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KODAI MATH. J.
38 (2015), 521–533

EXISTENCE AND MULTIPLE SOLUTIONS FOR NONAUTONOMOUS SECOND ORDER SYSTEMS WITH NONSMOOTH POTENTIALS

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Abstract

This paper is concerned with the nonautonomous second order Hamiltonian systems with nondifferetiable potentials. By using the nonsmooth least action principle and the nonsmooth local linking theorem, we obtain some new existence and multiplicity results for the periodic solutions.

1. Introduction and main results

In this paper we consider the following second order periodic system with a nonsmooth potential

$$(1)\qquad \begin{cases} \ddot{u}(t) \in \partial F(t, u(t)) \quad a.e.\ t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases}$$

where $T > 0$, the potential function $F : [0, T] \times \mathbf{R}^N \to \mathbf{R}$ is locally Lipschitz continuous in x and $\partial F(t, x)$ denotes the Clarke subdifferential of F for x.

The system (1) has been studied in the past decades and many excellent results appeared, for example, the work of D. Pasca [10]. Systems driven by the vector p-Laplacian or p-Laplacian-like operators were studied by E. H. Papageorgiou and N. S. Papageorgiou [9], S. Aizicovici and N. S. Papageorgiou [1], D. Pasca [11] and the reference therein. We only focus on the semilinear case (i.e., $p = 2$) in the present paper, and our approach is based on the nonsmooth least action principle by [4] and a nonsmooth local linking theorem by [6]. It should be noted that our results are different from that of those mentioned above even letting $p = 2$ in their theorems. Examples are given to show the difference.

2010 *Mathematics Subject Classification.* 34C25; 35B38; 49J52.

Key words and phrases. Periodic solutions; Second order systems; Generalized subdifferential; Nonsmooth Cerami condition; Locally Lipschitz function; Nonsmooth local linking theorem.

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Received November 13, 2014; revised December 19, 2014.

When $F(t,x)$ is continuously differentiable in x, the problem (1) becomes the second order Hamiltonian systems

$$\begin{cases} \ddot{u}(t) = \nabla F(t,u(t)) & a.e.\ t \in [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases} \tag{2}$$

There have been a lot of contributions on problem (2), and we can refer to Mawhin-Willem [7], Tang [12], Tang-Wu [13], Aizmahin-An [2] and so on. In these works, the following assumption is necessary:

(A) $F(t,x)$ is measurable in t for every $x \in \mathbf{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, and there exist $a \in C(\mathbf{R}^+, \mathbf{R}^+)$, $b \in L^1(0,T;\mathbf{R}^+)$ such that

$$|F(t,x)| \leq a(|x|)b(t), \quad |\nabla F(t,x)| \leq a(|x|)b(t)$$

for all $x \in \mathbf{R}^N$ and a.e. $t \in [0,T]$, where $\mathbf{R}^+$ is the set of all nonnegative real number.

Throughout this paper, we always suppose that $F = F_1 + F_2$ with F_1, F_2 satisfying the following assumption (A'):

- F_1, F_2 are integrable in t over $[0,T]$ for each $x \in \mathbf{R}^N$;
- F_1 is strictly differentiable and F_2 is locally Lipschitz continuous in x for each $t \in [0,T]$.

Let H_T^1 be the usual Sobolev space with norm

$$\|u\| = \left(\int_0^T |u(t)|^2\, dt + \int_0^T |\dot{u}(t)|^2\, dt \right)^{1/2}.$$

The corresponding functional $\varphi : H_T^1 \to R$ is given by

$$\varphi(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2\, dt + \int_0^T F(t,u(t))\, dt.$$

The main results of this paper are as follows:

THEOREM 1.1. *Assume that $F = F_1 + F_2$, where F_1, F_2 satisfy assumption (A') above and the following conditions*:

(i_1) *There exists $k \in L^2(0,T;\mathbf{R})$ such that for all $x_1, x_2 \in \mathbf{R}^N$ and all $t \in [0,T]$*

$$|F_1(t,x_1) - F_1(t,x_2)| \leq k(t)|x_1 - x_2|. \tag{3}$$

(i_2) *There exist $f, g \in L^\infty(0,T;\mathbf{R}^+)$ and $\alpha \in [0,1)$ such that for all $x \in \mathbf{R}^N$ and a.e. $t \in [0,T]$,*

$$\xi \in \partial F_2(t,x) \quad \Rightarrow \quad |\xi| \leq f(t)|x|^\alpha + g(t). \tag{4}$$

(i_3) *There exists $h \in L^1(0,T)$ such that for a.e. $t \in [0,T]$ and all $x \in \mathbf{R}^N$*

$$F_1(t,x) \geq h(t), \tag{5}$$

and

$$|x|^{-2\alpha}\int_0^T F_2(t,x)\,dt \to +\infty \quad \text{as } |x| \to \infty. \tag{6}$$

Then problem (1) *possesses at least one solution which minimizes the functional* φ *on* H_T^1.

