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UNIVERSAL INEQUALITIES FOR EIGENVALUES OF A SYSTEM OF SUB-ELLIPTIC EQUATIONS ON HEISENBERG GROUP[1]

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Abstract

In this paper, we study the eigenvalue problem of a system of sub-elliptic equations on abounded domain in the Heisenberg group and obtain some universal inequalities. Moreover, for the lower order eigenvalues of this eigenvalue problem, we also give some universal inequalities.

1. Introduction

It is well known that the $(2n+1)$-dimensional Heisenberg group $\mathbf{H}^n$ is the space R^{2n+1} equipped with the non-commutative group law

$$(x, y, t)(x', y', t') = \left(x + x', y + y', t + t' + \frac{1}{2}(\langle x', y\rangle_{\mathrm{R}^n} - \langle x, y'\rangle_{\mathrm{R}^n})\right),$$

where $x, y, x', y' \in \mathrm{R}^n$ and $t, t' \in \mathrm{R}$, and $\langle\,,\rangle_{R^n}$ denotes the inner product in R^n. The Lie algebra $\mathscr{H}^n$ of $\mathbf{H}^n$ has a basis formed by the following vector fields

$$X_p = \frac{\partial}{\partial x_p} - \frac{y_p}{2}\frac{\partial}{\partial t}, \quad Y_p = \frac{\partial}{\partial y_p} - \frac{x_p}{2}\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}, \quad p = 1, 2, \dots, n.$$

We note that the only nontrivial commutators are $[Y_p, X_q] = -T\delta_{pq}$. Thus the vector fields $X_1, \dots, X_n$, $Y_1, \dots, Y_n$ generate a vector bundle on $\mathbf{H}^n$, the so called horizontal vector bundle $H\mathbf{H}^n$, that is a vector subbundle of $T\mathbf{H}^n$, the tangent vector bundle on $\mathbf{H}^n$. Since each fiber of $H\mathbf{H}^n$ can be canonically identified with a vector subspace of $\mathbf{R}^{2n+1}$, each section φ of $H\mathbf{H}^n$ can be identified with a map $\varphi : \mathbf{H}^n \to \mathbf{R}^{2n+1}$. At each point $P \in \mathbf{H}^n$ the horizontal fiber is denoted as $H_P\mathbf{H}^n$ and each fiber endowed with the scalar product $\langle\cdot,\cdot\rangle_P$ and the associated norm

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$|\cdot|_P$ that make the vector fields $X_1,\dots,X_n$, $Y_1,\dots,Y_n$ orthonormal, hence we shall also identify a section of $H\mathbf{H}^n$ with it's canonical coordinates with respect to it's moving frame. In this way, a section φ will be identified with the function $\varphi=(\varphi_1,\dots,\varphi_{2n}):\mathbf{H}^n\to\mathbf{R}^{2n}$ such that $\varphi=\sum_{i=1}^{2n}\varphi_iX_i$. As it is common in Riemannian geometry, when dealing with two sections φ and φ' whose argument is not explicitly written, we shall drop the index P in the scalar product writing $\langle\varphi,\varphi'\rangle$ for $\langle\varphi(P),\varphi'(P)\rangle_P$. The same convention will be adopted for the norm. If Ω is an open domain of Heisenberg group $\mathbf{H}^n$ and $k\ge 0$ is a non-negative integer, the symbols $C^k_{\mathbf{H}}(\Omega)$, $C^\infty_{\mathbf{H}}(\Omega)$ denote the usual (Euclidean) spaces of real valued continuously differential functions. We denote by $C^k(\Omega,H\mathbf{H}^n)$ the all C^k-sections of $H\mathbf{H}^n$, where the C^k regularity is understood as regularity between smooth manifolds. The notations of $C^\infty(\Omega,H\mathbf{H}^n)$ is defined analogously. Now, let us introduce some differential operators on the Heisenberg group $\mathbf{H}^n$.

Let Ω be an open subset of $\mathbf{H}^n$, if $\varphi=(\varphi_1,\dots,\varphi_{2n})\in C^k(\Omega,H\mathbf{H}^n)$ and $k\ge 1$ is a positive integer, we define the *horizontal divergence* of φ as

$$\operatorname{div}_{\mathbf{H}}\varphi=\sum_{i=1}^n X_i\varphi_i+Y_i\varphi_{n+i}.$$

If $f\in C^k_{\mathbf{H}}(\Omega)$ and $k\ge 1$ is a positive integer, we define the *horizontal gradient* of f as

$$\operatorname{grad}_{\mathbf{H}}f=(X_1f,\dots,X_nf,Y_1f,\dots,Y_nf).$$

If $f\in C^k_{\mathbf{H}}(\Omega)$ and $k\ge 2$ is a positive integer, the real *Kohn Laplacian* of f is given by

$$\Delta_{\mathbf{H}}f=\operatorname{div}_{\mathbf{H}}\operatorname{grad}_{\mathbf{H}}f=\sum_{p=1}^n(X_p^2f+Y_p^2f).$$

