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MINIMAXNESS AND FINITENESS PROPERTIES OF FORMAL LOCAL COHOMOLOGY MODULES

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Abstract

Let $\mathfrak{a}$ be an ideal of local ring $(R, \mathfrak{m})$ and M a finitely generated R-module and n an integer. We prove some results concerning minimaxness and finiteness of formal local cohomology modules. We discuss the maximum and minimum integers such that $\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ is minimax and also we obtain the maximum and minimum integers such that $\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ is finitely gnerated.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring with identity, $\mathfrak{a}$ is an ideal of R and M is an R-module. Recall that the i-th local cohomology module of M with respect to $\mathfrak{a}$ is denoted by $\mathrm{H}_{\mathfrak{a}}^{i}(M)$. For basic facts about commutative algebra see [4], [6]; for local cohomology refer to [3]. Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module. For each $i \geq 0$; $\mathfrak{F}_{\mathfrak{a}}^{i}(M) := \varprojlim_{r} \mathrm{H}_{m}^{i}(M/\mathfrak{a}^{n}M)$ is called the i-th formal local cohomology of M with respect to $\mathfrak{a}$.

The basic properties of formal local cohomology modules are found in [7], [1], [5] and [2].

Recall that an R-module M is called minimax, if there is a finite submodule N of M, such that M/N is Artinian. The class of minimax modules was introduced by Zöschinger [10], and he has given in [10, 11] many equivalent conditions for a module to be minimax. The class of minimax modules includes all finite and all Artinian modules. Moreover it is closed under taking submodules, quotients and extensions, i.e., it is a Serre subcategory of the category of R-modules.

In this paper we investigate some Minimaxness and Finiteness properties of formal local cohomology modules. At first we obtain a result about cosupport

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of minimax formal local cohomology modules and then we determine the least and the largest integers i such that $\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ is minimax. Also we investigate the relation between Finiteness and cosupport of formal local cohomology modules. We will get that (see Theorem 2.8):

$$\begin{aligned}&\inf\{i \in \mathbf{N}_0 : \mathfrak{F}_{\mathfrak{a}}^{i}(M) \text{ is not finitely generated}\}\\&\quad = \inf\{i \in \mathbf{N}_0 : \operatorname{Cosupp}_R(\mathfrak{F}_{\mathfrak{a}}^{i}(M)) \not\subseteq \{\mathfrak{m}\}\}\end{aligned}$$

and by Theorem 2.10:

$$\begin{aligned}&\sup\{i \in \mathbf{N}_0 : \mathfrak{F}_{\mathfrak{a}}^{i}(M) \neq 0\} = \sup\{i \in \mathbf{N}_0 : \mathfrak{F}_{\mathfrak{a}}^{i}(M) \text{ is not finitely generated}\}\\&\quad = \sup\{i \in \mathbf{N}_0 : \operatorname{Coass}_R(\mathfrak{F}_{\mathfrak{a}}^{i}(M)) \not\subseteq \{\mathfrak{m}\}\}.\end{aligned}$$

2. Finiteness of formal local cohomology modules

We first recall the concept of coassociated primes and cosupport of an R-module M. A module is called cocyclic if it is a submodule of $E(R/\mathfrak{m})$ for some maximal ideal $\mathfrak{m}$ of R. A prime ideal $\mathfrak{p}$ is called coassociated to a non-zero R-module M if there is a cocyclic homomorphic image T of M with $\mathfrak{p} = \operatorname{Ann}_R T$ [8]. The set of coassociated primes of M is denoted by $\operatorname{Coass}_R(M)$. Also, Yassemi [8] defined the cosupport of an R-module M, denoted by $\operatorname{Cosupp}_R(M)$, to be the set of primes $\mathfrak{p}$ such that there exists a cocyclic homomorphic image L of M with $\operatorname{Ann}_R(L) \subseteq \mathfrak{p}$. In [8] we can see that $\operatorname{Coass}_R(M) \subseteq \operatorname{Cosupp}_R(M)$ and every minimal element of the set $\operatorname{Cosupp}_R(M)$ belongs to $\operatorname{Coass}_R(M)$.

The following lemma is used in the sequel.