Remark 1.1. The function $F_1(t,x)$ is globally Lipschitz continuous in x on H_T^1 provided the condition (3) holds. If $F(t,x)$ is measurable in t for every $x \in \mathbf{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, the inequality (4) becomes

$$|\nabla F_2(t,x)| \le f(t)|x|^{\alpha} + g(t).$$

Then our Theorem 1.1 generalizes Theorem 1 in [12].

Remark 1.2. There are functions F satisfying our Theorem 1.1 but not satisfying the results in [1, 2, 7, 9–13]. For example, let $F(t,x) = F_1(t,x) + F_2(t,x)$ with

$$F_1(t,x) = \frac{t^2}{2}|\sin x|, \quad F_2(t,x) = \begin{cases} -\dfrac{\theta(t)}{2}|x|^2, & |x| \le 1, \\ \dfrac{f(t)}{\alpha+1}|x|^{\alpha+1} - \dfrac{\theta(t)}{2} - \dfrac{f(t)}{\alpha+1}, & |x| > 1 \end{cases}$$

for all $(t,x) \in [0,T] \times \mathbf{R}^N$, where $\alpha \in [0,1)$, $\theta, f \in L^{\infty}(0,T;\mathbf{R}^+)$.

THEOREM 1.2. *Assume that* $F = F_1 + F_2$ *with* $\int_0^T F(t,0)\,dt = 0$ *and* F_1, F_2 *satisfy assumptions of Theorem* 1.1. *Suppose that there exist* $\delta > 0$ *and an integer* $k \ge 0$ *such that*

$$-\frac{1}{2}(k+1)^2\omega^2|x|^2 \le F(t,x) - F(t,0) \le -\frac{1}{2}k^2\omega^2|x|^2 \tag{7}$$

for all $|x| \le \delta$ *and a.e.* $t \in [0,T]$, *where* $\omega = \dfrac{2\pi}{T}$. *Then problem* (1) *has at least three distinct solutions in* H_T^1.

Remark 1.3. Theorem 1.2 generalizes Theorem 2 in [13] and Thorem 4 in [12]. Suppose that $F = F_1 + F_2$ and

$$F_1(t,x) = \frac{t^2}{2}|x|, \quad F_2(t,x) = \begin{cases} -\dfrac{1}{2}\omega^2|x|^2 - \dfrac{t^2}{2}|x|, & |x| \le 1, \\ \dfrac{t^2}{\alpha+1}|x|^{\alpha+1} - \dfrac{1}{2}\omega^2 - \dfrac{t^2}{2} - \dfrac{t^2}{\alpha+1}, & |x| > 1 \end{cases}$$

for all $(t,x) \in [0,T] \times \mathbf{R}^N$, where $\alpha \in [0,1)$, $\omega = \dfrac{2\pi}{T}$. Then the function F satisfies Theorem 1.2 but not satisfies Theorem 2 in [13] and Thorem 4 in [12].

THEOREM 1.3. *Assume that* $F = F_1 + F_2$ *with* $\int_0^T F(t,0)\,dt = 0$ *and* F_1, F_2 *satisfy assumptions of Theorem* 1.1. *Suppose that*

$$\liminf_{x\to 0} \frac{F(t,x)}{|x|^2} \geq -\frac{2\pi^2}{T^2} \tag{8}$$

for a.e. $t \in [0,T]$, *and there exist* $\hat{\delta} > 0$ *such that for all* $|x| \leq \hat{\delta}$

$$\int_0^T F(t,x)\,dt \leq 0. \tag{9}$$

Then problem (1) *has at least three distinct solutions in* H_T^1.

Remark 1.4. Theorem 1.3 is new even in the case that $F \in C^1$ for system (2). There are functions F satisfying Theorem 1.3 but not satisfying the results in [1, 2, 7, 9–13]. For example, let $F = F_1 + F_2$ and

$$F_1(t,x) \equiv 0, \quad F_2(t,x) = \begin{cases} -\dfrac{2\pi^2}{T^2}|x|^2, & |x| \leq 1, \\ \dfrac{2t^3}{\alpha+1}|x|^{\alpha+1} - \dfrac{2\pi^2}{T^2} - \dfrac{2t^3}{\alpha+1}, & |x| > 1 \end{cases}$$

for all $(t,x) \in [0,T] \times \mathbf{R}^N$, where $\alpha \in [0,1)$.