If $\varphi=(\varphi_1,\dots,\varphi_{2n})\in C^k(\Omega,H\mathbf{H}^n)$ and $k\ge 2$ is a positive integer, we define

$$X_i(\varphi)=(X_i\varphi_1,\dots,X_i\varphi_{2n}),\quad i=1,\dots,n;$$

$$Y_j(\varphi)=(Y_j\varphi_1,\dots,Y_j\varphi_{2n}),\quad j=1,\dots,n;$$

$$\Delta_{\mathbf{H}}(\varphi)=(\Delta_{\mathbf{H}}\varphi_1,\dots,\Delta_{\mathbf{H}}\varphi_{2n});$$

$$\operatorname{grad}_{\mathbf{H}}f\cdot\operatorname{grad}_{\mathbf{H}}(\varphi)=(\langle\operatorname{grad}_{\mathbf{H}}f,\operatorname{grad}_{\mathbf{H}}\varphi_1\rangle,\dots,\langle\operatorname{grad}_{\mathbf{H}}f,\operatorname{grad}_{\mathbf{H}}\varphi_{2n}\rangle).$$

Based on the above facts and more results [25, 23, 17, 16, 24] of eigenvalue problem in the Heisenberg group, in this paper, we consider the following eigenvalue problem of a system of sub-elliptic equations

$$\begin{cases}\Delta_{\mathbf{H}}\mathbf{u}+\alpha\operatorname{grad}_{\mathbf{H}}(\operatorname{div}_{\mathbf{H}}(\mathbf{u}))=-\sigma\mathbf{u}, & \text{in }\Omega,\\ \mathbf{u}=0, & \text{on }\partial\Omega,\end{cases}\tag{1.1}$$

where Ω is a bounded domain in $\mathbf{H}^n$, α is a non-negative constant, and $\mathbf{u}\in C^k(\Omega,H\mathbf{H}^n)$.

In [18], Ilias and Makhoul mentioned the following inverse spectral problem (see [9]): "*What kind of increasing sequences of non negative numbers can be the spectrum of the Laplacian of a compact Riemannian manifold* (*respectively*, *of the Dirichlet Laplacian on a domain of a fixed Euclidean space*)?". In the same paper, they proposed another question which is less difficult than the first one: "*Is there any restrictions on these spectral sequences*, *which are independent of the manifold* (*respectively*, *the domain*)?". Such restrictions will be called "universal". Thus, the universal inequalities are the useful tools to study the inverse spectral problems. In recent years, many universal inequalities for eigenvalues on Riemannian manifolds have been obtained by many mathematicians, we refer to [1, 4–8, 10–15, 18–20, 26–28, etc.] and the references therein. In sub-Riemannian geometry, it is natural to consider the following problem: *In sub-Riemannian manifolds*, *are there similar results for the eigenvalue problems as in the Riemannian case*? For the eigenvalue problem of a system of elliptic equations on a Euclidean space $\mathbf{R}^m$, Levine-Protter [21], Levitin-Parnovski [22], Cheng-Yang [3], and Chen-Cheng-Wang-Xia [2] gave some universal inequalities, respectively. Our purpose in this paper is to prove a universal inequality for the eigenvalue problem of a system of elliptic equations on a Heisenberg group $\mathbf{H}^n$ although the noncommutativity of vector fields $\{X_i, Y_i\}$ makes the discussion of this problem more complicated than the similar one on the Euclidean space.

Theorem 1.1. *Let Ω be a bounded domain in a $(2n+1)$-dimensional Heisenberg group $\mathbf{H}^n$ and let σ_i be the i-th eigenvalue of the eigenvalue problem* (1.1). *Then we have*

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2 \le \min\left\{\frac{(2n+\alpha)}{n^2}, A(n,\alpha)\right\}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\sigma_i, \tag{1.2}$$

where

$$A(n,\alpha) = \begin{cases} \dfrac{8+(2n+2)\alpha}{(2n+\alpha)(1+L)}, & \text{when } 0 \le \alpha < (n+1)+\sqrt{(n+1)^2+4}, \\ \dfrac{4+\alpha^2}{2n+\alpha}, & \text{when } \alpha \ge (n+1)+\sqrt{(n+1)^2+4}, \end{cases}$$

and $L = \dfrac{(4+(2n+2)\alpha-\alpha^2)n^2}{(2n+\alpha)^2}$.

From Theorem 1.1, we can easily get the following

Corollary 1.1. *Under the assumption of Theorem* 1.1, *we have*

$$\sigma_{k+1} \le \left(1+\min\left\{\frac{(2n+\alpha)}{n^2}, A(n,\alpha)\right\}\right)\frac{1}{k}\sum_{i=1}^{k}\sigma_i, \tag{1.3}$$

and the gap of any consecutive eigenvalues

$$\sigma_{k+1} - \sigma_k \leq \min\left\{\frac{(2n+\alpha)}{n^2}, A(n,\alpha)\right\}\frac{1}{k}\sum_{i=1}^{k}\sigma_i. \tag{1.4}$$

For the lower order eigenvalues, we can obtain

THEOREM 1.2. *Under the assumption of Theorem* 1.1, *we have*

$$\sum_{i=1}^{2n}(\sigma_{i+1} - \sigma_1) \leq 4(1+\alpha)\sigma_1. \tag{1.5}$$

2. Preliminaries

In this section, we will prove a lemma which will play a key role in the proof of Theorem 1.1.

