e_t}, (E_{tj})_{e_t}]$ for $1 \le i, j \le n$ with $i \ne j$,

(2) $\rho_t(y)$ for $y \in Y$,

(3) $\cdot$ $(E_{ij})_{e_t}^{-1} (E_{ti})_{e_t} (E_{ij})_{e_t} \cdot (E_{tj})_{e_t}^{-2} (E_{ti})_{e_t}^{-1}$ for $1 \le i, j \le n$ with $i \ne j$ and $i, j \ne t$,

     $\cdot$ $(E_{ij})_{e_t}^{-1} (E_{tj})_{e_t} (E_{ij})_{e_t} \cdot (E_{tj})_{e_t}^{-1}$ for $1 \le i, j \le n$ with $i \ne j$ and $i, j \ne t$,

     $\cdot$ $(E_{ij})_{e_t}^{-1} (E_{tk})_{e_t} (E_{ij})_{e_t} \cdot (E_{tk})_{e_t}^{-1}$ for $1 \le i, j, k \le n$ with $i, j, k \ne t$ and i, j, k are mutually different (when $n \ge 4$),

     $\cdot$ $(F_i)_{e_t}^{-1} (E_{ti})_{e_t} (F_i)_{e_t} \cdot (E_{ti})_{e_t}$ for $1 \le i \le n$ with $i \ne t$,

     $\cdot$ $(F_i)_{e_t}^{-1} (E_{tj})_{e_t} (F_i)_{e_t} \cdot (E_{tj})_{e_t}^{-1}$ for $1 \le i, j \le n$ with $i \ne j$ and $i, j \ne t$.

The relators (3) can be rephrased as follows.

- $[(E_{ij})_{e_t}, (E_{ti})_{e_t}](E_{tj})_{e_t}^2$ for $1 \le i, j \le n$ with $i \ne j$ and $i, j \ne t$,
- $[(E_{ij})_{e_t}, (E_{tj})_{e_t}]$ for $1 \le i, j \le n$ with $i \ne j$ and $i, j \ne t$,
- $[(E_{ij})_{e_t}, (E_{tk})_{e_t}]$ for $1 \le i, j, k \le n$ with $i, j, k \ne t$ and i, j, k are mutually different (when $n \ge 4$),
- $((E_{ti})_{e_t} (F_i)_{e_t})^2$ for $1 \le i \le n$ with $i \ne t$,
- $[(E_{tj})_{e_t}, (F_i)_{e_t}]$ for $1 \le i, j \le n$ with $i \ne j$ and $i, j \ne t$.

By Lemma 4.3, Remark 4.4 and Proposition 3.1, we have the following.

LEMMA 4.5. $\Gamma_2(3)_{v_1}$ *has a finite presentation with generators* $(E_{12})_{v_1}$, $(E_{13})_{v_1}$, $(E_{23})_{v_1}$, $(E_{32})_{v_1}$, $(F_2)_{v_1}$ *and* $(F_3)_{v_1}$, *and with the following relators*

(1.1) $((F_2)_{v_1})^2$, $((F_3)_{v_1})^2$,

(1.2) $((E_{12})_{v_1}(F_2)_{v_1})^2$, $((E_{13})_{v_1}(F_3)_{v_1})^2$, $((E_{23})_{v_1}(F_2)_{v_1})^2$, $((E_{23})_{v_1}(F_3)_{v_1})^2$, $((E_{32})_{v_1}(F_2)_{v_1})^2$, $((E_{32})_{v_1}(F_3)_{v_1})^2$, $((F_2)_{v_1}(F_3)_{v_1})^2$,

(1.3) $[(E_{12})_{v_1}, (E_{13})_{v_1}]$, $[(E_{12})_{v_1}, (E_{32})_{v_1}]$, $[(E_{12})_{v_1}, (F_3)_{v_1}]$, $[(E_{13})_{v_1}, (E_{23})_{v_1}]$, $[(E_{13})_{v_1}, (F_2)_{v_1}]$, $[(E_{23})_{v_1}, (E_{12})_{v_1}](E_{13})_{v_1}^2$, $[(E_{32})_{v_1}, (E_{13})_{v_1}](E_{12})_{v_1}^2$.

For $X \in GL(n; \mathbf{Z})$, let $\Phi_X : \Gamma_2(n) \to \Gamma_2(n)$ be the homomorphism defined by $\Phi_X(A) = XAX^{-1}$. Note that this definition is well-defined, since $\Gamma_2(n)$ is a normal subgroup of $GL(n; \mathbf{Z})$. For $1 \le i, j \le n$ with $i \ne j$, let T_{ij} denote the matrix whose (i, j) entry is 1, diagonal entries are 1 and others are 0, and let S_i denote the matrix whose (i, i) and $(i+1, i+1)$ entries are 0, other diagonal entries are 1, $(i, i+1)$ and $(i+1, i)$ entries are 1 and others are 0. Using homomorphisms Φ_X for some $X \in GL(n; \mathbf{Z})$, we provide presentations of $\Gamma_2(n)_{v_i}$ for all $2 \le i \le 7$.

First, considering $\Phi_{S_1} : \Gamma_2(3)_{v_1} \to \Gamma_2(3)_{v_2}$, it follows that $\Gamma_2(3)_{v_2}$ has a finite presentation with generators $(E_{21})_{v_2}$, $(E_{23})_{v_2}$, $(E_{13})_{v_2}$, $(E_{31})_{v_2}$, $(F_1)_{v_2}$ and $(F_3)_{v_2}$, and with the following relators

(2.1) $((F_1)_{v_2})^2$, $((F_3)_{v_2})^2$,

(2.2) $((E_{21})_{v_2}(F_1)_{v_2})^2$, $((E_{23})_{v_2}(F_3)_{v_2})^2$, $((E_{13})_{v_2}(F_1)_{v_2})^2$, $((E_{13})_{v_2}(F_3)_{v_2})^2$, $((E_{31})_{v_2}(F_1)_{v_2})^2$, $((E_{31})_{v_2}(F_3)_{v_2})^2$, $((F_1)_{v_2}(F_3)_{v_2})^2$,

(2.3) $[(E_{21})_{v_2},(E_{23})_{v_2}]$, $[(E_{21})_{v_2},(E_{31})_{v_2}]$, $[(E_{21})_{v_2},(F_3)_{v_2}]$, $[(E_{23})_{v_2},(E_{13})_{v_2}]$, $[(E_{23})_{v_2},(F_1)_{v_2}]$, $[(E_{13})_{v_2},(E_{21})_{v_2}](E_{23})_{v_2}^2$, $[(E_{31})_{v_2},(E_{23})_{v_2}](E_{21})_{v_2}^2$.

Next, considering $\Phi_{S_2S_1} : \Gamma_2(3)_{v_1} \to \Gamma_2(3)_{v_3}$, it follows that $\Gamma_2(3)_{v_3}$ has a finite presentation with generators $(E_{31})_{v_3}$, $(E_{32})_{v_3}$, $(E_{12})_{v_3}$, $(E_{21})_{v_3}$, $(F_1)_{v_3}$ and $(F_2)_{v_3}$, and with the following relators

(3.1) $((F_1)_{v_3})^2$, $((F_2)_{v_3})^2$,

(3.2) $((E_{31})_{v_3}(F_1)_{v_3})^2$, $((E_{32})_{v_3}(F_2)_{v_3})^2$, $((E_{12})_{v_3}(F_1)_{v_3})^2$, $((E_{12})_{v_3}(F_2)_{v_3})^2$, $((E_{21})_{v_3}(F_1)_{v_3})^2$, $((E_{21})_{v_3}(F_2)_{v_3})^2$, $((F_1)_{v_3}(F_2)_{v_3})^2$,

(3.3) $[(E_{31})_{v_3},(E_{32})_{v_3}]$, $[(E_{31})_{v_3},(E_{21})_{v_3}]$, $[(E_{31})_{v_3},(F_2)_{v_3}]$, $[(E_{32})_{v_3},(E_{12})_{v_3}]$, $[(E_{32})_{v_3},(F_1)_{v_3}]$, $[(E_{12})_{v_3},(E_{31})_{v_3}](E_{32})_{v_3}^2$, $[(E_{21})_{v_3},(E_{32})_{v_3}](E_{31})_{v_3}^2$.

Next, considering $\Phi_{T_{21}} : \Gamma_2(3)_{v_1} \to \Gamma_2(3)_{v_4}$, it follows that $\Gamma_2(3)_{v_4}$ has a finite presentation with generators $(E_{21}F_2E_{12}F_1)_{v_4}$, $(E_{13}E_{23})_{v_4}$, $(E_{23})_{v_4}$, $(E_{31}^{-1}E_{32})_{v_4}$, $(E_{21}F_2)_{v_4}$ and $(F_3)_{v_4}$, and with the following relators

(4.1) $((E_{21}F_2)_{v_4})^2$, $((F_3)_{v_4})^2$,

(4.2) $((E_{21}F_2E_{12}F_1)_{v_4}(E_{21}F_2)_{v_4})^2$, $((E_{13}E_{23})_{v_4}(F_3)_{v_4})^2$, $((E_{23})_{v_4}(E_{21}F_2)_{v_4})^2$, $((E_{23})_{v_4}(F_3)_{v_4})^2$, $((E_{31}^{-1}E_{32})_{v_4}(E_{21}F_2)_{v_4})^2$, $((E_{31}^{-1}E_{32})_{v_4}(F_3)_{v_4})^2$, $((E_{21}F_2)_{v_4}(F_3)_{v_4})^2$,

(4.3) $[(E_{21}F_2E_{12}F_1)_{v_4},(E_{13}E_{23})_{v_4}]$, $[(E_{21}F_2E_{12}F_1)_{v_4},(E_{31}^{-1}E_{32})_{v_4}]$, $[(E_{21}F_2E_{12}F_1)_{v_4},(F_3)_{v_4}]$, $[(E_{13}E_{23})_{v_4},(E_{23})_{v_4}]$, $[(E_{13}E_{23})_{v_4},(E_{21}F_2)_{v_4}]$, $[(E_{23})_{v_4},(E_{21}F_2E_{12}F_1)_{v_4}](E_{13}E_{23})_{v_4}^2$, $[(E_{31}^{-1}E_{32})_{v_4},(E_{13}E_{23})_{v_4}](E_{21}F_2E_{12}F_1)_{v_4}^2$.