2. Basic definitions and preliminary results

Let $(X, \|\cdot\|)$ be a real Banach space. We denote by X^* the dual space of X, while $\langle \cdot, \cdot \rangle$ stands for the duality pairing between X and X^*. A functional $h : X \to \mathbf{R}$ is called **locally Lipschitz continuous** if for every $u \in X$ there exist a neighborhood V_u of u and a constant $L_u \geq 0$ such that

$$|h(z) - h(w)| \leq L_u\|z - w\|, \quad \forall z, w \in V_u.$$

If $u, v \in X$, we write $h^o(u;v)$ for the **generalized directional derivative** of h at the point u along the direction v, i.e.,

$$h^\circ(u;v) := \limsup_{w\to u,\, t\to 0^+} \frac{h(w+tv) - h(w)}{t}.$$

It is well known that h° is upper semicontinuous on $X \times X$ [5, Proposition 2.1.1].

For locally Lipschitz continuous functionals $h_1, h_2 : X \to \mathbf{R}$, we have

$$(h_1 + h_2)^\circ(x;z) \leq h_1^\circ(x;z) + h_2^\circ(x;z), \quad \forall x, z \in X.$$

The **generalized gradient** of the function h in u, denoted by $\partial h(u)$, is the set defined by

$$\partial h(u) := \{u^* \in X^* : \langle u^*, v\rangle \leq h^\circ(u;v), \forall v \in X\}.$$

Proposition 2.1.2 of [5] ensures that $\partial h(u)$ turns out nonempty, convex, weak* compact, thus the function $\lambda(x) = \min_{w \in \partial h(x)} \|w\|_{X^*}$ exists and is lower semicontinuous.

We call $h : X \to R$ is **strictly differentiable** in u if there exist an element $\xi \in X^*$ such that for each v,

$$\lim_{w \to u,\, t \to 0^+} \frac{h(w + tv) - h(w)}{t} = \langle \xi, v \rangle,$$

and provided the convergence is uniform for v in compact sets (This last condition is automatic if h is Lipschitz near u, see [5], P30).

If $f, g : X \to X$ be locally Lipschitz continuous, then

$$\partial(f + g)(x) \subset \partial f(x) + \partial g(x) \tag{10}$$

for all $x \in X$. Further, if at least one of the functional f, g is strictly differentiable at x then equality holds, and $\partial f(x) = \{f'(x)\}$ when $f \in C^1(X)$.

A point $u \in X$ is said to be a **critical point** of h if

$$h^\circ(u; v) \geq 0, \quad \forall v \in X,$$

which clearly means $\theta \in \partial h(u)$.

We say the locally Lipschitz functional h satisfies the **nonsmooth Cerami condition** if any sequence $\{x_n\}$ in X such that $\{h(x_n)\}$ is bounded and $(1 + \|x_n\|)\lambda(x_n) \to 0$ possesses a strongly convergent subsequence.

For convenience to quote we state some well known results, for more details, we can refer to [3, 8].

LEMMA 2.1 ([5], Theorem 2.3.7). *Let x and y be points in X, and suppose that f is Lipschitz on open set containing the line segment $[x, y]$. Then there exists a point u in (x, y) such that*

$$f(y) - f(x) \in \langle \partial f(u), y - x \rangle.$$

LEMMA 2.2 ([6], Theorem 8). *If X is a reflexive Banach space, $X = Y \oplus V$ with dim $Y < +\infty$, $\phi : X \to R$ is a locally Lipschitz function which is bounded from below, satisfies the nonsmooth Cerami condition, $\phi(0) = 0$, $\inf_X \phi < 0$ and there exists $r > 0$ such that*

$$\phi(x) \leq 0, \quad \text{for } x \in Y,\ \|x\| \leq r,$$

$$\phi(x) \geq 0, \quad \text{for } x \in V,\ \|x\| \leq r.$$

Then ϕ has at least two nontrivial critical points.

3. Proof of Theorems

For every $u \in H_T^1$, let $\bar{u} = \frac{1}{T}\int_0^T u(t)\,dt$, $\tilde{u}(t) = u(t) - \bar{u}$. Then the following inequalities hold:

$$\|\tilde{u}\|_\infty^2 \le \frac{T}{12}\int_0^T |\dot{u}(t)|^2\,dt, \quad \text{(Sobolev's inequality)}$$

$$\int_0^T |\tilde{u}(t)|^2\,dt \le \frac{T^2}{4\pi^2}\int_0^T |\dot{u}(t)|^2\,dt. \quad \text{(Wirtinger's inequality)}$$

$$\|u\|_\infty \le C\|u\|, \tag{11}$$

where $C > 0$ is a constant and $\|u\|_\infty = \max_{t\in[0,T]}|u(t)|$.