LEMMA 2.1. *Let* Ω *be a bounded domain in an* $(2n+1)$*-dimensional Heisenberg group* $\mathbf{H}^n$. *Let* σ_i *denote the i-th eigenvalue of the eigenvalue problem* (1.1) *and let* $\mathbf{u}_i$ *be the i-th orthonormal eigenfunctions corresponding to* σ_i, *namely,* $\mathbf{u}_i$ *satisfies*

$$\begin{cases} \Delta_{\mathbf{H}}\mathbf{u}_i + \alpha\,\mathrm{grad}_{\mathbf{H}}(\mathrm{div}_{\mathbf{H}}(\mathbf{u}_i)) = -\sigma_i\Delta_{\mathbf{H}}\mathbf{u}_i, & \text{in } \Omega, \\ \mathbf{u}_i = 0, & \text{on } \partial\Omega, \\ \int_\Omega \langle \mathbf{u}_i, \mathbf{u}_j\rangle = \delta_{ij}, & i,j = 1,2,\ldots \end{cases} \tag{2.1}$$

Then for any $h \in C^2(\Omega) \cap C^1(\partial\Omega)$, *we have*

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2\left\{\int_\Omega |\mathrm{grad}_{\mathbf{H}}\, h|^2|\mathbf{u}_i|^2 + \alpha\int_\Omega \langle \mathrm{grad}_{\mathbf{H}}\, h, \mathbf{u}_i\rangle^2\right\} \leq \sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\|\mathbf{p}_i\|^2 \tag{2.2}$$

and, for any positive constant A,

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2\left\{(1-A)\int_\Omega |\mathrm{grad}_{\mathbf{H}}\, h|^2|\mathbf{u}_i|^2 - A\alpha\int_\Omega |\mathrm{grad}_{\mathbf{H}}\, h\cdot\mathbf{u}_i|^2\right\} \leq \frac{1}{A}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\left\|\mathrm{grad}_{\mathbf{H}}\, h\cdot\mathrm{grad}_{\mathbf{H}}(u_i) + \frac{1}{2}\Delta_{\mathbf{H}}h\mathbf{u}_i\right\|^2, \tag{2.3}$$

where $\mathbf{p}_i \in C^k(\Omega, H\mathbf{H}^n)$ *defined by*

$$\mathbf{p}_i = 2\,\mathrm{grad}_{\mathbf{H}}\, h\cdot\mathrm{grad}_{\mathbf{H}}(\mathbf{u}_i) + \Delta_{\mathbf{H}}h\mathbf{u}_i + \alpha\{\mathrm{grad}_{\mathbf{H}}\langle \mathrm{grad}_{\mathbf{H}}\, h, \mathbf{u}_i\rangle + \mathrm{div}_{\mathbf{H}}(\mathbf{u}_i)\,\mathrm{grad}_{\mathbf{H}}\, h\}.$$

Proof. Defining $\mathbf{v}_i \in C^k(\Omega, H\mathbf{H}^n)$ by

$$\mathbf{v}_i = h\mathbf{u}_i - \sum_{j=1}^{k} a_{ij}\mathbf{u}_j, \tag{2.4}$$

where $a_{ij} = \int_\Omega h\langle \mathbf{u}_i, \mathbf{u}_j\rangle = a_{ji}$, we have

$$\mathbf{v}_i|_{\partial\Omega} = \mathbf{0}, \quad \int_\Omega \langle \mathbf{v}_i, \mathbf{u}_j\rangle = 0 \quad \text{for any } i, j = 1, \dots, k. \tag{2.5}$$

From the Rayleigh-Ritz inequality, we have

$$\sigma_{k+1}\int_\Omega |\mathbf{v}_i|^2 \le \int_\Omega (\Delta_{\mathbf{H}}\mathbf{v}_i \cdot \mathbf{v}_i + \alpha(\mathrm{div}_{\mathbf{H}}(\mathbf{v}_i))^2). \tag{2.6}$$