Lemma 2.1. *Let $\mathfrak{a}$ be an ideal of local ring $(R, \mathfrak{m})$ and M an R-module. If $\mathfrak{a}^k M = 0$ for some $k \in \mathbf{N}$, then $\mathfrak{F}_{\mathfrak{a}}^{i}(M) \cong \mathrm{H}_{\mathfrak{m}}^{i}(M)$. Therefore $\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ is Artinian for all $i \geq 0$ and so it is also minimax.*

Proof. It is clear that $\mathfrak{F}_{\mathfrak{a}}^{i}(M) \cong \varprojlim_r \mathrm{H}_{m}^{i}(M/\mathfrak{a}^n M) \cong \mathrm{H}_{m}^{i}(M)$. But by [3, Theorem 7.1.3] $\mathrm{H}_{\mathfrak{m}}^{i}(M)$ is Artinian for all $i \geq 0$ and so the proof is complete. □

Theorem 2.2. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module. Let i be a natural number. If $\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ is minimax, then* $\operatorname{Cosupp}_R \mathfrak{F}_{\mathfrak{a}}^{i}(M) \subseteq V(\mathfrak{a})$.

Proof. Since $\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ is minimax, there exists a finitely generated submodule N of $\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ such that $\mathfrak{F}_{\mathfrak{a}}^{i}(M)/N$ is Artinian. If in [2, Theorem 2.3], we replace $\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ with $\mathfrak{F}_{\mathfrak{a}}^{i}(M)/N$ then with small changes in its proof we can deduce that $\mathrm{Att}_R\, \mathfrak{F}_{\mathfrak{a}}^i(M)/N \subseteq V(\mathfrak{a})$. On the other hand by [8, Theorem 1.14] $\mathrm{Coass}_R\, \mathfrak{F}_{\mathfrak{a}}^i(M)/N = \mathrm{Att}_R\, \mathfrak{F}_{\mathfrak{a}}^i(M)/N$. Hence $\mathrm{Coass}_R\, \mathfrak{F}_{\mathfrak{a}}^i(M)/N \subseteq V(\mathfrak{a})$ and so $\mathrm{Cosupp}_R\, \mathfrak{F}_{\mathfrak{a}}^i(M)/N \subseteq V(\mathfrak{a})$. Since N is finitely generated $\mathrm{Cosupp}_R\, N \subseteq V(\mathfrak{m})$ by [8, Theorem 2.10]. Thus $\mathrm{Cosupp}_R\, \mathfrak{F}_{\mathfrak{a}}^i(M) \subseteq V(\mathfrak{a})$ by [8, Theorem 2.7], as required. □

We need the following Lemma in the proof of the Next Theorem.

Lemma 2.3. *Let* $\mathfrak{a}$ *be an ideal of a local ring* $(R, \mathfrak{m})$ *and* M *a finitely generated R-module. Then* $\mathfrak{F}_{\mathfrak{a}}^0(M)/\mathfrak{a}^k.\mathfrak{F}_{\mathfrak{a}}^0(M)$ *is Artinian for all* $k \in \mathbf{N}$.

Proof. By [1, Theorem 3.8] $\mathfrak{F}_{\mathfrak{a}^k}^0(M)/\mathfrak{a}^k.\mathfrak{F}_{\mathfrak{a}^k}^0(M)$ is Artinian for all $k \in \mathbf{N}$. But $\mathfrak{F}_{\mathfrak{a}}^0(M) \simeq \mathfrak{F}_{\mathfrak{a}^k}^0(M)$ by [7, Lemma 3.8] and so we get the result. □

Theorem 2.4. *Let* $\mathfrak{a}$ *be an ideal of a local ring* $(R, \mathfrak{m})$ *and* M *a finitely generated R-module, and let* $n \in \mathbf{N}$. *Then the following statements are equivalent*:

i) $\mathfrak{F}_{\mathfrak{a}}^i(M)$ *is minimax for all* $i < n$.

ii) $\mathrm{Cosupp}_R(\mathfrak{F}_{\mathfrak{a}}^i(M)) \subseteq V(\mathfrak{a})$ *for all* $i < n$.

Proof. i) $\Rightarrow$ ii): By Theorem 2.2.

ii) $\Rightarrow$ i): We use induction on n. Since $\mathrm{Coass}_R(\mathfrak{F}_{\mathfrak{a}}^i(M)) \subseteq \mathrm{Cosupp}_R(\mathfrak{F}_{\mathfrak{a}}^i(M)) \subseteq V(\mathfrak{a})$, by [9, Satz 2.4] there exists an integer $k \geq 0$ such that $\mathfrak{a}^k \mathfrak{F}_{\mathfrak{a}}^i(M)$ is finitely generated. By Lemma 2.3, $\mathfrak{F}_{\mathfrak{a}}^0(M)/\mathfrak{a}^k \mathfrak{F}_{\mathfrak{a}}^0(M)$ is Artinian and so $\mathfrak{F}_{\mathfrak{a}}^0(M)$ is minimax.