Define two functionals $\varphi_1, \varphi_2 : H_T^1 \to \mathbf{R}$ as follows:

$$\varphi_1(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2\,dt, \quad \varphi_2(u) = \int_0^T F(t, u(t))\,dt.$$

It is easy to verify that φ_1 is continuously differentiable and weakly lower semi-continuous (w.l.s.c.), and φ_2 is locally Lipschitz continuous on H_T^1. Since H_T^1 is embedded compactly and densely in $L^2(0, T; \mathbf{R}^N)$, let $\hat{\varphi}_2 : L^2(0, T; \mathbf{R}^N) \to \mathbf{R}$ such that $\varphi_2 = \hat{\varphi}_2|_{H_T^1}$, then for every $u \in H_T^1$, $\xi \in \partial\varphi_2(u)$,

$$\partial\varphi_2(u) \subseteq \partial\hat{\varphi}_2(u) \subseteq (L^2(0, T; \mathbf{R}^N))^* = L^2(0, T; \mathbf{R}^N)$$

and $\xi(t) \in \partial F(t, u(t))$ a.e. on $[0, T]$. Moreover,

$$\langle \varphi_1'(u), v\rangle = \int_0^T (\dot{u}(t), \dot{v}(t))\,dt, \quad \forall u, v \in H_T^1.$$

Lemma 3.1. *Let $F : [0, T] \times \mathbf{R}^N \to \mathbf{R}$ such that $F = F_1 + F_2$, where F_1, F_2 satisfy assumption (A'), (3) and (4). Then the critical point of φ corresponds to the solutions of problem (1).*

Proof. From the condition (3), Obviously F_1 satisfies the Hypothesis A of Theorem 2.7.5 in [5]. Since $f, g \in L^\infty(0, T; \mathbf{R}^+)$, there exists a constant $c_0 > 0$ such that

$$\eta \in \partial F_2(t, x) \quad \Rightarrow \quad |\eta| \le f(t)|x|^\alpha + g(t) \le c_0(|x| + 1), \quad \forall x \in \mathbf{R}^N,\ t \in [0, T],$$

i.e., F_2 satisfies the Hypothesis B of Theorem 2.7.5 in [5]. Thus

$$\partial\left(\int_0^T F_1(t, u)\,dt\right) \subset \int_0^T \partial F_1(t, u)\,dt, \quad \partial\left(\int_0^T F_2(t, u)\,dt\right) \subset \int_0^T \partial F_2(t, u)\,dt.$$

Corollary 1 of Proposition 2.3.3 from [5] and (10) imply that, if at least one of the functions F_1, F_2 is strictly differentiable in x for all $t \in [0, T]$ then for all $u \in H_T^1$,

$$\begin{aligned}\partial\varphi_2(u) &\subset \partial\left(\int_0^T F_1(t, u)\,dt\right) + \partial\left(\int_0^T F_2(t, u)\,dt\right)\\ &\subset \int_0^T \partial F_1(t, u)\,dt + \int_0^T \partial F_2(t, u)\,dt = \int_0^T \partial F(t, u)\,dt.\end{aligned}$$

Therefore, from (10) one has

$$\begin{aligned}\partial\varphi(u) &\subset \partial\varphi_1(u) + \partial\varphi_2(u)\\ &\subset \partial\varphi_1(u) + \int_0^T \partial F(t,u)\,dt\\ &= \partial\varphi_1(u) + \int_0^T \partial F_1(t,u)\,dt + \int_0^T \partial F_2(t,u)\,dt,\end{aligned}$$

i.e., for every $\xi \in \partial\varphi(u)$, there exist mappings $t \mapsto q(t)$ and $t \mapsto q_i(t)$ $(i = 1, 2)$ from $[0, T]$ to $(H_T^1)^*$ with $q(t) \in \partial F(t, u(t))$ and $q_i(t) \in \partial F_i(t, u(t))$ a.e. $t \in [0, T]$ such that for every $v \in H_T^1$,

$$\begin{aligned}\langle \xi, v\rangle &= \int_0^T (\dot u(t), \dot v(t))\,dt + \int_0^T (q(t), v(t))\,dt\\ &= \int_0^T (\dot u(t), \dot v(t))\,dt + \int_0^T (q_1(t), v(t))\,dt + \int_0^T (q_2(t), v(t))\,dt.\end{aligned}$$

If $u \in H_T^1$ is a critical point of φ, then there exists $q_0(t) \in \partial F(t, u)$ such that for all $v \in H_T^1$,

$$0 = \langle \theta, v\rangle = \int_0^T (\dot u, \dot v)\,dt + \int_0^T (q_0(t), v(t))\,dt.$$

It follows easily that $q_0(t) = \ddot u(t)$ a.e. $t \in [0, T]$, thus

$$\ddot u(t) \in \partial F(t, u(t)) \quad \textit{a.e.} \text{ on } [0, T],$$

which means that the critical point of φ corresponds to the solutions of problem (1), which completes the proof. □