From the definition of $\mathbf{v}_i$, we derive

$$\begin{aligned}
\Delta_{\mathbf{H}}\mathbf{v}_i &= \Delta_{\mathbf{H}}(h\mathbf{u}_i) - \sum_{j=1}^{k} a_{ij}\Delta_{\mathbf{H}}\mathbf{u}_j \\
&= h\Delta_{\mathbf{H}}(\mathbf{u}_i) + 2\ \mathrm{grad}_{\mathbf{H}}\ h \cdot \mathrm{grad}_{\mathbf{H}}(\mathbf{u}_i) + \Delta_{\mathbf{H}}h\mathbf{u}_i - \sum_{j=1}^{k} a_{ij}\Delta_{\mathbf{H}}\mathbf{u}_j \\
&= h(-\sigma_i\mathbf{u}_i - \alpha\ \mathrm{grad}_{\mathbf{H}}(\mathrm{div}_{\mathbf{H}}(\mathbf{u}_i))) + 2\ \mathrm{grad}_{\mathbf{H}}\ h \cdot \mathrm{grad}_{\mathbf{H}}(\mathbf{u}_i) + \Delta_{\mathbf{H}}h\mathbf{u}_i \\
&\quad - \sum_{j=1}^{k} a_{ij}(-\sigma_i\mathbf{u}_i - \alpha\ \mathrm{grad}_{\mathbf{H}}(\mathrm{div}_{\mathbf{H}}(\mathbf{u}_i))) \\
&= -\sigma_i h\mathbf{u}_i + \sum_{j=1}^{k} a_{ij}\sigma_i\mathbf{u}_i + 2\ \mathrm{grad}_{\mathbf{H}}\ h \cdot \mathrm{grad}_{\mathbf{H}}(\mathbf{u}_i) + \Delta_{\mathbf{H}}h\mathbf{u}_i \\
&\quad - \alpha\ \mathrm{grad}_{\mathbf{H}}(\mathrm{div}_{\mathbf{H}}(\mathbf{u}_i)) + \alpha\sum_{j=1}^{k} a_{ij}\ \mathrm{grad}_{\mathbf{H}}(\mathrm{div}_{\mathbf{H}}(\mathbf{u}_i)).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\int_\Omega \langle -\Delta_{\mathbf{H}}\mathbf{v}_i, \mathbf{v}_i\rangle &= \sigma_i\|\mathbf{v}_i\|^2 - \int_\Omega \langle (2\ \mathrm{grad}_{\mathbf{H}}\ h \cdot \mathrm{grad}_{\mathbf{H}}(\mathbf{u}_i) + \Delta_{\mathbf{H}}h\mathbf{u}_i), \mathbf{v}_i\rangle \\
&\quad - \alpha\int_\Omega \langle \mathrm{grad}_{\mathbf{H}}(\mathrm{div}_{\mathbf{H}}(\mathbf{u}_i)), \mathbf{v}_i\rangle \\
&\quad + \alpha\sum_{j=1}^{k} a_{ij}\int_\Omega \langle \mathrm{grad}_{\mathbf{H}}(\mathrm{div}_{\mathbf{H}}(\mathbf{u}_i)), \mathbf{v}_i\rangle.
\end{aligned} \tag{2.7}$$

Using integration by parts, we have

$$-\alpha\int_\Omega \langle \mathrm{grad}_{\mathbf{H}}(\mathrm{div}_{\mathbf{H}}(\mathbf{u}_i)), \mathbf{v}_i\rangle + \alpha\sum_{j=1}^k a_{ij}\int_\Omega \langle \mathrm{grad}_{\mathbf{H}}(\mathrm{div}_{\mathbf{H}}(\mathbf{u}_j)), \mathbf{v}_i\rangle$$
$$= \alpha\int_\Omega (\mathrm{div}_{\mathbf{H}}\, \mathbf{v}_i)^2 - \alpha\int_\Omega (\mathrm{div}_{\mathbf{H}}(\mathbf{v}_i)\langle \mathrm{grad}_{\mathbf{H}}\, h, \mathbf{u}_i\rangle + \mathrm{div}_{\mathbf{H}}(\mathbf{u}_i)\langle \mathrm{grad}_{\mathbf{H}}\, h, \mathbf{v}_i\rangle)$$
$$= \alpha\int_\Omega (\mathrm{div}_{\mathbf{H}}\, \mathbf{v}_i)^2 + \alpha\int_\Omega (\langle \mathbf{v}_i, \mathrm{grad}_{\mathbf{H}}\langle \mathrm{grad}_{\mathbf{H}}\, h, \mathbf{u}_i\rangle\rangle + \mathrm{div}_{\mathbf{H}}(\mathbf{u}_i)\langle \mathrm{grad}_{\mathbf{H}}\, h, \mathbf{v}_i\rangle).$$

By (2.6)–(2.7) and the above equality, we have

$$(2.8)\quad (\sigma_{k+1}-\sigma_i)\|\mathbf{v}_i\|^2 \le -\int_\Omega \langle (2\ \mathrm{grad}_{\mathbf{H}}\, h\cdot \mathrm{grad}_{\mathbf{H}}(\mathbf{u}_i) + \Delta_{\mathbf{H}} h\mathbf{u}_i), \mathbf{v}_i\rangle$$
$$+ \alpha\int_\Omega (\langle \mathbf{v}_i, \mathrm{grad}_{\mathbf{H}}\langle \mathrm{grad}_{\mathbf{H}}\, h, \mathbf{u}_i\rangle\rangle + \mathrm{div}_{\mathbf{H}}(\mathbf{u}_i)\langle \mathrm{grad}_{\mathbf{H}}\, h, \mathbf{v}_i\rangle).$$

Let us set

$$b_{ij} = \int_\Omega \left\langle \mathrm{grad}_{\mathbf{H}}\, h\cdot \mathrm{grad}_{\mathbf{H}}(\mathbf{u}_i) + \frac{1}{2}\Delta_{\mathbf{H}} h\mathbf{u}_i, \mathbf{u}_j\right\rangle = -b_{ji},$$

and

$$\mathbf{p}_i = 2\ \mathrm{grad}_{\mathbf{H}}\, h\cdot \mathrm{grad}_{\mathbf{H}}(\mathbf{u}_i) + \Delta_{\mathbf{H}} h\mathbf{u}_i + \alpha\{\mathrm{grad}_{\mathbf{H}}(\mathrm{grad}_{\mathbf{H}}\, h\cdot \mathbf{u}_i) + \mathrm{div}_{\mathbf{H}}(\mathbf{u}_i)\ \mathrm{grad}_{\mathbf{H}}\, h\}.$$