Now suppose, inductively, that $n > 0$ and we have established the result for smaller values of n. Thus $\mathfrak{F}_{\mathfrak{a}}^i(M)$ is minimax for all $i \leq n-2$. It is enough to show that $\mathfrak{F}_{\mathfrak{a}}^{n-1}(M)$ is minimax. By [7, Theorem 3.11], the short exact sequence

$$0 \to \Gamma_{\mathfrak{a}}(M) \to M \to M/\Gamma_{\mathfrak{a}}(M) \to 0$$

implies the long exact sequence

$$\cdots \to \mathfrak{F}_{\mathfrak{a}}^{i-1}(\Gamma_{\mathfrak{a}}(M)) \to \mathfrak{F}_{\mathfrak{a}}^{i-1}(M) \to \mathfrak{F}_{\mathfrak{a}}^{i-1}(M/\Gamma_{\mathfrak{a}}(M)) \to \mathfrak{F}_{\mathfrak{a}}^{i}(\Gamma_{\mathfrak{a}}(M)) \to \cdots$$

Since by Lemma 2.1, $\mathfrak{F}_{\mathfrak{a}}^i(\Gamma_{\mathfrak{a}}(M))$ is minimax for all $i \geq 0$, using the above long exact sequence, we can see that $\mathfrak{F}_{\mathfrak{a}}^i(M)$ is minimax if and only if $\mathfrak{F}_{\mathfrak{a}}^i(M/\Gamma_{\mathfrak{a}}(M))$ is minimax for all $i \geq 0$. On the other hand, since $\mathfrak{F}_{\mathfrak{a}}^i(\Gamma_{\mathfrak{a}}(M))$ is minimax for all $i \geq 0$, $\mathrm{Cosupp}_R\, \mathfrak{F}_{\mathfrak{a}}^i(\Gamma_{\mathfrak{a}}(M)) \subseteq V(\mathfrak{a})$ for all $i \geq 0$ by Theorem 2.2. Now the above long exact sequance and our assumption that $\mathrm{Cosupp}_R\, \mathfrak{F}_{\mathfrak{a}}^i(M) \subseteq V(\mathfrak{a})$ for all $i < n$ imply that $\mathrm{Cosupp}\, \mathfrak{F}_{\mathfrak{a}}^i(M/\Gamma_{\mathfrak{a}}(M)) \subseteq V(\mathfrak{a})$ for all $i < n$. Therefore we can and do assume that M is an $\mathfrak{a}$-torsion-free R-module.

By [3, 2.1.1 (ii)], $\mathfrak{a}$ contains an element r which is a non-zerodivisor on M. Since $\mathrm{Cosupp}_R\, \mathfrak{F}_{\mathfrak{a}}^{n-1}(M) \subseteq V(\mathfrak{a})$, by [9, Satz 2.4] there is an integer $k \geq 1$ such that $r^k \mathfrak{F}_{\mathfrak{a}}^{n-1}(M)$ is finitely generated. The short exact sequence

$$0 \to M \xrightarrow{r^k} M \to M/r^k M \to 0$$

induces a long exact sequence

$$0 \to \mathfrak{F}_{\mathfrak{a}}^{0}(M) \xrightarrow{r^k} \mathfrak{F}_{\mathfrak{a}}^{0}(M) \to \mathfrak{F}_{\mathfrak{a}}^{0}(M/r^kM) \to \cdots \to \mathfrak{F}_{\mathfrak{a}}^{i}(M)$$
$$\xrightarrow{r^k} \mathfrak{F}_{\mathfrak{a}}^{i}(M) \to \mathfrak{F}_{\mathfrak{a}}^{i}(M/r^kM) \to \cdots$$