Next, considering $\Phi_{S_2T_{21}} : \Gamma_2(3)_{v_1} \to \Gamma_2(3)_{v_5}$, it follows that $\Gamma_2(3)_{v_5}$ has a finite presentation with generators $(E_{31}F_3E_{13}F_1)_{v_5}$, $(E_{12}E_{32})_{v_5}$, $(E_{32})_{v_5}$, $(E_{21}^{-1}E_{23})_{v_5}$, $(E_{31}F_3)_{v_5}$ and $(F_2)_{v_5}$, and with the following relators

(5.1) $((E_{31}F_3)_{v_5})^2$, $((F_2)_{v_5})^2$,

(5.2) $((E_{31}F_3E_{13}F_1)_{v_5}(E_{31}F_3)_{v_5})^2$, $((E_{12}E_{32})_{v_5}(F_2)_{v_5})^2$, $((E_{32})_{v_5}(E_{31}F_3)_{v_5})^2$, $((E_{32})_{v_5}(F_2)_{v_5})^2$, $((E_{21}^{-1}E_{23})_{v_5}(E_{31}F_3)_{v_5})^2$, $((E_{21}^{-1}E_{23})_{v_5}(F_2)_{v_5})^2$, $((E_{31}F_3)_{v_5}(F_2)_{v_5})^2$,

(5.3) $[(E_{31}F_3E_{13}F_1)_{v_5},(E_{12}E_{32})_{v_5}]$, $[(E_{31}F_3E_{13}F_1)_{v_5},(E_{21}^{-1}E_{23})_{v_5}]$, $[(E_{31}F_3E_{13}F_1)_{v_5},(F_2)_{v_5}]$, $[(E_{12}E_{32})_{v_5},(E_{32})_{v_5}]$, $[(E_{12}E_{32})_{v_5},(E_{31}F_3)_{v_5}]$, $[(E_{32})_{v_5},(E_{31}F_3E_{13}F_1)_{v_5}](E_{12}E_{32})_{v_5}^2$, $[(E_{21}^{-1}E_{23})_{v_5},(E_{12}E_{32})_{v_5}](E_{31}F_3E_{13}F_1)_{v_5}^2$.

Next, considering $\Phi_{S_1S_2T_{21}} : \Gamma_2(3)_{v_1} \to \Gamma_2(3)_{v_6}$, it follows that $\Gamma_2(3)_{v_6}$ has a finite presentation with generators $(E_{32}F_3E_{23}F_2)_{v_6}$, $(E_{21}E_{31})_{v_6}$, $(E_{31})_{v_6}$, $(E_{12}^{-1}E_{13})_{v_6}$, $(E_{32}F_3)_{v_6}$ and $(F_1)_{v_6}$, and with the following relators

(6.1) $((E_{32}F_3)_{v_6})^2$, $((F_1)_{v_6})^2$,

(6.2) $((E_{32}F_3E_{23}F_2)_{v_6}(E_{32}F_3)_{v_6})^2$, $((E_{21}E_{31})_{v_6}(F_1)_{v_6})^2$, $((E_{31})_{v_6}(E_{32}F_3)_{v_6})^2$,

$$((E_{31})_{v_6}(F_1)_{v_6})^2, \quad ((E_{12}^{-1}E_{13})_{v_6}(E_{32}F_3)_{v_6})^2, \quad ((E_{12}^{-1}E_{13})_{v_6}(F_1)_{v_6})^2,$$
$$((E_{32}F_3)_{v_6}(F_1)_{v_6})^2,$$
(6.3) $[(E_{32}F_3E_{23}F_2)_{v_6}, (E_{21}E_{31})_{v_6}], \quad [(E_{32}F_3E_{23}F_2)_{v_6}, (E_{12}^{-1}E_{13})_{v_6}],$
$$[(E_{32}F_3E_{23}F_2)_{v_6}, (F_1)_{v_6}], \quad [(E_{21}E_{31})_{v_6}, (E_{31})_{v_6}], \quad [(E_{21}E_{31})_{v_6}, (E_{32}F_3)_{v_6}],$$
$$[(E_{31})_{v_6}, (E_{32}F_3E_{23}F_2)_{v_6}](E_{21}E_{31})_{v_6}^2,$$
$$[(E_{12}^{-1}E_{13})_{v_6}, (E_{21}E_{31})_{v_6}](E_{32}F_3E_{23}F_2)_{v_6}^2.$$

Finally, considering $\Phi_{T_{31}T_{21}} : \Gamma_2(3)_{v_1} \to \Gamma_2(3)_{v_7}$, it follows that $\Gamma_2(3)_{v_7}$ has a finite presentation with generators $(E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32})_{v_7}$, $(E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23})_{v_7}$, $(E_{21}^{-1}E_{23})_{v_7}$, $(E_{31}^{-1}E_{32})_{v_7}$, $(E_{21}F_2)_{v_7}$ and $(E_{31}F_3)_{v_7}$, and with the following relators

(7.1) $((E_{21}F_2)_{v_7})^2$, $((E_{31}F_3)_{v_7})^2$,

(7.2) $((E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32})_{v_7}(E_{21}F_2)_{v_7})^2$, $((E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23})_{v_7}(E_{31}F_3)_{v_7})^2$,
$$((E_{21}^{-1}E_{23})_{v_7}(E_{21}F_2)_{v_7})^2, \quad ((E_{21}^{-1}E_{23})_{v_7}(E_{31}F_3)_{v_7})^2, \quad ((E_{31}^{-1}E_{32})_{v_7}(E_{21}F_2)_{v_7})^2,$$
$$((E_{31}^{-1}E_{32})_{v_7}(E_{31}F_3)_{v_7})^2, \quad ((E_{21}F_2)_{v_7}(E_{31}F_3)_{v_7})^2,$$

(7.3) $[(E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32})_{v_7}, (E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23})_{v_7}]$, $[(E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32})_{v_7}$,
$(E_{31}^{-1}E_{32})_{v_7}]$, $[(E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32})_{v_7}, (E_{31}F_3)_{v_7}]$, $[(E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23})_{v_7}$,
$(E_{21}^{-1}E_{23})_{v_7}]$, $[(E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23})_{v_7}, (E_{21}F_2)_{v_7}]$,
$[(E_{21}^{-1}E_{23})_{v_7}, (E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32})_{v_7}](E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23})_{v_7}^2$,
$[(E_{31}^{-1}E_{32})_{v_7}, (E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23})_{v_7}](E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32})_{v_7}^2$.

4.3. On edge relations

Note that

$$\Gamma_2(3)_{(v_1, v_2)} = \Gamma_2(3)_{(v_1, v_4)} = \Gamma_2(3)_{(v_2, v_4)},$$
$$\Gamma_2(3)_{(v_1, v_3)} = \Gamma_2(3)_{(v_1, v_5)} = \Gamma_2(3)_{(v_3, v_5)},$$
$$\Gamma_2(3)_{(v_2, v_3)} = \Gamma_2(3)_{(v_2, v_6)} = \Gamma_2(3)_{(v_3, v_6)},$$
$$\Gamma_2(3)_{(v_1, v_6)} = \Gamma_2(3)_{(v_1, v_7)} = \Gamma_2(3)_{(v_6, v_7)},$$
$$\Gamma_2(3)_{(v_2, v_5)} = \Gamma_2(3)_{(v_2, v_7)} = \Gamma_2(3)_{(v_5, v_7)},$$
$$\Gamma_2(3)_{(v_3, v_4)} = \Gamma_2(3)_{(v_3, v_7)} = \Gamma_2(3)_{(v_4, v_7)}.$$

It follows that $\Gamma_2(3)_{(v_1, v_2)}$, $\Gamma_2(3)_{(v_1, v_4)}$ and $\Gamma_2(3)_{(v_2, v_4)}$ are generated by

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then we have the following edge relations
- $(E_{13})_{v_1} = (E_{13})_{v_2} = (E_{13}E_{23})_{v_4}(E_{23})_{v_4}^{-1}$,
- $(E_{23})_{v_1} = (E_{23})_{v_2} = (E_{23})_{v_4}$,
- $(F_3)_{v_1} = (F_3)_{v_2} = (F_3)_{v_4}$.

Next, considering $\Phi_{S_2} : \Gamma_2(3)_{(v_1, v_2)} \to \Gamma_2(3)_{(v_1, v_3)}$, it follows that $\Gamma_2(3)_{(v_1, v_3)}$, $\Gamma_2(3)_{(v_1, v_5)}$ and $\Gamma_2(3)_{(v_3, v_5)}$ are generated by

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we have the following edge relations

- $(E_{12})_{v_1} = (E_{12})_{v_3} = (E_{12}E_{32})_{v_5}(E_{32})_{v_5}^{-1}$,
- $(E_{32})_{v_1} = (E_{32})_{v_3} = (E_{32})_{v_5}$,
- $(F_2)_{v_1} = (F_2)_{v_3} = (F_2)_{v_5}$.

Next, considering $\Phi_{S_1 S_2} : \Gamma_2(3)_{(v_1, v_2)} \to \Gamma_2(3)_{(v_2, v_3)}$, it follows that $\Gamma_2(3)_{(v_2, v_3)}$, $\Gamma_2(3)_{(v_2, v_6)}$ and $\Gamma_2(3)_{(v_3, v_6)}$ are generated by

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we have the following edge relations

- $(E_{21})_{v_2} = (E_{21})_{v_3} = (E_{21}E_{31})_{v_6}(E_{31})_{v_6}^{-1}$,
- $(E_{31})_{v_2} = (E_{31})_{v_3} = (E_{31})_{v_6}$,
- $(F_1)_{v_2} = (F_1)_{v_3} = (F_1)_{v_6}$.

Next, considering $\Phi_{T_{32}} : \Gamma_2(3)_{(v_1, v_2)} \to \Gamma_2(3)_{(v_1, v_6)}$, it follows that $\Gamma_2(3)_{(v_1, v_6)}$, $\Gamma_2(3)_{(v_1, v_7)}$ and $\Gamma_2(3)_{(v_6, v_7)}$ are generated by

$$\begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & -2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix}.$$

Then we have the following edge relations

- $(E_{12})_{v_1}^{-1}(E_{13})_{v_1} = (E_{12}^{-1}E_{13})_{v_6}$
  $= (E_{31}F_3)_{v_7}(E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23})_{v_7}(E_{21}^{-1}E_{23})_{v_7}^{-1}(E_{21}F_2)_{v_7}$
  $\cdot (E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32})_{v_7}(E_{31}^{-1}E_{32})_{v_7}^{-1}$,
- $(E_{32})_{v_1}(F_3)_{v_1}(E_{23})_{v_1}(F_2)_{v_1} = (E_{32}F_3E_{23}F_2)_{v_6}$
  $= (E_{31}^{-1}E_{32})_{v_7}(E_{31}F_3)_{v_7}(E_{21}^{-1}E_{23})_{v_7}(E_{21}F_2)_{v_7}$,
- $(E_{32})_{v_1}(F_3)_{v_1} = (E_{32}F_3)_{v_6} = (E_{31}^{-1}E_{32})_{v_7}(E_{31}F_3)_{v_7}$.