By the similar computation such as (2.9)–(2.14) in [2], we have

$$\sum_{i=1}^k (\sigma_{k+1}-\sigma_i)^2\left\{\int_\Omega |\mathrm{grad}_{\mathbf{H}}\, h|^2|\mathbf{u}_i|^2 + \alpha\int_\Omega \langle \mathrm{grad}_{\mathbf{H}}\, h, \mathbf{u}_i\rangle^2\right\} \le \sum_{i=1}^k (\sigma_{k+1}-\sigma_i)\|\mathbf{p}_i\|^2,$$

hence, (2.2) is true.

By the similar computation such as (2.15) in [2], we have

$$(\sigma_{k+1}-\sigma_i)^2\left(\int_\Omega |\mathrm{grad}_{\mathbf{H}}\, h|^2|\mathbf{u}_i|^2 + 2\sum_{j=1}^k a_{ij}b_{ij}\right)$$
$$\le (\sigma_{k+1}-\sigma_i)^2 A\left(\int_\Omega |\mathrm{grad}_{\mathbf{H}}\, h|^2|\mathbf{u}_i|^2 + \sum_{j=1}^k (\sigma_i-\sigma_j)a_{ij}^2 + \alpha\int_\Omega \langle \mathrm{grad}_{\mathbf{H}}\, h, \mathbf{u}_i\rangle^2\right)$$
$$+ \frac{(\sigma_{k+1}-\sigma_i)}{A}\left(\left\|\mathrm{grad}_{\mathbf{H}}\, h\cdot \mathrm{grad}_{\mathbf{H}}(\mathbf{u}_i) + \frac{1}{2}\Delta_{\mathbf{H}} h\mathbf{u}_i\right\|^2 - \sum_{j=1}^k b_{ij}^2\right).$$

In the above inequality, summing over i from 1 to k and noticing $a_{ij} = a_{ji}$, $b_{ij} = -b_{ji}$, we obtain

$$
\begin{aligned}
&\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2\int_\Omega |\mathrm{grad}_{\mathbf{H}}\, h|^2|\mathbf{u}_i|^2 - 2\sum_{i,j=1}^{k}(\sigma_{k+1}-\sigma_i)(\sigma_i-\sigma_j)a_{ij}b_{ij}\\
&\qquad\le A\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2\left(\int_\Omega |\mathrm{grad}_{\mathbf{H}}\, h|^2|\mathbf{u}_i|^2 + \alpha\int_\Omega \langle \mathrm{grad}_{\mathbf{H}}\, h, \mathbf{u}_i\rangle^2\right)\\
&\qquad\quad+\sum_{i=1}^{k}\frac{(\sigma_{k+1}-\sigma_i)}{A}\left\|\mathrm{grad}_{\mathbf{H}}\, h\cdot \mathrm{grad}_{\mathbf{H}}(\mathbf{u}_i)+\frac{1}{2}\Delta_{\mathbf{H}}h\mathbf{u}_i\right\|^2\\
&\qquad\quad-A\sum_{i,j=1}^{k}(\sigma_{k+1}-\sigma_i)(\sigma_i-\sigma_j)^2a_{ij}^2-\sum_{i,j=1}^{k}\frac{(\sigma_{k+1}-\sigma_i)}{A}b_{ij}^2,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2\int_\Omega |\mathrm{grad}_{\mathbf{H}}\, h|^2|\mathbf{u}_i|^2\\
&\qquad\le A\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2\left(\int_\Omega |\mathrm{grad}_{\mathbf{H}}\, h|^2|\mathbf{u}_i|^2 + \alpha\int_\Omega \langle \mathrm{grad}_{\mathbf{H}}\, h, \mathbf{u}_i\rangle^2\right)\\
&\qquad\quad+\sum_{i=1}^{k}\frac{(\sigma_{k+1}-\sigma_i)}{A}\left\|\mathrm{grad}_{\mathbf{H}}\, h\cdot \mathrm{grad}_{\mathbf{H}}(\mathbf{u}_i)+\frac{1}{2}\Delta_{\mathbf{H}}h\mathbf{u}_i\right\|^2,
\end{aligned}
$$

Thus, (2.3) is true. This completes the proof of Lemma 2.1.

3. Proofs of main results

In this section, we will prove our main results by using Lemma 2.1.