Proof of Theorem 1.1. It follows from conditions (3), Hölder inequality and Wirtinger's inequality that

$$\begin{aligned}&\left|\int_0^T F_1(t, u(t))\,dt - \int_0^T F_1(t, \bar u)\,dt\right|\\ &\quad\le \int_0^T |F_1(t, u(t)) - F_1(t, \bar u)|\,dt \le \int_0^T k(t)|\tilde u(t)|\,dt\\ &\quad\le \left(\int_0^T |k(t)|^2\,dt\right)^{1/2}\left(\int_0^T |\tilde u(t)|^2\,dt\right)^{1/2}\\ &\quad\le \frac{T}{2\pi}\|k\|_{L^2}\left(\int_0^T |\dot u(t)|^2\,dt\right)^{1/2}\end{aligned}$$

for all $u \in H_T^1$. From Lemma 2.1 it follows that for each $t \in [0, T]$, there exist $s \in [0, 1]$ and $\xi \in \partial F_2(t, \bar{u} + s\tilde{u})$ such that $F_2(t, u(t)) - F_2(t, \bar{u}) = (\xi(t), \tilde{u}(t))_{\mathbf{R}^N}$. By conditions (4) and Sobolev's inequality one has

$$\begin{aligned}
&\left| \int_0^T F_2(t, u(t))\, dt - \int_0^T F_2(t, \bar{u})\, dt \right| \\
&\quad \le \int_0^T |F_2(t, u(t)) - F_2(t, \bar{u})|\, dt = \int_0^T |(\xi(t), \tilde{u}(t))|\, dt \\
&\quad \le \int_0^T |\xi|\, |\tilde{u}(t)|\, dt \le \int_0^T (f(t)|\bar{u} + s\tilde{u}|^\alpha + g(t))|\tilde{u}(t)|\, dt \\
&\quad \le 2|\bar{u}|^\alpha \int_0^T f(t)|\tilde{u}(t)|\, dt + 2\|\tilde{u}\|_\infty^{\alpha+1} \int_0^T f(t)\, dt + \|\tilde{u}\|_\infty \int_0^T g(t)\, dt \\
&\quad \le \frac{3}{T}\|\tilde{u}\|_\infty^2 + \frac{T}{3}|\bar{u}|^{2\alpha} \left(\int_0^T f(t)\, dt \right)^2 + 2\|\tilde{u}\|_\infty^{\alpha+1} \int_0^T f(t)\, dt + \|\tilde{u}\|_\infty \int_0^T g(t)\, dt \\
&\quad \le \frac{1}{4}\|\dot{u}\|_{L^2}^2 + C_1\|\dot{u}\|_{L^2}^{\alpha+1} + C_2\|\dot{u}\|_{L^2} + C_3|\bar{u}|^{2\alpha}
\end{aligned}$$

for all $u \in H_T^1$ and some positive constants C_1, C_2 and C_3.

Hence from (5) we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T |\dot{u}(t)|^2\, dt + \left(\int_0^T F_1(t, u(t))\, dt - \int_0^T F_1(t, \bar{u})\, dt \right) + \int_0^T F_1(t, \bar{u})\, dt \\
&\quad + \left(\int_0^T F_2(t, u(t))\, dt - \int_0^T F_2(t, \bar{u})\, dt \right) + \int_0^T F_2(t, \bar{u})\, dt \\
&\ge \frac{1}{2}\int_0^T |\dot{u}(t)|^2\, dt - \frac{T}{2\pi}\|k\|_{L^2} \left(\int_0^T |\dot{u}(t)|^2\, dt \right)^{1/2} - \int_0^T |h(t)| dt - \frac{1}{4}\|\dot{u}\|_{L^2}^2 \\
&\quad - C_1\|\dot{u}\|_{L^2}^{\alpha+1} - C_2\|\dot{u}\|_{L^2} - C_3|\bar{u}|^{2\alpha} + \int_0^T F_2(t, \bar{u})\, dt \\
&\ge \frac{1}{4}\|\dot{u}\|_{L^2}^2 - C_1\|\dot{u}\|_{L^2}^{\alpha+1} - \left(C_2 + \frac{T}{2\pi}\|k\|_{L^2} \right)\|\dot{u}\|_{L^2} \\
&\quad + \int_0^T F_2(t, \bar{u})\, dt - C_3|\bar{u}|^{2\alpha} - C_4 \\
&= \frac{1}{4}\|\dot{u}\|_{L^2}^2 - C_1\|\dot{u}\|_{L^2}^{\alpha+1} - \left(C_2 + \frac{T}{2\pi}\|k\|_{L^2} \right)\|\dot{u}\|_{L^2} - C_4 \\
&\quad + |\bar{u}|^{2\alpha} \left(|\bar{u}|^{-2\alpha} \int_0^T F_2(t, \bar{u})\, dt - C_3 \right)
\end{aligned}$$

for all $u \in H_T^1$ and some positive constant C_4, which implies that

$$\varphi(u) \to +\infty \quad as \ \|u\| \to \infty$$

by (6) because $\alpha \in [0, 1)$ and the norm $\|\cdot\|$ given by $\|u\| = (|\bar{u}|^2 + \|\dot{u}\|_{L^2}^2)^{1/2}$ is an equivalent norm on H_T^1.