Next, considering $\Phi_{S_1 T_{32}} : \Gamma_2(3)_{(v_1, v_2)} \to \Gamma_2(3)_{(v_2, v_5)}$, it follows that $\Gamma_2(3)_{(v_2, v_5)}$, $\Gamma_2(3)_{(v_2, v_7)}$ and $\Gamma_2(3)_{(v_5, v_6)}$ are generated by

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ -2 & 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}.$$

Then we have the following edge relations

- $(E_{21})_{v_2}^{-1}(E_{23})_{v_2} = (E_{21}^{-1}E_{23})_{v_5} = (E_{21}^{-1}E_{23})_{v_7}$,
- $(E_{31})_{v_2}(F_3)_{v_2}(E_{13})_{v_2}(F_1)_{v_2} = (E_{31}F_3E_{13}F_1)_{v_5}$
  $= (E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23})_{v_7}(E_{21}^{-1}E_{23})_{v_7}^{-1}$,
- $(E_{31})_{v_2}(F_3)_{v_2} = (E_{31}F_3)_{v_5} = (E_{31}F_3)_{v_7}$.

Next, considering $\Phi_{S_2 S_1 T_{32}} : \Gamma_2(3)_{(v_1, v_2)} \to \Gamma_2(3)_{(v_3, v_4)}$, it follows that $\Gamma_2(3)_{(v_3, v_4)}$, $\Gamma_2(3)_{(v_3, v_7)}$ and $\Gamma_2(3)_{(v_4, v_7)}$ are generated by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we have the following edge relations
- $(E_{31})_{v_3}^{-1}(E_{32})_{v_3} = (E_{31}^{-1}E_{32})_{v_4} = (E_{31}^{-1}E_{32})_{v_7}$,
- $(E_{21})_{v_3}(F_2)_{v_3}(E_{12})_{v_3}(F_1)_{v_3} = (E_{21}F_2E_{12}F_1)_{v_4}$
  $= (E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32})_{v_7}(E_{31}^{-1}E_{32})_{v_7}^{-1}$,
- $(E_{21})_{v_3}(F_2)_{v_3} = (E_{21}F_2)_{v_4} = (E_{21}F_2)_{v_7}$.

Next, considering $\Phi_{T_{31}T_{32}} : \Gamma_2(3)_{(v_1, v_2)} \to \Gamma_2(3)_{(v_5, v_6)}$, it follows that $\Gamma_2(3)_{(v_5, v_6)}$ is generated by

$$\begin{pmatrix} -1 & -2 & 2 \\ 0 & 1 & 0 \\ -2 & -2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ -2 & -1 & 2 \\ -2 & -2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & -1 \end{pmatrix}.$$

Then we have the following edge relations
- $(E_{12}E_{32})_{v_5}^{-1}(E_{31}F_3E_{13}F_1)_{v_5} = (E_{31})_{v_6}(F_1)_{v_6}(E_{32}F_3)_{v_6}(E_{12}^{-1}E_{13})_{v_6}^{-1}$,
- $(E_{32})_{v_5}(F_2)_{v_5}(E_{31}F_3)_{v_5}(E_{21}^{-1}E_{23})_{v_5}^{-1} = (E_{21}E_{31})_{v_6}^{-1}(E_{32}F_3E_{23}F_2)_{v_6}$,
- $(E_{32})_{v_5}(E_{31}F_3)_{v_5} = (E_{31})_{v_6}(E_{32}F_3)_{v_6}$.

Next, considering $\Phi_{S_2 T_{31}T_{32}} : \Gamma_2(3)_{(v_1, v_2)} \to \Gamma_2(3)_{(v_4, v_6)}$, it follows that $\Gamma_2(3)_{(v_4, v_6)}$ is generated by

$$\begin{pmatrix} -1 & 2 & -2 \\ -2 & 3 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ -2 & 3 & -2 \\ -2 & 2 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we have the following edge relations
- $(E_{13}E_{23})_{v_4}^{-1}(E_{21}F_2E_{12}F_1)_{v_4}$
  $= (E_{21}E_{31})_{v_6}(E_{31})_{v_6}^{-1}(F_1)_{v_6}(E_{32}F_3)_{v_6}(E_{32}F_3E_{23}F_2)_{v_6}(E_{12}^{-1}E_{13})_{v_6}$,
- $(E_{23})_{v_4}(F_3)_{v_4}(E_{21}F_2)_{v_4}(E_{31}^{-1}E_{32})_{v_4}^{-1} = (E_{21}E_{31})_{v_6}^{-1}(E_{32}F_3E_{23}F_2)_{v_6}^{-1}$,
- $(E_{23})_{v_4}(E_{21}F_2)_{v_4} = (E_{21}E_{31})_{v_6}(E_{31})_{v_6}^{-1}(E_{32}F_3)_{v_6}(E_{32}F_3E_{23}F_2)_{v_6}$.

Finally, considering $\Phi_{S_1 S_2 T_{31}T_{32}} : \Gamma_2(3)_{(v_1, v_2)} \to \Gamma_2(3)_{(v_4, v_5)}$, it follows that $\Gamma_2(3)_{(v_4, v_5)}$ is generated by

$$\begin{pmatrix} 3 & -2 & -2 \\ 2 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & -2 & -2 \\ 0 & 1 & 0 \\ 2 & -2 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we have the following edge relations
- $(E_{13}E_{23})_{v_4}^{-1}(E_{21}F_2E_{12}F_1)_{v_4}^{-1}$
  $= (E_{12}E_{32})_{v_5}(E_{32})_{v_5}^{-1}(F_2)_{v_5}(E_{31}F_3)_{v_5}(E_{31}F_3E_{13}F_1)_{v_5}(E_{21}^{-1}E_{23})_{v_5}$,

- $(E_{13}E_{23})_{v_4}(E_{23})_{v_4}^{-1}(F_3)_{v_4}(E_{21}F_2)_{v_4}(E_{21}F_2E_{12}F_1)_{v_4}(E_{31}^{-1}E_{32})_{v_4}$
  $= (E_{12}E_{32})_{v_5}^{-1}(E_{31}F_3E_{13}F_1)_{v_5}^{-1}$,
- $(E_{13}E_{23})_{v_4}(E_{23})_{v_4}^{-1}(E_{21}F_2)_{v_4}(E_{21}F_2E_{12}F_1)_{v_4}$
  $= (E_{12}E_{32})_{v_5}(E_{32})_{v_5}^{-1}(E_{31}F_3)_{v_5}(E_{31}F_3E_{13}F_1)_{v_5}$.

Therefore, using Tietze transformations, by Lemma 4.2, we obtain the presentation for Proposition 4.1 (For more details see Appendix A). Thus, Theorem 1.1 is valid when $n = 3$.

5. A simplicial complex on which $\Gamma_2(n)$ acts

Let $\Gamma_2\mathscr{B}_n(\mathbf{Z})$ denote the subcomplex of $\mathscr{B}_n(\mathbf{Z})$ whose $(k-1)$-simplex $\{x_1, x_2, \ldots, x_k\}$ is the set of k-vectors $x_i \in \mathbf{Z}^n$ such that $x_1, x_2, \ldots, x_k$ are mutually different column vectors of a matrix $A \in \Gamma_2(n)$. Note that for a vertex v, we have $v \equiv e_i \bmod 2$ for some $1 \le i \le n$, where $e_1, e_2, \ldots, e_n$ are canonical normal vectors in $\mathbf{Z}^n$. For a $(k-1)$-simplex $\Delta = \{x_1, x_2, \ldots, x_k\}$, $A \in \Gamma_2(n)$ is an *extension* of Δ if each x_i is a column vector of A.

In this section, we prove the following proposition.

PROPOSITION 5.1. *For $n \ge 4$, the simplicial complex $\Gamma_2\mathscr{B}_n(\mathbf{Z})$ is simply connected.*

In a proof of this proposition, we will use the idea of Day-Putman [2] for proving that $\mathscr{B}_n(\mathbf{Z})$ is $(n-2)$-connected.

5.1. Preparation
Let X be a simplicial complex. Then we define followings.
- For a simplex $\Delta \in X$, $\mathrm{star}_X(\Delta)$ is the subcomplex of X whose simplex $\Delta' \in X$ satisfies that $\Delta, \Delta' \subset \Delta''$ for some simplex $\Delta'' \in X$. We also define $\mathrm{star}_X(\emptyset) = X$.
- For a simplex $\Delta \in X$, $\mathrm{link}_X(\Delta)$ is the subcomplex of $\mathrm{star}_X(\Delta)$ whose simplex $\Delta' \in \mathrm{star}_X(\Delta)$ does not intersect Δ. We also define $\mathrm{link}_X(\emptyset) = X$.

Here, we prove followings.

LEMMA 5.2. *For $n \ge 2$, $\Gamma_2\mathscr{B}_n(\mathbf{Z})$ is path connected.*

Proof. We first consider the case $n = 2$. Let $v_0 = v_{01}e_1 + v_{02}e_2 \in \Gamma_2\mathscr{B}_2(\mathbf{Z})$ be a vertex. Then there exists a vertex $v_1 = v_{11}e_1 + v_{12}e_2 \in \Gamma_2\mathscr{B}_2(\mathbf{Z})$ such that $\{v_0, v_1\} \in \Gamma_2\mathscr{B}_2(\mathbf{Z})$. Note that $v_{01}v_{12} - v_{02}v_{11} = \pm 1$. By Euclidean algorithm, we can suppose that $|v_{01}| > |v_{11}|$. Similarly, there exist vertices $v_2 = v_{21}e_1 + v_{22}e_2, \ldots,$ $v_k = v_{k1}e_1 + v_{k2}e_2 \in \Gamma_2\mathscr{B}_2(\mathbf{Z})$ such that $\{v_i, v_{i+1}\} \in \Gamma_2\mathscr{B}_2(\mathbf{Z})$, $|v_{i1}| > |v_{i+1\,1}|$ for $1 \le i \le k-1$ and $v_k = e_1$ or e_2, for some positive integer k. Hence, $\Gamma_2\mathscr{B}_2(\mathbf{Z})$ is path connected.