Proof of Theorem 1.1. Let $Y_p = X_{n+p}$, $y_p = x_{n+p}$, $p = 1, \dots, n$, we know that

$$X_\gamma(x_\beta) = \delta_{\gamma\beta}, \quad \beta, \gamma = 1, \dots, 2n.$$

Then we can get

$$\sum_{\beta=1}^{2n}|\mathrm{grad}_{\mathbf{H}}\, x_\beta|^2 = 2n, \quad \sum_{\beta=1}^{2n}\langle \mathrm{grad}_{\mathbf{H}}\, x_\beta, \mathbf{u}_i\rangle^2 = |\mathbf{u}_i|^2; \tag{3.1}$$

$$\mathrm{grad}_{\mathbf{H}}\, x_\beta\cdot \mathrm{grad}_{\mathbf{H}}(\mathbf{u}_i) = X_\beta(\mathbf{u}_i), \quad \Delta_{\mathbf{H}}x_\beta = 0. \tag{3.2}$$

Using integration by parts, we have

$$(3.3)\qquad \sum_{\beta=1}^{2n}\int_\Omega |\mathrm{grad}_{\mathbf{H}}\, x_\beta\cdot \mathrm{grad}_{\mathbf{H}}(\mathbf{u}_i)|^2 = \int_\Omega \sum_{\beta=1}^{2n}\langle X_\beta(\mathbf{u}_i), X_\beta(\mathbf{u}_i)\rangle$$
$$= \int_\Omega -\left\langle \sum_{\beta=1}^{2n} X_\beta^2(\mathbf{u}_i), \mathbf{u}_i\right\rangle$$
$$= \int_\Omega -\langle \Delta_{\mathbf{H}}\mathbf{u}_i, \mathbf{u}_i\rangle$$
$$= \int_\Omega \langle \sigma_i\mathbf{u}_i + \alpha\, \mathrm{grad}_{\mathbf{H}}(\mathrm{div}_{\mathbf{H}}(\mathbf{u}_i)), \mathbf{u}_i\rangle$$
$$= \sigma_i - \alpha\|\mathrm{div}_{\mathbf{H}}(\mathbf{u}_i)\|^2.$$

Taking $h = x_\alpha$ in (2.3) and summing over α from 1 to $2n$, we have

$$(3.4)\qquad \sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2\sum_{\beta=1}^{2n}\left\{(1-A)\int_\Omega |\mathrm{grad}_{\mathbf{H}}\, x_\beta|^2|\mathbf{u}_i|^2 - A\alpha\int_\Omega |\mathrm{grad}_{\mathbf{H}}\, x_\beta\cdot \mathbf{u}_i|^2\right\}$$
$$\le \frac{1}{A}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\sum_{\beta=1}^{2n}\left\|\mathrm{grad}_{\mathbf{H}}\, x_\beta\cdot \mathrm{grad}_{\mathbf{H}}(u_i) + \frac{1}{2}\Delta_{\mathbf{H}}x_\beta\mathbf{u}_i\right\|^2,$$

Taking (3.1)–(3.2) into (3.4), we have

$$(3.5)\qquad \sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2(2n - A(2n+\alpha)) \le \frac{1}{A}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)(\sigma_i - \alpha\|\mathrm{div}_{\mathbf{H}}(\mathbf{u}_i)\|^2).$$

Taking

$$(3.6)\qquad A = \left\{\frac{\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)(\sigma_i-\alpha\|\mathrm{div}_{\mathbf{H}}(\mathbf{u}_i)\|^2)}{(2n+\alpha)\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2}\right\},$$

we have

$$(3.7)\qquad \sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2 \le \frac{(2n+\alpha)}{n^2}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)(\sigma_i-\alpha\|\mathrm{div}_{\mathbf{H}}(\mathbf{u}_i)\|^2),$$

since $\alpha\|\mathrm{div}_{\mathbf{H}}(\mathbf{u}_i)\|^2 \ge 0$, we have

$$(3.8)\qquad \sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2 \le \frac{(2n+\alpha)}{n^2}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\sigma_i.$$

Taking $h = x_\alpha$ into (2.2) and using (3.1) and (3.2), we have

$$(3.9)\qquad \sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2\left\{1+\alpha\int_\Omega \langle \mathrm{grad}_{\mathbf{H}}\, x_\beta, \mathbf{u}_i\rangle^2\right\} \le \sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\|\mathbf{p}_i\|^2,$$

and

$$\mathbf{p}_i = 2\,\mathrm{grad}_{\mathbf{H}}\, x_\beta \cdot \mathrm{grad}_{\mathbf{H}}(\mathbf{u}_i) + \alpha\{\mathrm{grad}_{\mathbf{H}}\langle \mathrm{grad}_{\mathbf{H}}\, x_\beta, \mathbf{u}_i\rangle + \mathrm{div}_{\mathbf{H}}(\mathbf{u}_i)\,\mathrm{grad}_{\mathbf{H}}\, x_\beta\}. \tag{3.10}$$

Taking (3.10) into (3.9) and summing over β from 1 to $2n$, we derive from (3.3) that

$$\begin{aligned}(2n+\alpha)\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2 &\le \sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i) \\ &\quad \times \|2X_\beta(\mathbf{u}_i) + \alpha\{\mathrm{grad}_{\mathbf{H}}\langle \mathrm{grad}_{\mathbf{H}}\, x_\beta, \mathbf{u}_i\rangle + \mathrm{div}_{\mathbf{H}}(\mathbf{u}_i)\,\mathrm{grad}_{\mathbf{H}}\, x_\beta\}\|^2 \\ &= \sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)((4+\alpha^2)\sigma_i - \alpha(\alpha^2-(2n+2)\alpha-4)\|\mathrm{div}_{\mathbf{H}}(\mathbf{u}_i)\|^2).\end{aligned} \tag{3.11}$$