Next we show that the functional φ is w.l.s.c. on H_T^1. Since φ_1 is w.l.s.c., we only have to prove that φ_2 is w.l.s.c. on H_T^1. Suppose there are a sequence $\{u_n\} \subseteq H_T^1$ and $u \in H_T^1$ such that $u_n \rightharpoonup u$ in H_T^1. Since the embedding $H_T^1 \hookrightarrow L^2([0, T], \mathbf{R}^N)$ is compact, one has $u_n \to u$ in $L^2([0, T], \mathbf{R}^N)$.

On account of (3), one has

$$\begin{aligned}
&\left| \int_0^T F_1(t, u_n)\, dt - \int_0^T F_1(t, u)\, dt \right| \\
&\quad \leq \int_0^T |F_1(t, u_n) - F_1(t, u)|\, dt \leq \int_0^T k(t)|u_n - u|\, dt \\
&\quad \leq \left(\int_0^T |k(t)|^2\, dt \right)^{1/2} \left(\int_0^T |u_n - u|^2\, dt \right)^{1/2} \to 0,
\end{aligned}$$

Since $\alpha \in [0, 1)$, there exists $C_5 \in \mathbf{R}$ such that

$$|x|^2 \geq |x|^{2\alpha} + C_5, \quad \forall x \in \mathbf{R}^N.$$

Due to (4) and Lemma 2.1, there exist $\eta \in \partial F_2(t, su_n + (1 - s)u)$ and positive constants C_6, C_7 such that

$$\begin{aligned}
&\left| \int_0^T F_2(t, u_n)\, dt - \int_0^T F_2(t, u)\, dt \right| \\
&\quad \leq \int_0^T |F_2(t, u_n) - F_2(t, u)|\, dt = \int_0^T |(\eta(t), u_n(t) - u(t))|\, dt \\
&\quad \leq \int_0^T |\eta|\, |u_n - u|\, dt \leq \int_0^T (f(t)|su_n + (1 - s)u|^\alpha + g(t))|u_n - u|\, dt \\
&\quad \leq \int_0^T (2f(t)(|u_n|^\alpha + |u|^\alpha) + g(t))|u_n - u|\, dt \\
&\quad \leq \left(2\|f\|_\infty \left(\left(\int_0^T |u_n|^{2\alpha}\, dt \right)^{1/2} + \left(\int_0^T |u|^{2\alpha}\, dt \right)^{1/2} \right) + \sqrt{T}\|g\|_\infty \right) \\
&\qquad \times \left(\int_0^T |u_n - u|^2\, dt \right)^{1/2} \\
&\quad \leq (C_6(\|u_n\|_{L^2} + \|u\|_{L^2}) + C_7)\|u_n - u\|_{L^2} \to 0,
\end{aligned}$$

which implies $\varphi_2(u_n) \to \varphi_2(u)$ in H_T^1. Thus φ_2 is sequentially weakly continuous; therefore, φ is w.l.s.c. on H_T^1. Thanks to Theorem 1.1 and Corollary 1.1 in [7], φ

has a minimum u_0 on H_T^1. Proposition 2.3.2 in [5] implies that u_0 is a critical point of φ. Consequently, by Lemma 3.1, u_0 is a solution of problem (1), which completes the proof. □

Proof of Theorem 1.2. Let us first note that φ satisfies the nonsmooth Cerami condition. Pick a sequence $\{u_n\} \subset H_T^1$ such that $\{\varphi(u_n)\}$ is bounded and $(1 + \|u_n\|)\lambda(u_n) \to 0$ *as* $n \to \infty$. By the weak* compactness of $\partial\varphi(u_n)$ and the weak lower semicontinuity of the norm, one can find $u_n^* \in \partial\varphi(u_n)$ such that $\lambda(u_n) = \|u_n^*\| = o(1)$, then there exists an integer n_0 such that for each $n \geq n_0$, we have

$$|\langle u_n^*, v\rangle| \leq \|v\|, \quad \forall v \in H_T^1.$$

Since F_1, F_2 satisfy the conditions of Theorem 2.7.5 in [5], one has $\partial\varphi_2(u) \subset \int_0^T \partial F(t,u)\,dt$ and $\partial\varphi(u) \subset \partial\varphi_1(u) + \int_0^T \partial F(t,u)\,dt$. Thus to every $u_n^* \in \partial\varphi(u_n)$, there corresponds a mapping $t \mapsto q_n(t)$ from $[0,T]$ to $(H_T^1)^*$ with $q_n(t) \in \partial F(t, u_n(t))$ such that

$$\langle u_n^*, v\rangle = \int_0^T (\dot{u}_n(t), \dot{v}(t))\,dt + \int_0^T (q_n(t), v(t))\,dt, \quad \forall v \in H_T^1.$$

From the proof of Theorem 1.1 we know that φ is coercive, which implies that the sequence $\{u_n\}$ turns out bounded. Thus there exists an $u \in H_T^1$ such that $u_n \rightharpoonup u$ in H_T^1 and $u_n \to u$ in $C([0,T], \mathbf{R}^N)$, where a subsequence is considered when necessary.