Next, we suppose $n \ge 3$. Let $v, w \in \Gamma_2\mathscr{B}_n(\mathbf{Z})$ be vertices. Without loss of generality, we suppose $v \equiv e_1$ and $w \equiv e_2 \bmod 2$. Then there is an extension

$A \in \Gamma_2(n)$ of v. We write $A^{-1}w = \sum_{i=1}^n a_i e_i$. Let $S_{A^{-1}w} = \sum_{i=3}^n |a_i|$. For $3 \leq i \leq n$, if $|a_2| < |a_i|$, there is an integer $u \in \mathbf{Z}$ such that $|a_2| > |a_i + 2ua_2|$. Then we have that $S_{E_{i2}^u A^{-1}w} < S_{A^{-1}w}$ and $E_{i2}^u A^{-1}v = e_1$. If $|a_2| > |a_i| \neq 0$, there is an integer $u' \in \mathbf{Z}$ such that $|a_2 + 2u'a_i| < |a_i|$. In addition, there is an integer $u'' \in \mathbf{Z}$ such that $|a_2 + 2u'a_1| > |a_i + 2u''(a_2 + 2u'a_1)|$. Then we have that $S_{E_{i2}^{u''} E_{2i}^{u'} A^{-1}w} < S_{A^{-1}w}$ and $E_{i2}^{u''} E_{2i}^{u'} A^{-1}v = e_1$. Repeating this operation, we conclude that there exists $B \in \Gamma_2(n)$ such that $S_{Bw} = 0$ and $Bv = e_1$. Note that Bw can be regarded as a vertex in $\Gamma_2\mathscr{B}_2(\mathbf{Z})$. Hence, Bw is joined to e_1, that is, Bw is joined to Bv. The action of B^{-1} brings the path joining Bw with Bv to the path joining w with v. Thus, $\Gamma_2\mathscr{B}_n(\mathbf{Z})$ is path connected. $\square$

LEMMA 5.3. *Let $\Delta \in \Gamma_2\mathscr{B}_n(\mathbf{Z})$ be a $(k-1)$-simplex. Then we have followings.*

- $\mathrm{star}_{\Gamma_2\mathscr{B}_n(\mathbf{Z})}(\Delta)$ *is isomorphic to* $\mathrm{star}_{\Gamma_2\mathscr{B}_n(\mathbf{Z})}(\{e_1, e_2, \ldots, e_k\})$ *as a simplicial complex.*
- $\mathrm{link}_{\Gamma_2\mathscr{B}_n(\mathbf{Z})}(\Delta)$ *is isomorphic to* $\mathrm{link}_{\Gamma_2\mathscr{B}_n(\mathbf{Z})}(\{e_1, e_2, \ldots, e_k\})$ *as a simplicial complex.*

Proof. For $\Delta = \{x_1, x_2, \ldots, x_k\}$, suppose $x_j \equiv e_{i(j)} \bmod 2$. Let $A \in \Gamma_2(n)$ be an extension of Δ. Then restrictions of the action of A^{-1} on $\Gamma_2\mathscr{B}_n(\mathbf{Z})$

$$A^{-1}|_{\mathrm{star}_{\Gamma_2\mathscr{B}_n(\mathbf{Z})}(\Delta)} : \mathrm{star}_{\Gamma_2\mathscr{B}_n(\mathbf{Z})}(\Delta) \to \mathrm{star}_{\Gamma_2\mathscr{B}_n(\mathbf{Z})}(\{e_{i(1)}, e_{i(2)}, \ldots, e_{i(k)}\}),$$

$$A^{-1}|_{\mathrm{link}_{\Gamma_2\mathscr{B}_n(\mathbf{Z})}(\Delta)} : \mathrm{link}_{\Gamma_2\mathscr{B}_n(\mathbf{Z})}(\Delta) \to \mathrm{link}_{\Gamma_2\mathscr{B}_n(\mathbf{Z})}(\{e_{i(1)}, e_{i(2)}, \ldots, e_{i(k)}\})$$

are isomorphisms as a simplicial map. It is clear that $\mathrm{star}_{\Gamma_2\mathscr{B}_n(\mathbf{Z})}(\{e_{i(1)}, e_{i(2)}, \ldots, e_{i(k)}\})$ and $\mathrm{link}_{\Gamma_2\mathscr{B}_n(\mathbf{Z})}(\{e_{i(1)}, e_{i(2)}, \ldots, e_{i(k)}\})$ are respectively isomorphic to $\mathrm{star}_{\Gamma_2\mathscr{B}_n(\mathbf{Z})}(\{e_1, e_2, \ldots, e_k\})$ and $\mathrm{link}_{\Gamma_2\mathscr{B}_n(\mathbf{Z})}(\{e_1, e_2, \ldots, e_k\})$. Thus, we obtain the claim. $\square$

COROLLARY 5.4. *Let $\Delta \in \Gamma_2\mathscr{B}_n(\mathbf{Z})$ be a $(k-1)$-simplex. If $n - k \geq 2$, then $\mathrm{link}_{\Gamma_2\mathscr{B}_n(\mathbf{Z})}(\Delta)$ is path connected.*

Proof. By an argument similar to the proof of Lemma 5.2, we have that $\mathrm{link}_{\Gamma_2\mathscr{B}_n(\mathbf{Z})}(\{e_1, e_2, \ldots, e_k\})$ is path connected. By Lemma 5.3, $\mathrm{link}_{\Gamma_2\mathscr{B}_n(\mathbf{Z})}(\Delta)$ is also path connected. $\square$

5.2. Proof of Proposition 5.1

We suppose $n \geq 4$. Let $\alpha = \{x_i, \{x_i, x_{i+1}\} \mid 1 \leq i \leq k, x_{k+1} = x_1\}$ be a loop on $\Gamma_2\mathscr{B}_n(\mathbf{Z})$. We show that α is null-homotopic.

For $v = \sum_{i=1}^n v_i e_i \in \mathbf{Z}^n$, we define $\mathrm{Rank}(v) = |v_n|$. Let $R_\alpha = \max \mathrm{Rank}(x_i)$. We first prove the next lemma.

LEMMA 5.5. *For a 1-simplex $\{v, w\} \in \Gamma_2\mathscr{B}_n(\mathbf{Z})$ with $\mathrm{Rank}(v) = \mathrm{Rank}(w) = 0$, we have $\{v, w\} \in \mathrm{link}_{\Gamma_2\mathscr{B}_n(\mathbf{Z})}(e_n)$.*

Proof. Note that $v \not\equiv w$ mod 2. Suppose that $v \equiv e_i$, $w \equiv e_j$ mod 2 and $i < j$. Since $\mathrm{Rank}(v) = \mathrm{Rank}(w) = 0$, we have that $v, w \not\equiv e_n$ mod 2. There exists an extension $A = (a_1 a_2 \cdots a_n) \in \Gamma_2(n)$ of $\{v, w\}$. Let $S_A = \sum_{l=1}^{n} \mathrm{Rank}(a_l)$. Note that S_A is odd.

First, we consider the case $S_A = 1$. Note that $\mathrm{Rank}(a_l) = 0$ for $1 \le l \le n-1$ and $\mathrm{Rank}(a_n) = 1$. Put $a_n = \sum_{i=1}^{n-1} 2b_i e_i + \varepsilon e_n$, where $\varepsilon = \pm 1$. Let $B = E_{1n}^{b_1} E_{2n}^{b_2} \cdots E_{n-1n}^{b_{n-1}} F_n^{(\varepsilon-1)/2}$. Then we have $BA = (a_1 \cdots a_{n-1} e_n)$. Hence, we have that $\{v, w\} = \{a_i, a_j\} \in \mathrm{link}_{\Gamma_2 \mathscr{B}_n(\mathbf{Z})}(e_n)$.

Next, we suppose $S_A \ge 3$. Note that there exists $1 \le l \le n-1$ with $l \ne i, j$ such that $\mathrm{Rank}(a_l) \ne 0$. If $\mathrm{Rank}(a_l) > \mathrm{Rank}(a_n)$, there exists an integer $u \in \mathbf{Z}$ such that $\mathrm{Rank}(a_l + 2u a_n) < \mathrm{Rank}(a_n)$. Then we have that $A E_{nl}^{u}$ is an extension of $\{v, w\}$ and that $S_{AE_{nl}^{u}} < S_A$. Similarly, if $\mathrm{Rank}(a_l) < \mathrm{Rank}(a_n)$, there exists an integer $u' \in \mathbf{Z}$ such that $\mathrm{Rank}(a_l) > \mathrm{Rank}(a_n + 2u' a_l)$. Then we have that $AE_{ln}^{u'}$ is an extension of $\{v, w\}$ and that $S_{AE_{ln}^{u'}} < S_A$. Repeating this operation, we conclude that there exists an extension $A' \in \Gamma_2(n)$ of $\{v, w\}$ such that $S_{A'} = 1$. Therefore, we have $\{v, w\} \in \mathrm{link}_{\Gamma_2 \mathscr{B}_n(\mathbf{Z})}(e_n)$. Thus, we obtain the claim. $\square$

When $R_\alpha = 0$, by this lemma, we have $\{x_i, x_{i+1}\} \in \mathrm{link}_{\Gamma_2 \mathscr{B}_n(\mathbf{Z})}(e_n)$. Namely, the loop α is in $\mathrm{link}_{\Gamma_2 \mathscr{B}_n(\mathbf{Z})}(e_n)$. Since $\mathrm{link}_{\Gamma_2 \mathscr{B}_n(\mathbf{Z})}(e_n)$ is the subcomplex of $\mathrm{star}_{\Gamma_2 \mathscr{B}_n(\mathbf{Z})}(e_n)$ and $\mathrm{star}_{\Gamma_2 \mathscr{B}_n(\mathbf{Z})}(e_n)$ is contractible, α is null-homotopic. Therefore, we next assume $R_\alpha > 0$.