We set $L = \dfrac{(4+(2n+2)\alpha-\alpha^2)n^2}{(2n+\alpha)^2}$, and

$$A(n,\alpha) = \begin{cases} \dfrac{8+(2n+2)\alpha}{(2n+\alpha)(1+L)}, & \text{when } 0 \le \alpha < (n+1)+\sqrt{(n+1)^2+4}, \\ \dfrac{4+\alpha^2}{2n+\alpha}, & \text{when } \alpha \ge (n+1)+\sqrt{(n+1)^2+4}. \end{cases} \tag{3.12}$$

It then follows from the similar discussion as in [2] that

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2 \le A(n,\alpha)\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\sigma_i \tag{3.13}$$

By (3.8) and (3.13), we have

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2 \le \min\left\{\frac{(2n+\alpha)}{n^2}, A(n,\alpha)\right\}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\sigma_i. \tag{3.14}$$

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Defining a $(2n \times 2n)$-matrix $C := (c_{ij})$, where

$$c_{ij} = \int_\Omega x_i\langle \mathbf{u}_1, \mathbf{u}_{j+1}\rangle, \quad i,j = 1,\dots,2n.$$

From the orthogonalization of Gram and Schimidt, there exist an upper triangle matrix $R = (R_{ij})$ and an orthogonal matrix $T = (T_{ij})$ such that $R = TC$. Thus, we have

$$R_{ij} = \sum_{k=1}^{2n} T_{ik} C_{kj} = \sum_{k=1}^{2n} \int_\Omega T_{ik} x_k \langle \mathbf{u}_1, \mathbf{u}_{j+1} \rangle = 0, \quad \text{for } 1 \le j < i \le 2n.$$

Setting $g_i = \sum_{k=1}^{2n} T_{ik} x_k$, we have

$$\int_\Omega g_i \langle \mathbf{u}_1, \mathbf{u}_{j+1} \rangle = 0, \quad \text{for } 1 \le j < i \le 2n.$$

We put

$$\mathbf{w}_i = (g_i - a_i)\mathbf{u}_1, \tag{3.15}$$

where $a_i = \int_\Omega g_i |\mathbf{u}_1|^2$, then it follows that

$$\mathbf{w}_i|_\Omega = 0, \quad \int_\Omega \langle \mathbf{w}_i, \mathbf{u}_{j+1} \rangle = 0, \quad \text{for } 0 \le j < i \le 2n. \tag{3.16}$$

From the Rayleigh-Ritz inequality, we have

$$\sigma_{i+1} \int_\Omega |\mathbf{w}_i|^2 \le \int_\Omega (-\langle \mathbf{w}_i, \Delta_{\mathbf{H}} \mathbf{w}_i \rangle + \alpha(\mathrm{div}_{\mathbf{H}}(\mathbf{w}_i))^2) \tag{3.17}$$

It follows from the definition of g_i, the fact that T is an orthogonal matrix, (3.1) and (3.2) that

$$\sum_{i=1}^{2n} |\mathrm{grad}_{\mathbf{H}}\, g_i|^2 = 2n, \quad \sum_{i=1}^{2n} \langle \mathrm{grad}_{\mathbf{H}}\, g_i, \mathbf{u}_1 \rangle^2 = |\mathbf{u}_1|^2; \tag{3.18}$$

$$\mathrm{grad}_{\mathbf{H}}\, g_i \cdot \mathrm{grad}_{\mathbf{H}}(\mathbf{u}_1) = \sum_{k=1}^{2n} T_{ik} X_k(\mathbf{u}_1), \quad \Delta_{\mathbf{H}} g_i = 0. \tag{3.19}$$

Hence, we have

$$\begin{aligned} \Delta_{\mathbf{H}} \mathbf{w}_i &= (g_i - a_i)\Delta_{\mathbf{H}} \mathbf{u}_1 + 2\, \mathrm{grad}_{\mathbf{H}}\, g_i \cdot \mathrm{grad}_{\mathbf{H}}(\mathbf{u}_1) \\ &= (g_i - a_i)(-\sigma_1 \mathbf{u}_1 - \alpha\, \mathrm{grad}_{\mathbf{H}}(\mathrm{div}_{\mathbf{H}}(\mathbf{u}_1))) + 2\, \mathrm{grad}_{\mathbf{H}}\, g_i \cdot \mathrm{grad}_{\mathbf{H}}(\mathbf{u}_1) \\ &= -\sigma_1 \mathbf{w}_i + (g_i - a_i)(-\alpha\, \mathrm{grad}_{\mathbf{H}}(\mathrm{div}_{\mathbf{H}}(\mathbf{u}_1))) + 2 \sum_{k=1}^{2n} T_{ik} X_k(\mathbf{u}_1) \end{aligned}$$

and

$$\begin{aligned} &\int_\Omega -\langle \Delta_{\mathbf{H}} \mathbf{w}_i, \mathbf{w}_i \rangle \\ &\quad = \sigma_1 \int_\Omega |\mathbf{w}_i|^2 + \int_\Omega \left\langle (g_i - a_i)(\alpha\, \mathrm{grad}_{\mathbf{H}}(\mathrm{div}_{\mathbf{H}}(\mathbf{u}_1))) - 2 \sum_{k=1}^{2n} T_{ik} X_k(\mathbf{u}_1), \mathbf{w}_i \right\rangle. \end{aligned}$$