From this long exact sequence, it turns out that $\operatorname{Cosupp}_R \mathfrak{F}_{\mathfrak{a}}^{i}(M/r^kM) \subseteq V(\mathfrak{a})$ for all $i < n-1$. Hence, by the inductive hypothesis, $\mathfrak{F}_{\mathfrak{a}}^{i}(M/r^kM)$ is minimax for all $i < n-1$, and so $\mathfrak{F}_{\mathfrak{a}}^{n-2}(M/r^kM)$ is minimax. Since $r^k\mathfrak{F}_{\mathfrak{a}}^{n-1}(M)$ is finitely generated and so is minimax, the above long exact sequence implies that $\mathfrak{F}_{\mathfrak{a}}^{n-2}(M/r^kM) \to \mathfrak{F}_{\mathfrak{a}}^{n-1}(M) \to r^k\mathfrak{F}_{\mathfrak{a}}^{n-1}(M)$ is exact. Thus $\mathfrak{F}_{\mathfrak{a}}^{n-1}(M)$ is minimax, as required. □

THEOREM 2.5. *Let* $\mathfrak{a}$ *be an ideal of a local ring* $(R, \mathfrak{m})$ *and* M *a finitely generated R-module, and let* $t \in \mathbf{N}$. *Then the following statements are equivalent*:

(i) $\mathfrak{F}_{\mathfrak{a}}^{j}(M)$ *is minimax for all* $j > t$.

(ii) $\operatorname{Cosupp}_R \mathfrak{F}_{\mathfrak{a}}^{j}(M) \subseteq V(\mathfrak{a})$ *for all* $j > t$.

Proof. (i) $\Rightarrow$ (ii): By Theorem 2.2.

(ii) $\Rightarrow$ (i): We argue by induction on $n := \dim M$. If $n = 0$, then $\mathfrak{F}_{\mathfrak{a}}^{i}(M) = 0$ for all $i > 0$, so that the result has been proved in this case. Now assume, inductively, that $n > 0$ and that the result has been proved for all R-modules of dimensions smaller than n. By an argument analogue to that used in the proof of Theorem 2.4, we can and do assume that M is an $\mathfrak{a}$-torsion-free R-module. By [3, 2.1.1 (ii)], $\mathfrak{a}$ contains an element r which is a non-zerodivisor on M. Let $j > t$ be an integer. By assumption and [9, Satz 2.4] there exists an integer u_j such that $\mathfrak{a}^{u_j}\mathfrak{F}_{\mathfrak{a}}^{j}(M)$ is finitely generated. But $\mathfrak{F}_{\mathfrak{a}}^{i}(M) = 0$ for all $i > \dim M$. Thus we can find an integer u such that $\mathfrak{a}^{u}\mathfrak{F}_{\mathfrak{a}}^{j}(M)$ is finitely generated for all $j > t$. The exact sequence

$$0 \to M \xrightarrow{r^u} M \to M/r^uM \to 0$$

induces a long exact sequence of formal local cohomology modules

$$0 \to \mathfrak{F}_{\mathfrak{a}}^{0}(M) \xrightarrow{r^u} \mathfrak{F}_{\mathfrak{a}}^{0}(M) \to \mathfrak{F}_{\mathfrak{a}}^{0}(M/r^uM) \to \cdots \to \mathfrak{F}_{\mathfrak{a}}^{i}(M)$$
$$\xrightarrow{r^u} \mathfrak{F}_{\mathfrak{a}}^{i}(M) \to \mathfrak{F}_{\mathfrak{a}}^{i}(M/r^uM) \to \cdots$$

From this long exact sequence, we get $\operatorname{Cosupp}_R \mathfrak{F}_{\mathfrak{a}}^{j}(M/r^uM) \subseteq V(\mathfrak{a})$ for all $j > t$. Since $\dim(M/r^uM) = n-1$, it follows from the inductive hypothesis that $\mathfrak{F}_{\mathfrak{a}}^{j}(M/r^uM)$ is minimax for all $j > t$. The exact sequence

$$0 \to M \xrightarrow{r^u} M \to M/r^uM \to 0$$

provides the following exact sequence

$$\to r^u\mathfrak{F}_{\mathfrak{a}}^{j}(M) \to \mathfrak{F}_{\mathfrak{a}}^{j}(M) \to \mathfrak{F}_{\mathfrak{a}}^{j}(M/r^uM) \to \cdots,$$

for all $j > t$. Since $\mathfrak{a}^u \mathfrak{F}_{\mathfrak{a}}^j(M)$ is finitely generated for all $j > t$, $r^u \mathfrak{F}_{\mathfrak{a}}^j(M)$ is minimax for all $j > t$. Thus $\mathfrak{F}_{\mathfrak{a}}^j(M)$ is minimax for all $j > t$. This completes the proof. □