Suppose that R_α is odd. There exists $1 \le i \le k$ such that $\mathrm{Rank}(x_i) = R_\alpha$. Since R_α is odd, we have that $x_i \equiv e_n$, $x_{i\pm 1} \not\equiv e_n$ mod 2 and $\mathrm{Rank}(x_{i\pm 1}) < R_\alpha$. By Corollary 5.4, we have that $\mathrm{link}_{\Gamma_2 \mathscr{B}_n(\mathbf{Z})}(x_i)$ is path connected. Since $x_{i+1} \in \mathrm{link}_{\Gamma_2 \mathscr{B}_n(\mathbf{Z})}(x_i)$, there exists a path $\{y_j, y_l, \{y_j, y_{j+1}\} \mid 1 \le j \le l-1\}$ on $\mathrm{link}_{\Gamma_2 \mathscr{B}_n(\mathbf{Z})}(x_i)$ between x_{i-1} and x_{i+1} such that $y_1 = x_{i-1}$ and $y_l = x_{i+1}$ (see Figure 1). Since R_α is odd and $\mathrm{Rank}(y_j)$ is even for each y_j, there exists an
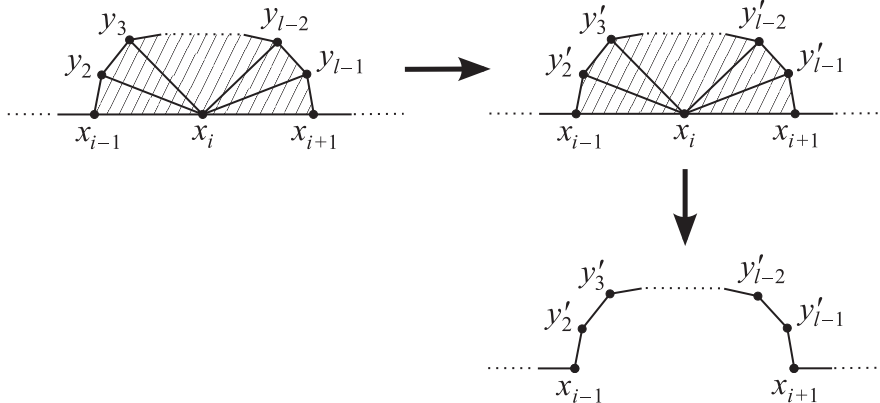

FIGURE 1.   The case R_α is odd.

integer $s_j \in \mathbf{Z}$ such that $\mathrm{Rank}(y'_j) < R_\alpha$, where $y'_j = y_j + 2s_j x_i$. We choose $s_j = 0$ if $\mathrm{Rank}(y_j) < R_\alpha$. When $y_j \equiv e_t$, $y_{j+1} \equiv e_u$ mod 2, for an extension $A \in \Gamma_2(n)$ of $\{x_i, y_j, y_{j+1}\}$, we have that $\{x_i, y'_j, y'_{j+1}\} = \{AE_{nt}^{s_j} E_{nu}^{s_{j+1}} e_n, AE_{nt}^{s_j} E_{nu}^{s_{j+1}} e_t, AE_{nt}^{s_j} E_{nu}^{s_{j+1}} e_u\}$. Hence $\{x_i, y'_j, y'_{j+1}\}$ is a 2-simplex which has an extension $AE_{nt}^{s_j} E_{nu}^{s_{j+1}}$. Therefore we have that the path $\{y'_j, y'_l, \{y'_j, y'_{j+1}\} \mid 1 \le j \le l-1\}$ between x_{i-1} and x_{i+1} is in $\mathrm{link}_{\Gamma_2 \mathscr{B}_n(\mathbf{Z})}(x_i)$ (see Figure 1). Let $\alpha' = \alpha \cup \{y'_j, y'_l, \{y'_j, y'_{j+1}\} \mid 1 \le j \le l-1\} \setminus \{x_i, \{x_i, x_{i\pm 1}\}\}$. Then α' is homotopic to α (see Figure 1). For all x_i with $\mathrm{Rank}(x_i) = R_\alpha$, applying the same operation, we conclude that $R_\beta < R_\alpha$, where β is a resulting loop which is homotopic to α.

Next, suppose that R_α is even. There exists $1 \le i \le k$ such that $\mathrm{Rank}(x_i) = R_\alpha$. Since R_α is even, we have $x_i \not\equiv e_n$ mod 2.

Remark 5.6. Under the assumption $n \ge 4$, we may suppose that α satisfies all of the following conditions.
- $\mathrm{Rank}(x_{i\pm 1}) < R_\alpha$,
- $x_{i\pm 1} \not\equiv e_n$ mod 2,
- $x_{i-1} \not\equiv x_{i+1}$ mod 2.

Proof. Without loss of generality, we suppose that $x_i \equiv e_1$ mod 2.
- Suppose that $\mathrm{Rank}(x_{i-1}) = R_\alpha$. Since R_α is even we have $x_{i-1} \not\equiv e_n$ mod 2. Without loss of generality, we suppose that $x_{i-1} \equiv e_2$ mod 2. There exists an extension $A \in \Gamma_2(n)$ of $\{x_i, x_{i-1}\}$ such that $\mathrm{Rank}(Ae_n) < R_\alpha$. In fact, if $\mathrm{Rank}(Ae_n) > R_\alpha$, there is an integer $u \in \mathbf{Z}$ such that $\mathrm{Rank}(AE_{1n}^u e_n) < R_\alpha$. Then we choose AE_{1n}^u in place of A as an extension of $\{x_i, x_{i-1}\}$. (Note that $\mathrm{Rank}(Ae_n)$ and $\mathrm{Rank}(AE_{1n}^u e_n)$ are not equal to R_α, since these are odd.) Let $y = Ae_n$, and let $\alpha' = \alpha \cup \{y, \{x_{i-1}, y\}, \{y, x_i\}\} \setminus \{\{x_{i-1}, x_i\}\}$. Then α' is homotopic to α. Hence, considering α' in place of α, we may suppose $\mathrm{Rank}(x_{i-1}) < R_\alpha$. Similarly, we may suppose $\mathrm{Rank}(x_{i+1}) < R_\alpha$.
- Suppose that $x_{i-1} \equiv e_n$ mod 2. Since $\mathrm{Rank}(x_{i-1})$ is odd we have $\mathrm{Rank}(x_{i-1}) < R_\alpha$. There exists an extension $A \in \Gamma_2(n)$ of $\{x_i, x_{i-1}\}$ such that $\mathrm{Rank}(Ae_2) < \mathrm{Rank}(x_{i-1})(< R_\alpha)$. In fact, if $\mathrm{Rank}(Ae_2) > \mathrm{Rank}(x_{i-1})$, there is an integer $u \in \mathbf{Z}$ such that $\mathrm{Rank}(AE_{n2}^u e_2) < \mathrm{Rank}(x_{i-1})$. Then we choose AE_{n2}^u in place of A as an extension of $\{x_i, x_{i-1}\}$. (Note that $\mathrm{Rank}(Ae_2)$ and $\mathrm{Rank}(AE_{n2}^u e_2)$ are not equal to $\mathrm{Rank}(x_{i-1})$, since these are even.) Let $y = Ae_2$, and let $\alpha' = \alpha \cup \{y, \{x_{i-1}, y\}, \{y, x_i\}\} \setminus \{\{x_{i-1}, x_i\}\}$. Then α' is homotopic to α. Hence, considering α' in place of α, we may suppose $\mathrm{Rank}(x_{i-1}) < R_\alpha$ and $x_{i-1} \not\equiv e_n$ mod 2. Similarly, we may suppose $\mathrm{Rank}(x_{i+1}) < R_\alpha$ and $x_{i+1} \not\equiv e_n$ mod 2.
- Suppose that $\mathrm{Rank}(x_{i\pm 1}) < R_\alpha$, $x_{i\pm 1} \not\equiv e_n$ mod 2 and $x_{i-1} \equiv x_{i+1}$ mod 2. Without loss of generality, we suppose that $x_{i\pm 1} \equiv e_2$ mod 2. There exists an extension $A \in \Gamma_2(n)$ of $\{x_i, x_{i-1}\}$ such that $\mathrm{Rank}(Ae_3) \le \mathrm{Rank}(x_{i-1})(< R_\alpha)$. In fact, if $\mathrm{Rank}(Ae_3) > \mathrm{Rank}(x_{i-1})$, there is an integer $u \in \mathbf{Z}$ such that $\mathrm{Rank}(AE_{23}^u e_3) \le \mathrm{Rank}(x_{i-1})$. Then we choose AE_{23}^u in

place of A as an extension of $\{x_i, x_{i-1}\}$. (Since $Ae_3 \not\equiv x_i, x_{i\pm1}, e_n \bmod 2$, we need the assumption $n \geq 4$.) Let $y = Ae_3$, and let $\alpha' = \alpha \cup \{y, \{x_{i-1}, y\}, \{y, x_i\}\} \setminus \{\{x_{i-1}, x_i\}\}$. Then α' is homotopic to α. Hence, considering α' in place of α, we may suppose that $\mathrm{Rank}(x_{i\pm1}) < R_\alpha$, $x_{i\pm1} \not\equiv e_n \bmod 2$ and $x_{i-1} \not\equiv x_{i+1} \bmod 2$. $\qquad\square$

We now suppose that α satisfies the conditions of the above remark. Suppose that $x_i \equiv e_s$, $x_{i-1} \equiv e_t$ and $x_{i+1} \equiv e_u \bmod 2$, where s, t and u are mutually different and not equal to n. Since $\{x_{i-1}, x_i\}$ is a 1-simplex in $\Gamma_2\mathscr{B}_n(\mathbf{Z})$, there is an extension $B \in \Gamma_2(n)$ of $\{x_{i-1}, x_i\}$. We write $B^{-1}x_{i+1} = \sum_{j=1}^n a_j e_j$. It follows that there exist an even integer b_u and an odd integer b_n such that $a_u b_n - a_n b_u = \gcd(a_u, a_n)$. Then we have that

$$\begin{pmatrix} a_u/\gcd(a_u, a_n) & b_u \\ a_n/\gcd(a_u, a_n) & b_n \end{pmatrix}^{-1} \begin{pmatrix} a_u \\ a_n \end{pmatrix} = \begin{pmatrix} \gcd(a_u, a_n) \\ 0 \end{pmatrix}.$$

Let $C \in \Gamma_2(n)$ be the matrix whose (u, u) entry is $a_u/\gcd(a_u, a_n)$, (n, u) entry is $a_n/\gcd(a_u, a_n)$, (u, n) entry is b_u, (n, n) entry is b_n, other diagonal entries are 1 and other entries are 0. Then if we set $A = C^{-1}B^{-1}$, it follows that $Ax_i = e_s$, $Ax_{i-1} = e_t$ and $\mathrm{Rank}(Ax_{i+1}) = 0$.