Since H_T^1 is reflexive while $\partial\varphi(u)$ is weak* compact, and the set-valued mapping $u \to \partial\varphi(u)$ is upper semicontinuous, we can find an $u^* \in \partial\varphi(u)$ such that

$$\langle u_n^* - u^*, u_n - u\rangle \to 0, \quad \textit{as } n \to \infty.$$

Moreover

$$\langle u_n^* - u^*, u_n - u\rangle = \int_0^T |\dot{u}_n(t) - \dot{u}|^2\,dt + \int_0^T (q_n(t) - q(t), u_n(t) - u(t))\,dt,$$

where $q_n(t) \in \partial F(t, u_n(t))$ and $q(t) \in \partial F(t, u(t))$. Similarly, by the upper semicontinuity of the set-valued mapping $u \to \partial F(u)$, one has $q_n(t) \to q(t)$ in w^* topology and $\int_0^T (q_n(t) - q(t), u_n(t) - u(t))\,dt \to 0$ *as* $n \to +\infty$. Thus $\int_0^T |\dot{u}_n - \dot{u}|^2\,dt \to 0$ and $u_n \to u$ in H_T^1. Therefore φ satisfies the nonsmooth Cerami condition.

Now let Y be a finite-dimensional subspace of $X = H_T^1$ given by

$$Y = \left\{ \sum_{j=0}^{k} (a_j \cos j\omega t + b_j \sin j\omega t) \mid a_j, b_j \in \mathbf{R}^N, j = 0, \dots, k \right\},$$

and let $V = Y^\perp$. Then from (7) we have

$$\varphi(u) \leq \frac{1}{2}\int_0^T |\dot{u}(t)|^2\,dt - \frac{1}{2}k^2\omega^2 \int_0^T |u(t)|^2\,dt \leq 0, \tag{12}$$

for all $u \in Y$ with $\|u\| \le C^{-1}\delta$, and

$$\varphi(u) \ge \frac{1}{2}\int_0^T |\dot{u}(t)|^2\, dt - \frac{1}{2}(k+1)^2\omega^2 \int_0^T |u(t)|^2\, dt \ge 0,$$

for all $u \in V$ with $\|u\| \le C^{-1}\delta$, where C is the positive constant given by (11). Clearly $\varphi(0) = 0$ because $\int_0^T F(t,0)\, dt = 0$, and φ is bounded from below for it is coercive.

In the case $\inf_X \varphi < 0$, Theorem 1.2 follows from Lemma 2.2.

In the case $\inf_X \varphi \ge 0$, according to (12) one has

$$\varphi(u) = \inf_X \varphi = 0$$

for all $u \in Y$ with $\|u\| \le C^{-1}\delta$, which implies that all $u \in Y$ with $\|u\| \le C^{-1}\delta$ are minimum points of φ. Hence by Lemma 3.1, all $u \in Y$ with $\|u\| \le C^{-1}\delta$ are solutions of problem (1). Therefore Theorem 1.2 is proved. □

Proof of Theorem 1.3. Similar as in the proof of Theorem 1.2, we know φ is bounded from below, satisfies the nonsmooth Cerami condition, and $\varphi(0) = 0$.

According to the condition (8), we know for every $\varepsilon > 0$ there exists $\delta_1 > 0$ such that

$$F(t,x) > -\left(\frac{2\pi^2}{T^2} + \varepsilon\right)|x|^2$$

for a.e. $t \in [0,T]$ and $|x| \le \delta_1$. Let $\delta_2 = \min\{\delta_1, \hat{\delta}\}$, then from (9) one has

$$-\left(\frac{2\pi^2}{T^2} + \varepsilon\right)|x|^2 < F(t,x) \quad \text{and} \quad \int_0^T F(t,x)\, dt \le 0 \quad \forall |x| \le \delta_2,\ t \in [0,T].$$

Let $H_T^1 = \tilde{H}_T^1 \oplus \mathbf{R}^N$ with $\tilde{H}_T^1 = \{u \in H_T^1 \mid \int_0^T u(t)\, dt = 0\}$.