THEOREM 2.6. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module, and let $t \in \mathbf{N}$. If $\mathfrak{F}_{\mathfrak{a}}^i(M)$ is minimax for all $i > t$ then* $\operatorname{Supp}_R \mathfrak{F}_{\mathfrak{a}}^i(M) \cap V(\mathfrak{a}) \subseteq \{\mathfrak{m}\}$ *for all $i > t$.*

Proof. We use induction on $n := \dim M$. If $n = 0$, then $\mathfrak{F}_{\mathfrak{a}}^i(M) = 0$ for all $i > 0$, so that the result has been proved in this case. Now assume, inductively, that $n > 0$ and that the result has been proved for all R-modules of dimensions smaller than n.

The short exact sequence

$$0 \to \Gamma_{\mathfrak{a}}(M) \to M \to M/\Gamma_{\mathfrak{a}}(M) \to 0$$

induces a long exact sequence

$$\cdots \to \mathfrak{F}_{\mathfrak{a}}^i(\Gamma_{\mathfrak{a}}(M)) \to \mathfrak{F}_{\mathfrak{a}}^i(M) \to \mathfrak{F}_{\mathfrak{a}}^i(M/\Gamma_{\mathfrak{a}}(M)) \to \mathfrak{F}_{\mathfrak{a}}^{i+1}(\Gamma_{\mathfrak{a}}(M)) \to \cdots$$

Since $\mathfrak{F}_{\mathfrak{a}}^i(\Gamma_{\mathfrak{a}}(M))$ is Artinian for all $i \geq 0$ we obtain the following exact sequence for all prime ideal $\mathfrak{p} \subsetneq \mathfrak{m}$:

$$0 \to (\mathfrak{F}_{\mathfrak{a}}^i(M))_{\mathfrak{p}} \to (\mathfrak{F}_{\mathfrak{a}}^i(M/\Gamma_{\mathfrak{a}}(M)))_{\mathfrak{p}} \to 0$$

From the above exact sequence we conclude that $\operatorname{Supp}_R \mathfrak{F}_{\mathfrak{a}}^i(M) \cap V(\mathfrak{a}) \subseteq \{\mathfrak{m}\}$ if and only if $\operatorname{Supp}_R \mathfrak{F}_{\mathfrak{a}}^i(M/\Gamma_{\mathfrak{a}}(M)) \cap V(\mathfrak{a}) \subseteq \{\mathfrak{m}\}$. Hence we can and do assume that M is an $\mathfrak{a}$-torsion-free R-module. Thus, there exists an element $x \in \mathfrak{a}$ which is a non-zerodivisor on M. Now the exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

induces a long exact sequence

$$\cdots \to \mathfrak{F}_{\mathfrak{a}}^{i-1}(M/xM) \to \mathfrak{F}_{\mathfrak{a}}^i(M) \to \mathfrak{F}_{\mathfrak{a}}^i(M) \to \mathfrak{F}_{\mathfrak{a}}^i(M/xM) \to \cdots$$

From the above exact sequance we see that $\mathfrak{F}_{\mathfrak{a}}^i(M/xM)$ is minimax for all $i > t$. But $\dim(M/xM) = n - 1$ and so from the inductive hypothesis $\operatorname{Supp}_R(\mathfrak{F}_{\mathfrak{a}}^i(M/xM)) \cap V(\mathfrak{a}) \subseteq \{\mathfrak{m}\}$ for all $i > t$. Thus if $\mathfrak{p}$ is a prime ideal such that $\mathfrak{a} \subseteq \mathfrak{p} \subsetneq \mathfrak{m}$ then $(\mathfrak{F}_{\mathfrak{a}}^i(M/xM))_{\mathfrak{p}} = 0$ for all $i > t$. Hence we get the following exact sequance:

$$\to (\mathfrak{F}_{\mathfrak{a}}^i(M))_{\mathfrak{p}} \xrightarrow{x/1} (\mathfrak{F}_{\mathfrak{a}}^i(M))_{\mathfrak{p}} \to 0.$$