Since $\{e_s, Ax_{i+1}\}$ is a 1-simplex and $\mathrm{Rank}(e_s) = \mathrm{Rank}(Ax_{i+1}) = 0$, by Lemma 5.5, we have that $\{e_s, Ax_{i+1}\} \in \mathrm{link}_{\Gamma_2\mathscr{B}_n(\mathbf{Z})}(e_n)$. Therefore, we have that $e_n \in \mathrm{link}_{\Gamma_2\mathscr{B}_n(\mathbf{Z})}(\{e_s, Ax_{i+1}\})$. In addition, it is clear that $e_n \in \mathrm{link}_{\Gamma_2\mathscr{B}_n(\mathbf{Z})}(\{e_s, e_t\})$. Hence, we have that $A^{-1}e_n \in \mathrm{link}_{\Gamma_2\mathscr{B}_n(\mathbf{Z})}(\{x_i, x_{i\pm1}\})$ (see Figure 2). Then, there exists an integer l such that $\mathrm{Rank}(x_i') < R_\alpha$, where $x_i' = A^{-1}e_n + 2lx_i$. We have also that $x_i' \in \mathrm{link}_{\Gamma_2\mathscr{B}_n(\mathbf{Z})}(\{x_i, x_{i\pm1}\})$ (see Figure 2). Let $\alpha' = \alpha \cup \{\{x_i'\}, \{x_i', x_{i\pm1}\}\} \setminus \{x_i, \{x_i, x_{i\pm1}\}\}$. Then α' is homotopic to α (see Figure 2). Similar to the case R_α is odd, for all x_i with $\mathrm{Rank}(x_i) = R_\alpha$, applying
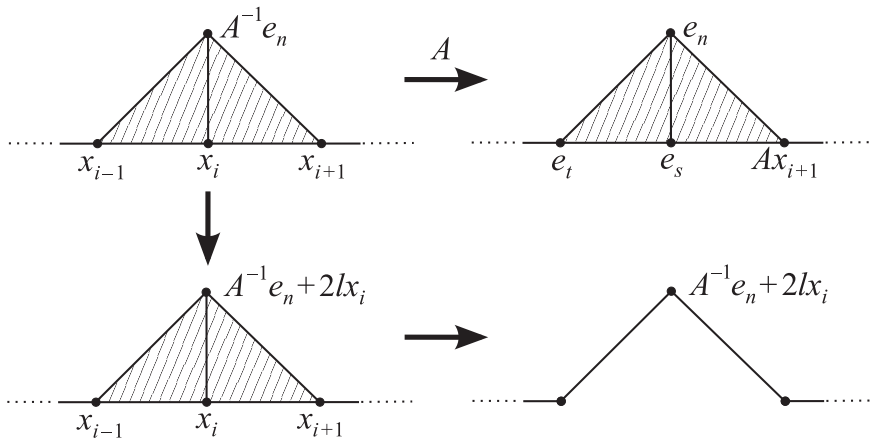

FIGURE 2. The case R_α is even.

the same operation, we conclude that $R_\beta < R_\alpha$, where β is a resulting loop which is homotopic to α.

Repeating this operation until $R_\alpha = 0$, we conclude that the loop α on $\Gamma_2 \mathscr{B}_n(\mathbf{Z})$ is null homotopic. Thus, $\Gamma_2 \mathscr{B}_n(\mathbf{Z})$ is simply connected.

6. Proof of Theorem 1.1

We first prove the next proposition.

LEMMA 6.1. *For any $n \geq 4$, $\Gamma_2(n)$ is isomorphic to the quotient of $\underset{1 \leq i \leq n}{*} \Gamma_2(n)_{e_i}$ by the normal subgroup generated by edge relators.*

Proof. For a $(k-1)$-simplex $\Delta = \{x_1, x_2, \ldots, x_k\} \in \Gamma_2 \mathscr{B}_n(\mathbf{Z})$ with $x_j \equiv e_{i(j)} \bmod 2$, let $A \in \Gamma_2(n)$ be an extension of Δ. Then we have $A^{-1} \cdot \Delta = \{e_{i(1)}, e_{i(2)}, \ldots, e_{i(k)}\}$. Therefore, we have

$$\Gamma_2(n) \backslash \Gamma_2 \mathscr{B}_n(\mathbf{Z})$$
$$= \{\{e_{i(1)}, e_{i(2)}, \ldots, e_{i(k)}\} \mid 1 \leq k \leq n, 1 \leq i(1) < i(2) < \cdots < i(k) \leq n\}.$$

It is clear that $\Gamma_2(n) \backslash \Gamma_2 \mathscr{B}_n(\mathbf{Z})$ is contractible. Note that the action of $\Gamma_2(n)$ on $\Gamma_2 \mathscr{B}_n(\mathbf{Z})$ is without rotation.

We first set followings.

- $T = \{(e_1, e_i) \mid 2 \leq i \leq n\}$.
- $E = \{(e_i, e_j) \mid 1 \leq i < j \leq n\}$.
- $F = \{(e_i, e_j, e_k) \mid 1 \leq i < j < k \leq n\}$.
- For $e \in E$, we choose $g_e = 1$, and write $g_e = g_{ij}$ when $e = (e_i, e_j)$.
- For $\tau = (e_i, e_j, e_k) \in F$, let $g_\tau = g_{ij} g_{jk} g_{ik}^{-1}$.

Then, since $\Gamma_2 \mathscr{B}_n(\mathbf{Z})$ is simply connected, it follows from Theorem 2.1 that $\Gamma_2(n)$ is isomorphic to the quotient of $\left(\underset{1 \leq i \leq n}{*} \Gamma_2(n)_{e_i} \right) \left(\underset{1 \leq i < j \leq n}{*} \langle \hat{g}_{ij} \rangle \right)$ by the normal subgroup generated by followings

(1) $\hat{g}_{1i}$, where $2 \leq i \leq n$,
(2) $\hat{g}_{ij}^{-1} X_{e_i} \hat{g}_{ij} X_{e_j}^{-1}$, where $1 \leq i < j \leq n$ and $X \in \Gamma_2(n)_{(e_i, e_j)}$,
(3) $\hat{g}_\tau g_\tau^{-1}$, where $\tau \in F$.

Since $g_\tau = 1$, the relation $\hat{g}_\tau g_\tau^{-1}$ is equivalent to the relation $\hat{g}_{ij} \hat{g}_{jk} = \hat{g}_{ik}$ if $\tau = (e_i, e_j, e_k)$. By relations $\hat{g}_{1i} = 1$, we have the relation $\hat{g}_{ij} = 1$ for $1 \leq i < j \leq n$. Thus, we obtain the claim. $\square$

Note that for $e = (e_s, e_t)$, $\Gamma_2(n)_e$ is generated by $(E_{ij})_e$ and $(F_j)_e$ for $1 \leq i, j \leq n$ with $j \neq s, t$. Hence, we have edge relations

- $(E_{ij})_{e_s} = (E_{ij})_{e_t}$,
- $(F_j)_{e_s} = (F_j)_{e_t}$.

Since we already obtained presentations of $\Gamma_2(2)$ and $\Gamma_2(3)$, from Lemma 6.1 and Remark 4.4, we obtain the presentation of $\Gamma_2(n)$ for $n \geq 4$, by induction on n.

Thus, we complete the proof of Theorem 1.1.

Appendix A

In this section, we check Tietze transformations of Subsection 4.3.

Let $\hat{\Gamma}$ denote the quotient of $\underset{1 \leq i \leq 7}{*} \Gamma_2(3)_{v_i}$ by the normal subgroup generated by edge relators. By the edge relations of Subsection 4.3, we have the following relations, in $\hat{\Gamma}$,

(1) $\cdot$ $(E_{23})_{v_2} = (E_{23})_{v_1}$,
   $\cdot$ $(E_{13})_{v_2} = (E_{13})_{v_1}$,
   $\cdot$ $(F_3)_{v_2} = (F_3)_{v_1}$,

(2) $\cdot$ $(E_{31})_{v_3} = (E_{31})_{v_2}$,
   $\cdot$ $(E_{32})_{v_3} = (E_{32})_{v_1}$,
   $\cdot$ $(E_{12})_{v_3} = (E_{12})_{v_1}$,
   $\cdot$ $(E_{21})_{v_3} = (E_{21})_{v_2}$,
   $\cdot$ $(F_1)_{v_3} = (F_1)_{v_2}$,
   $\cdot$ $(F_2)_{v_3} = (F_2)_{v_1}$,

(3) $\cdot$ $(E_{21}F_2E_{12}F_1)_{v_4} = (E_{21})_{v_2}(F_2)_{v_1}(E_{12})_{v_1}(F_1)_{v_2}$,
   $\cdot$ $(E_{13}E_{23})_{v_4} = (E_{13})_{v_1}(E_{23})_{v_1}$,
   $\cdot$ $(E_{23})_{v_4} = (E_{23})_{v_1}$,
   $\cdot$ $(E_{31}^{-1}E_{32})_{v_4} = (E_{31})_{v_2}^{-1}(E_{32})_{v_1}$,
   $\cdot$ $(E_{21}F_2)_{v_4} = (E_{21})_{v_2}(F_2)_{v_1}$,
   $\cdot$ $(F_3)_{v_4} = (F_3)_{v_1}$,

(4) $\cdot$ $(E_{31}F_3E_{13}F_1)_{v_5} = (E_{31})_{v_2}(F_3)_{v_1}(E_{13})_{v_1}(F_1)_{v_2}$,
   $\cdot$ $(E_{12}E_{32})_{v_5} = (E_{12})_{v_1}(E_{32})_{v_1}$,
   $\cdot$ $(E_{32})_{v_5} = (E_{32})_{v_1}$,
   $\cdot$ $(E_{21}^{-1}E_{23})_{v_5} = (E_{21})_{v_2}^{-1}(E_{23})_{v_1}$,
   $\cdot$ $(E_{31}F_3)_{v_5} = (E_{31})_{v_2}(F_3)_{v_1}$,
   $\cdot$ $(F_2)_{v_5} = (F_2)_{v_1}$,

(5) $\cdot$ $(E_{32}F_3E_{23}F_2)_{v_6} = (E_{32})_{v_1}(F_3)_{v_1}(E_{23})_{v_1}(F_2)_{v_1}$,
   $\cdot$ $(E_{21}E_{31})_{v_6} = (E_{21})_{v_2}(E_{31})_{v_2}$,
   $\cdot$ $(E_{31})_{v_6} = (E_{31})_{v_2}$,
   $\cdot$ $(E_{12}^{-1}E_{13})_{v_6} = (E_{12})_{v_1}^{-1}(E_{13})_{v_1}$,
   $\cdot$ $(E_{32}F_3)_{v_6} = (E_{32})_{v_1}(F_3)_{v_1}$,
   $\cdot$ $(F_1)_{v_6} = (F_1)_{v_2}$,

(6) $\cdot$ $(E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32})_{v_7} = (E_{21})_{v_2}(F_2)_{v_1}(E_{12})_{v_1}(F_1)_{v_2}(E_{31})_{v_2}^{-1}(E_{32})_{v_1}$,
   $\cdot$ $(E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23})_{v_7} = (E_{31})_{v_2}(F_3)_{v_1}(E_{13})_{v_1}(F_1)_{v_2}(E_{21})_{v_2}^{-1}(E_{23})_{v_1}$,
   $\cdot$ $(E_{21}^{-1}E_{23})_{v_7} = (E_{21})_{v_2}^{-1}(E_{23})_{v_1}$,
   $\cdot$ $(E_{31}^{-1}E_{32})_{v_7} = (E_{31})_{v_2}^{-1}(E_{32})_{v_1}$,
   $\cdot$ $(E_{21}F_2)_{v_7} = (E_{21})_{v_2}(F_2)_{v_1}$,
   $\cdot$ $(E_{31}F_3)_{v_7} = (E_{31})_{v_2}(F_3)_{v_1}$.