By (3.17), we have

$$(3.20)\qquad (\sigma_{i+1}-\sigma_1)\int_\Omega |\mathbf{w}_i|^2 \le \int_\Omega \left\langle (g_i-a_i)(\alpha\,\mathrm{grad}_{\mathbf{H}}(\mathrm{div}_{\mathbf{H}}(\mathbf{u}_1))) - 2\sum_{k=1}^{2n} T_{ik}X_k(\mathbf{u}_1), \mathbf{w}_i \right\rangle + \alpha\int_\Omega (\mathrm{div}_{\mathbf{H}}(\mathbf{w}_i))^2.$$

Since

$$\int_\Omega \left\langle \sum_{k=1}^{2n} T_{ik}X_k(\mathbf{u}_1), \mathbf{w}_i \right\rangle = \int_\Omega \sum_{k=1}^{2n} T_{ik}\langle X_k(\mathbf{u}_1), (g_1-a_1)\mathbf{u}_1\rangle = -\int_\Omega \sum_{k=1}^{2n} T_{ik}\langle X_k(\mathbf{u}_1), (g_1-a_1)\mathbf{u}_1\rangle - \int_\Omega \sum_{k=1}^{2n} T_{ik}^2|\mathbf{u}_1|^2,$$

we have

$$(3.21)\qquad -2\int_\Omega \left\langle \sum_{k=1}^{2n} T_{ik}X_k(\mathbf{u}_1), \mathbf{w}_i \right\rangle = 1$$

and

$$(3.22)\qquad \int_\Omega \langle (g_i-a_i)(\alpha\,\mathrm{grad}_{\mathbf{H}}(\mathrm{div}_{\mathbf{H}}(\mathbf{u}_1))), \mathbf{w}_i\rangle = -\alpha\int_\Omega \mathrm{div}_{\mathbf{H}}(\mathbf{u}_1)\,\mathrm{div}_{\mathbf{H}}((g_i-a_i)\mathbf{w}_i) = -\alpha\int_\Omega \mathrm{div}_{\mathbf{H}}(\mathbf{u}_1)((g_i-a_i)\,\mathrm{div}_{\mathbf{H}}(\mathbf{w}_i) + \langle \nabla_{\mathbf{H}} g_i, \mathbf{w}_i\rangle) = -\alpha\int_\Omega (\mathrm{div}_{\mathbf{H}}(\mathbf{w}_i))^2 + \alpha\int_\Omega \mathrm{div}_{\mathbf{H}}(\mathbf{w}_i)\langle \nabla_{\mathbf{H}} g_i, \mathbf{u}_1\rangle - \alpha\int_\Omega \mathrm{div}_{\mathbf{H}}(\mathbf{u}_1)\langle \nabla_{\mathbf{H}} g_i, \mathbf{w}_i\rangle = -\alpha\int_\Omega (\mathrm{div}_{\mathbf{H}}(\mathbf{w}_i))^2 + \alpha\int_\Omega \langle \nabla_{\mathbf{H}} g_i, \mathbf{u}_1\rangle^2.$$

Substituting (3.21) and (3.22) into (3.20), we have

$$(3.23)\qquad (\sigma_{i+1}-\sigma_1)\int_\Omega |\mathbf{w}_i|^2 \le 1 + \alpha\int_\Omega \langle \nabla_{\mathbf{H}} g_i, \mathbf{u}_1\rangle^2.$$

On the other hand, for any positive constant δ_i, we have

$$1 = -2\int_\Omega \left\langle \sum_{k=1}^{2n} T_{ik}X_k(\mathbf{u}_1), \mathbf{w}_i \right\rangle \tag{3.24}$$
$$\leq \delta_i \int_\Omega |\mathbf{w}_i|^2 + \frac{1}{\delta_i}\int_\Omega \left|\sum_{k=1}^{2n} T_{ik}X_k(\mathbf{u}_1)\right|^2.$$

Then by the similar discussion as in [3], we have

$$\sigma_{i+1} - \sigma_1 \leq 4(1+\alpha)\int_\Omega \left|\sum_{k=1}^{2n} T_{ik}X_k(\mathbf{u}_1)\right|^2.$$

Summing over i from 1 to $2n$, we have

$$\sum_{i=1}^{2n}(\sigma_{i+1} - \sigma_1) \leq 4(1+\alpha)\sum_{i=1}^{2n}\int_\Omega \left|\sum_{k=1}^{2n} T_{ik}X_k(\mathbf{u}_1)\right|^2$$
$$= 4(1+\alpha)\sum_{k=1}^{2n}\int_\Omega |X_k(\mathbf{u}_1)|^2 \leq 4(1+\alpha)\sigma_1.$$

Hence, (1.5) is true. This completes the proof of Theorem 1.2.

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