Since for every $u \in \tilde{H}_T^1$

$$|u(t)|^2 \le \|u\|_\infty^2 \le \frac{T}{12}\int_0^T |\dot{u}(t)|^2\, dt \le \frac{T}{12}\|u\|^2.$$

Put $\delta_3 = \min\left\{\sqrt{\frac{12}{T}}\delta_2, \hat{\delta}\right\}$, then for every $u \in \tilde{H}_T^1$ with $\|u\| \le \delta_3$, one has $|u(t)| \le \delta_2$ for all $t \in [0,T]$ and

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T |\dot{u}(t)|^2\, dt + \int_0^T F(t,u(t))\, dt \\
&> \frac{1}{2}\int_0^T |\dot{u}(t)|^2\, dt - \left(\frac{2\pi^2}{T^2} + \varepsilon\right)\int_0^T |u(t)|^2\, dt \\
&\ge \frac{1}{2}\int_0^T |\dot{u}(t)|^2\, dt - \left(\frac{2\pi^2}{T^2} + \varepsilon\right)\frac{T^2}{4\pi^2}\int_0^T |\dot{u}(t)|^2\, dt \\
&= -\frac{\varepsilon T^2}{4\pi^2}\int_0^T |\dot{u}(t)|^2\, dt,
\end{aligned}$$

thus

$$\varphi(u) \geq -\frac{\varepsilon T^2}{4\pi^2}\int_0^T |\dot{u}(t)|^2\,dt \geq -\frac{\varepsilon T^2}{4\pi^2}\|u\|^2$$
$$\geq -\frac{\varepsilon T^2}{4\pi^2}\delta_3 \geq -\frac{\varepsilon T^2}{4\pi^2}\hat{\delta},$$

which implies that $\varphi(u) \geq 0$ for all $\|u\| \leq \delta_3$ in $\tilde{H}_T^1$ by the arbitrariness of ε.

On the other hand for every $u \in \mathbf{R}^N$ with $\|u\| \leq \delta_3$, it follows from (9) that

$$\varphi(u) = \int_0^T F(t, u(t))\,dt \leq 0.$$

Therefore, φ satisfies the conditions of Lemma 2.2 and has at least two nontrivial critical points. With the critical point (global minima) obtained by Theorem 1.1 and taking Lemma 3.1 into account, problem (1) has at least three distinct solutions in H_T^1. The proof is completed. □

Acknowledgement. The authors sincerely thank the referee for valuable suggestions and comments on the manuscript of this paper. The work was supported by the Fundamental Research Funds for the Central Universities (No: 2014B38214).

References

[1] S. Aizicovici, N. S. Papageorgiou and V. Staicu, Multiple nontrivial solutions for nonlinear periodic problems with the p-Laplacian, J. Differential Equations **243** (2007), 504–535.

[2] N. Aizmahin and T. An, The existence of periodic solutions of non-autonomous second-order Hamiltonian systems, Nonliear Anal. **74** (2011), 4862–4867.

[3] G. Barletta and S. A. Marano, Some remarks on critical point theory for locally Lipschitz functions, Glasgow Math. J. **45** (2003), 131–141.

[4] K. C. Chang, Variational methods for non-differentiable functionals and their applications to partial differential equations, J. Math. Anal. Appl. **80** (1981), 102–129.

[5] F. H. Clarke, Optimization and nonsmooth analysis, Classics Appl. Math. **5**, SIAM, Philadephia, 1990.

[6] D. Kandilakis, N. Kourogenis and N. S. Papageorgiou, Two nontrivial critical points for nonsmooth functionals via local linking and applications, J. Global Optim. **34** (2006), 219–244.

[7] J. Mawhin and M. Willem, Critical point theory and Hamiltonian systems, Springer Verlag, New York, 1989.

[8] D. Motreanu and P. D. Panagiotopoulos, Minimax theorems and qualitative properties of the solutions of hemivariational inequalities, Nonconvex Optim. Appl. **29**, Kluwer Acad., Dordrecht, 1998.

[9] E. H. Papageorgiou and N. S. Papageorgiou, Existence of solutions and of multiple solutions for nonlinear nonsmooth periodic systems, Czechoslovak Math. J. **54** (2004), 347–371.

[10] D. Pasca, Periodic solutions of a class of non-autonomous second-order differential inclusions systems, Abstr. Appl. Anal. **6** (2001), 151–161.

[11] D. Pasca, Periodic solutions of second-order differential inclusions systems with p-Laplacian, J. Math. Anal. Appl. **325** (2007), 90–100.

[12] C. L. Tang, Periodic solutions for nonautonomous second systems with not sublinear nonlinearity, Proc. Amer. Math. Soc. **126** (1998), 3263–3270.

[13] C. L. Tang and X. P. Wu, Periodic solutions for second order Hamiltonian systems with not uniformly coercive potential, J. Math. Anal. Appl. **259** (2001), 386–397.

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