Using Tietze transformations, we obtain a presentation of $\hat{\Gamma}$ whose generators are $(E_{12})_{v_1}$, $(E_{13})_{v_1}$, $(E_{23})_{v_1}$, $(E_{32})_{v_1}$, $(F_2)_{v_1}$, $(F_3)_{v_1}$, $(E_{21})_{v_2}$, $(E_{31})_{v_2}$ and $(F_1)_{v_2}$. To

avoid complication of notations, we rewrite $X = X_{v_i}$. Then we have a finite presentation of $\hat{\Gamma}$ with generators $E_{12}, E_{13}, E_{23}, E_{32}, F_2, F_3, E_{21}, E_{31}$ and F_1, and with the following relators

(1.1) $F_2^2,\ F_3^2$,

(1.2) $(E_{12}F_2)^2,\ (E_{13}F_3)^2,\ (E_{23}F_2)^2,\ (E_{23}F_3)^2,\ (E_{32}F_2)^2,\ (E_{32}F_3)^2,\ (F_2F_3)^2$,

(1.3) $[E_{12}, E_{13}],\ [E_{12}, E_{32}],\ [E_{12}, F_3],\ [E_{13}, E_{23}],\ [E_{13}, F_2],\ [E_{23}, E_{12}]E_{13}^2,$ $[E_{32}, E_{13}]E_{12}^2$,

(2.1) F_1^2,

(2.2) $(E_{13}F_1)^2,\ (E_{21}F_1)^2,\ (E_{31}F_1)^2,\ (E_{31}F_3)^2,\ (F_1F_3)^2$,

(2.3) $[E_{21}, E_{23}],\ [E_{21}, E_{31}],\ [E_{21}, F_3],\ [E_{23}, F_1],\ [E_{13}, E_{21}]E_{23}^2,\ [E_{31}, E_{23}]E_{21}^2$,

(3.2) $(E_{12}F_1)^2,\ (E_{21}F_2)^2,\ (F_1F_2)^2$,

(3.3) $[E_{31}, E_{32}],\ [E_{31}, F_2],\ [E_{32}, F_1],\ [E_{12}, E_{31}]E_{32}^2,\ [E_{21}, E_{32}]E_{31}^2$,

(4.3) $[E_{31}^{-1}E_{32}, E_{13}E_{23}](E_{21}F_2E_{12}F_1)^2$,

(5.3) $[E_{21}^{-1}E_{23}, E_{12}E_{32}](E_{31}F_3E_{13}F_1)^2$,

(6.3) $[E_{12}^{-1}E_{13}, E_{21}E_{31}](E_{32}F_3E_{23}F_2)^2$,

(7.3) (a) $[E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32}, E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23}]$,

(b) $[E_{21}^{-1}E_{23}, E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32}](E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23})^2$,

(c) $[E_{31}^{-1}E_{32}, E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23}](E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32})^2$.

Let X, Y and Z be

$$X = \{(F_iF_j)^2, (E_{ij}F_i)^2, (E_{ij}F_j)^2, [E_{ij}, F_k] \mid \{i, j, k\} = \{1, 2, 3\}\},$$

$$Y = \{[E_{ij}, E_{ik}], [E_{ij}, E_{kj}] \mid \{i, j, k\} = \{1, 2, 3\}\},$$

$$Z = \{[E_{ij}, E_{ki}]E_{kj}^2 \mid \{i, j, k\} = \{1, 2, 3\}\}.$$

We show that relators (4.3), (5.3), (6.3) and (b), (c) of (7.3) are obtained from relators X, Y, Z and (a) of (7.3). In transformation, the notation "$\equiv$" means conjugation. An underline means applying relators Y, Z or (a) of (7.3).

LEMMA A.1. *Under relators (1.-), (2.-), (3.-) and conjugation,*

(1) *the relator* (a) *of* (7.3) *is equivalent to the relator* $(E_{j1}E_{1j}^{-1}E_{kj}^{-1}E_{jk}E_{1k}E_{k1}^{-1})^2$,

(2) *relators* (b) *and* (c) *of* (7.3) *are equivalent to the relator* $E_{kj}^{-1}E_{1j}E_{j1}^{-1}E_{jk}^{-1} \cdot E_{kj}E_{1j}^{-1}E_{j1}E_{jk}E_{1k}^{-1}E_{k1}E_{1k}^{-1}E_{k1}$,

where $(j, k) = (2, 3)$ *or* $(3, 2)$.

Proof. (1) At first, we delete words F_1, F_2 and F_3, using relators X, and then transform as follows.

$$[E_{j1}F_jE_{1j}F_1E_{k1}^{-1}E_{kj}, E_{k1}F_kE_{1k}F_1E_{j1}^{-1}E_{jk}]$$

$$= (E_{j1}F_jE_{1j}F_1\underbrace{E_{k1}^{-1}E_{kj}}_{Y})(E_{k1}\ F_kE_{1k}F_1E_{j1}^{-1}E_{jk})$$

$$\cdot (E_{kj}^{-1}E_{k1}F_1E_{1j}^{-1}F_j\underbrace{E_{j1}^{-1}}_{Y})(\underbrace{E_{jk}^{-1}E_{j1}}_{Y}F_1E_{1k}^{-1}F_kE_{k1}^{-1})$$

$$\underset{X}{=} E_{j1}E_{1j}^{-1}E_{kj}^{-1} \cdot \underline{E_{1k}E_{j1}E_{jk}}_{\,Y} \cdot E_{kj}^{-1}E_{k1}^{-1}E_{1j}^{-1} \cdot E_{jk}E_{1k}E_{k1}^{-1}$$

$$= E_{j1}E_{1j}^{-1}E_{kj}^{-1} \cdot E_{jk}E_{1k}\underline{E_{j1} \cdot E_{kj}^{-1}E_{k1}^{-1}}_{\,Y}E_{1j}^{-1} \cdot E_{jk}E_{1k}E_{k1}^{-1}$$

$$= E_{j1}E_{1j}^{-1}E_{kj}^{-1} \cdot E_{jk}E_{1k}E_{k1}^{-1}E_{j1} \cdot \underline{E_{kj}^{-1}E_{1j}^{-1}}_{\,Y} \cdot E_{jk}E_{1k}E_{k1}^{-1}$$

$$= E_{j1}E_{1j}^{-1}E_{kj}^{-1}E_{jk}E_{1k}E_{k1}^{-1} \cdot E_{j1}E_{1j}^{-1}E_{kj}^{-1}E_{jk}E_{1k}E_{k1}^{-1}$$

$$= (E_{j1}E_{1j}^{-1}E_{kj}^{-1}E_{jk}E_{1k}E_{k1}^{-1})^2.$$

Thus, we obtain the claim.

(2) Similarly, we delete words F_1, F_2 and F_3 as follows.

$$[E_{j1}^{-1}E_{jk}, E_{j1}F_jE_{1j}F_1E_{k1}^{-1}E_{kj}]$$

$$= E_{jk}^{-1}E_{j1} \cdot \underline{E_{kj}^{-1}E_{k1}}_{\,Y}F_1E_{1j}^{-1}F_jE_{j1}^{-1} \cdot \underline{E_{j1}^{-1}E_{jk} \cdot E_{j1}}_{\,Y}F_jE_{1j}F_1E_{k1}^{-1}E_{kj}$$

$$\underset{X}{=} E_{jk}^{-1}E_{j1} \cdot E_{k1}E_{kj}^{-1}E_{1j}E_{j1}^{-1} \cdot E_{jk}^{-1} \cdot E_{1j}^{-1}E_{k1}^{-1}E_{kj},$$

$$(E_{k1}F_kE_{1k}F_1E_{j1}^{-1}E_{jk})^2$$

$$= E_{k1}F_kE_{1k}F_1E_{j1}^{-1}E_{jk} \cdot E_{k1}F_kE_{1k}F_1E_{j1}^{-1}E_{jk}$$

$$\underset{X}{=} E_{k1}E_{1k}^{-1}E_{j1}E_{jk}^{-1} \cdot E_{k1}E_{1k}^{-1}E_{j1}^{-1}E_{jk}.$$

We next calculate

$$[E_{j1}^{-1}E_{jk}, E_{j1}F_jE_{1j}F_1E_{k1}^{-1}E_{kj}](E_{k1}F_kE_{1k}F_1E_{j1}^{-1}E_{jk})^2$$

$$= E_{jk}^{-1}E_{j1}E_{k1}E_{kj}^{-1}E_{1j}E_{j1}^{-1}E_{jk}^{-1} \underline{E_{1j}^{-1}E_{k1}^{-1}E_{kj} \cdot E_{k1}}_{\,Y} \underline{E_{1k}^{-1}E_{j1}E_{jk}^{-1}}_{\,Z}$$

$$\qquad \cdot E_{k1}E_{1k}^{-1}E_{j1}^{-1}E_{jk}$$

$$\equiv E_{kj}^{-1}E_{1j}E_{j1}^{-1}E_{jk}^{-1}E_{kj}E_{1j}^{-1}E_{j1}E_{jk}E_{1k}^{-1}E_{k1}E_{1k}^{-1}E_{k1}.$$

Thus, we obtain the claim.   □

PROPOSITION A.2. *Each of relators* (b) *and* (c) *of* (7.3) *is obtained from relators* X, Y, Z *and* (a) *of* (7.3).

Proof. Let $(j,k) = (2,3)$ or $(3,2)$. We calculate

$$1 = E_{j1}E_{1j}^{-1}E_{kj}^{-1}\underline{E_{jk}E_{1k}}_{\,Y}E_{k1}^{-1} \cdot E_{j1}E_{1j}^{-1}E_{kj}^{-1}E_{jk}E_{1k}E_{k1}^{-1}$$

$$= E_{j1}\underline{E_{1j}^{-1}E_{kj}^{-1}E_{1k}}_{\,Z}\underline{E_{jk}E_{k1}^{-1}E_{j1}}_{\,Z}E_{1j}^{-1}E_{kj}^{-1}E_{jk}E_{1k}E_{k1}^{-1}$$

$$= E_{j1}E_{1k}\underbrace{E_{1j}E_{kj}^{-1}E_{k1}^{-1}}_{Z}E_{j1}^{-1}E_{jk}E_{1j}^{-1}E_{kj}^{-1}E_{jk}E_{1k}E_{k1}^{-1}$$

$$= E_{j1}E_{1k}E_{k1}^{-1}E_{kj}E_{1j}E_{j1}^{-1}E_{jk}E_{1j}^{-1}E_{kj}^{-1}E_{jk}E_{1k}E_{k1}^{-1}$$

$$\equiv E_{kj}E_{1j}E_{j1}^{-1}E_{jk}\underbrace{E_{1j}^{-1}E_{kj}^{-1}}_{Y}E_{jk}E_{1k}\underbrace{E_{k1}^{-1}E_{j1}}_{Y}E_{1k}E_{k1}^{-1}$$

$$= E_{kj}E_{1j}E_{j1}^{-1}E_{jk}E_{kj}^{-1}E_{1j}^{-1}\underbrace{E_{jk}E_{1k}E_{j1}}_{Z}E_{k1}^{-1}E_{1k}E_{k1}^{-1}$$

$$= (E_{jk}E_{1k}E_{k1}^{-1}\underbrace{E_{k1}E_{1k}^{-1}E_{jk}^{-1})E_{kj}E_{1j}E_{j1}^{-1}}_{\text{(a) of (7.3)}}E_{jk}E_{kj}^{-1}E_{1j}^{-1}E_{j1}E_{jk}^{-1}E_{1k}E_{k1}^{-1}E_{1k}E_{k1}^{-1}$$

$$= E_{jk}E_{1k}E_{k1}^{-1}E_{j1}E_{1j}^{-1}E_{kj}^{-1}E_{jk}E_{1k}E_{k1}^{-1}E_{jk}E_{kj}^{-1}E_{1j}^{-1}E_{j1}E_{jk}^{-1}E_{1k}E_{k1}^{-1}E_{1k}E_{k1}^{-1}$$

$$\equiv E_{kj}^{-1}\cdot E_{jk}E_{1k}E_{k1}^{-1}E_{j1}E_{1j}^{-1}E_{kj}^{-1}E_{jk}E_{1k}E_{k1}^{-1}(E_{j1}E_{1j}^{-1}E_{1j}E_{j1}^{-1})$$

$$\cdot E_{jk}E_{kj}^{-1}E_{1j}^{-1}E_{j1}E_{jk}^{-1}E_{1k}E_{k1}^{-1}E_{1k}E_{k1}^{-1}\cdot E_{kj}$$

$$= \underbrace{(E_{kj}^{-1}E_{jk}E_{1k}E_{k1}^{-1}E_{j1}E_{1j}^{-1})^2}_{\text{(a) of (7.3)}}E_{1j}E_{j1}^{-1}E_{jk}E_{kj}^{-1}E_{1j}^{-1}E_{j1}E_{jk}^{-1}E_{1k}E_{k1}^{-1}E_{1k}E_{k1}^{-1}E_{kj}$$

$$= E_{1j}E_{j1}^{-1}E_{jk}E_{kj}^{-1}E_{1j}^{-1}E_{j1}E_{jk}^{-1}E_{1k}E_{k1}^{-1}E_{1k}E_{k1}^{-1}E_{kj}$$

$$\equiv F_k\cdot E_{kj}E_{1j}E_{j1}^{-1}E_{jk}E_{kj}^{-1}E_{1j}^{-1}E_{j1}E_{jk}^{-1}E_{1k}E_{k1}^{-1}E_{1k}E_{k1}^{-1}\cdot F_k$$

$$\underset{X}{=}E_{kj}^{-1}E_{1j}E_{j1}^{-1}E_{jk}^{-1}E_{kj}E_{1j}^{-1}E_{j1}E_{jk}E_{1k}^{-1}E_{k1}E_{1k}^{-1}E_{k1}.$$

By Lemma A.1, we obtain the claim. $\square$

PROPOSITION A.3. *Each of relators* (4.3), (5.3) *and* (6.3) *is obtained from other relators and conjugation.*

Proof. We first consider relators (4.3) and (5.3). Let $(j,k) = (2,3)$ or $(3,2)$.

$$[E_{j1}^{-1}E_{jk}, E_{1j}E_{kj}](E_{k1}F_kE_{1k}F_1)^2$$

$$= E_{jk}^{-1}E_{j1}\cdot\underbrace{E_{kj}^{-1}E_{1j}^{-1}}_{Y}\cdot\underbrace{E_{j1}^{-1}E_{jk}}_{Y}\cdot E_{1j}E_{kj}\cdot E_{k1}F_kE_{1k}F_1\cdot E_{k1}F_kE_{1k}F_1$$

$$\underset{X}{=}E_{jk}^{-1}E_{j1}E_{1j}^{-1}E_{kj}^{-1}E_{jk}E_{j1}^{-1}E_{1j}E_{kj}E_{k1}E_{1k}^{-1}E_{k1}E_{1k}^{-1}$$

$$\equiv F_1(E_{k1}E_{1k}^{-1}E_{k1}E_{1k}^{-1}E_{jk}^{-1}E_{j1}E_{1j}^{-1}E_{kj}^{-1}E_{jk}E_{j1}^{-1}E_{1j}E_{kj})F_1$$

$$\underset{X}{=}E_{k1}^{-1}E_{1k}E_{k1}^{-1}E_{1k}E_{jk}^{-1}E_{j1}^{-1}E_{1j}E_{kj}^{-1}E_{jk}E_{j1}E_{1j}^{-1}E_{kj}$$

$$= (E_{kj}^{-1}E_{1j}E_{j1}^{-1}E_{jk}^{-1}E_{kj}E_{1j}^{-1}E_{j1}E_{jk}E_{1k}^{-1}E_{k1}E_{1k}^{-1}E_{k1})^{-1}.$$

We next consider the relator (6.3).

$$[E_{12}^{-1}E_{13}, E_{21}E_{31}](E_{32}F_3E_{23}F_2)^2$$

$$= E_{13}^{-1}E_{12} \cdot E_{31}^{-1}E_{21}^{-1} \cdot E_{12}^{-1}E_{13} \cdot \underset{Z}{\underline{E_{21}E_{31} \cdot E_{32}}} F_3E_{23}F_2 \cdot E_{32}F_3E_{23}F_2$$

$$\underset{X}{=} E_{13}^{-1}E_{12}E_{31}^{-1}E_{21}^{-1}E_{12}^{-1}E_{13}E_{31}^{-1}E_{32} \underset{Y}{\underline{E_{21}E_{23}^{-1}}} E_{32}E_{23}^{-1}$$

$$\equiv \underset{Z}{\underline{E_{23}^{-1}E_{13}^{-1}E_{12}}} E_{31}^{-1}E_{21}^{-1}E_{12}^{-1}E_{13}E_{31}^{-1}E_{32}E_{23}^{-1}E_{21}E_{32}$$

$$= E_{13}E_{12} \underset{Z}{\underline{E_{23}^{-1}E_{31}^{-1}E_{21}^{-1}}} E_{12}^{-1}E_{13}E_{31}^{-1}E_{32}E_{23}^{-1}E_{21}E_{32}$$

$$= E_{13}E_{12}E_{31}^{-1}E_{21} \underset{Z}{\underline{E_{23}^{-1}E_{12}^{-1}E_{13}}} E_{31}^{-1}E_{32}E_{23}^{-1}E_{21}E_{32}$$

$$= E_{13}E_{12}E_{31}^{-1}E_{21}E_{12}^{-1}E_{23}^{-1}E_{13}^{-1}E_{31}^{-1}E_{32}E_{23}^{-1}E_{21}E_{32}$$

$$\equiv \underset{Z}{\underline{E_{12}^{-1}E_{23}^{-1}E_{13}^{-1}}} E_{31}^{-1}E_{32}E_{23}^{-1}E_{21} \underset{Z}{\underline{E_{32}E_{13}E_{12}}} E_{31}^{-1}E_{21}$$

$$= E_{23}^{-1}E_{13} \underset{Z}{\underline{E_{12}^{-1}E_{31}^{-1}E_{32}}} \underset{Z}{\underline{E_{23}^{-1}E_{21}E_{13}}} \underset{Z}{\underline{E_{32}E_{12}^{-1}E_{31}^{-1}}} E_{21}$$

$$= E_{23}^{-1}E_{13}E_{31}^{-1}E_{32}^{-1} \underset{Z}{\underline{E_{12}^{-1}E_{23}E_{13}}} \underset{Z}{\underline{E_{21}E_{32}^{-1}E_{31}^{-1}}} E_{12}^{-1}E_{21}$$

$$= E_{23}^{-1}E_{13}E_{31}^{-1}E_{32}^{-1}E_{23}E_{13}^{-1} \underset{Z}{\underline{E_{12}^{-1}E_{31}E_{32}^{-1}}} E_{21}E_{12}^{-1}E_{21}$$

$$= E_{23}^{-1}E_{13}E_{31}^{-1}E_{32}^{-1}E_{23}E_{13}^{-1}E_{31}E_{32}E_{12}^{-1}E_{21}E_{12}^{-1}E_{21}.$$

By Lemma A.1, each of relators (4.3), (5.3) and (6.3) is obtained from relators (1.-), (2.-), (3.-) and (b), (c) of (7.3). Thus, we obtain the claim. $\square$

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