Let $i > t$ be an integer. Since $\mathfrak{F}_{\mathfrak{a}}^i(M)$ is minimax, there exists a finitely generated submodule N of $\mathfrak{F}_{\mathfrak{a}}^i(M)$ such that $\mathfrak{F}_{\mathfrak{a}}^i(M)/N$ is Artinian. Thus $(\mathfrak{F}_{\mathfrak{a}}^i(M)/N)_{\mathfrak{p}} = 0$ and so $(\mathfrak{F}_{\mathfrak{a}}^i(M))_{\mathfrak{p}}$ is a finitely generated R-module. Hence $(\mathfrak{F}_{\mathfrak{a}}^i(M))_{\mathfrak{p}}$ is finitely generated for all $i > t$. Now the above exact sequance and Nakayama Lemma imply that $(\mathfrak{F}_{\mathfrak{a}}^i(M))_{\mathfrak{p}} = 0$. Therefore $\mathfrak{p} \notin \operatorname{Supp}_R(\mathfrak{F}_{\mathfrak{a}}^i(M))$ for all $\mathfrak{a} \subseteq \mathfrak{p} \subsetneq \mathfrak{m}$. This completes the proof. □

COROLLARY 2.7. *Let* $\mathfrak{a}$ *be an ideal of a local ring* $(R, \mathfrak{m})$ *and* M *a finitely generated* R*-module, and let* $t \in \mathbf{N}$. *If* $\mathfrak{F}_{\mathfrak{a}}^i(M)$ *is minimax for all* $i > t$ *then* $\operatorname{Hom}_R(R/\mathfrak{a}, \mathfrak{F}_{\mathfrak{a}}^i(M))$ *is Artinian for all* $i > t$.

Proof. By Theorem 2.6, $\operatorname{Ass}_R(\operatorname{Hom}_R(R/\mathfrak{a}, \mathfrak{F}_{\mathfrak{a}}^i(M))) = \operatorname{Ass}_R \mathfrak{F}_{\mathfrak{a}}^i(M) \cap V(\mathfrak{a}) \subseteq \{\mathfrak{m}\}$ for all $i > t$. But $\operatorname{Hom}_R(R/\mathfrak{a}, \mathfrak{F}_{\mathfrak{a}}^i(M)) \simeq (0 :_{\mathfrak{F}_{\mathfrak{a}}^i(M)} \mathfrak{a})$ is isomorphic to a submodule of $\mathfrak{F}_{\mathfrak{a}}^i(M)$ and so is minimax for all $i > t$. Now it is easy to see that, by the definition of minimax modules, $\operatorname{Hom}_R(R/\mathfrak{a}, \mathfrak{F}_{\mathfrak{a}}^i(M))$ is Artinian for all $i > t$, as required. □

THEOREM 2.8. *Let* $\mathfrak{a}$ *be an ideal of a local ring* $(R, \mathfrak{m})$ *and* M *a finitely generated* R*-module, and let* $n \in \mathbf{N}$. *Then the following statements are equivalent*:

i) $\mathfrak{F}_{\mathfrak{a}}^i(M)$ *is finitely generated for all* $i < n$.

ii) $\operatorname{Cosupp}_R(\mathfrak{F}_{\mathfrak{a}}^i(M)) \subseteq \{\mathfrak{m}\}$ *for all* $i < n$.

Proof. i) $\Rightarrow$ ii): By [8, Theorem 2.10].

ii) $\Rightarrow$ i): Since $\operatorname{Cosupp}_R \mathfrak{F}_{\mathfrak{a}}^i(M) \subseteq \{\mathfrak{m}\}$ for all $i < n$, $\operatorname{Cosupp}_R \mathfrak{F}_{\mathfrak{a}}^i(M) \subseteq V(\mathfrak{a})$ for all $i < n$ and so $\mathfrak{F}_{\mathfrak{a}}^i(M)$ is minimax for all $i < n$ by Theorem 2.4. Let $i < n$ be an integer. Then there exists a finitely generated submodule N such that $\mathfrak{F}_{\mathfrak{a}}^i(M)/N$ is Artinian. Hence $\operatorname{Att}_R \mathfrak{F}_{\mathfrak{a}}^i(M)/N = \operatorname{Coass}_R \mathfrak{F}_{\mathfrak{a}}^i(M)/N$. But $\operatorname{Coass}_R \mathfrak{F}_{\mathfrak{a}}^i(M)/N \subseteq \operatorname{Cosupp}_R \mathfrak{F}_{\mathfrak{a}}^i(M)/N \subseteq \operatorname{Cosupp}_R \mathfrak{F}_{\mathfrak{a}}^i(M) \subseteq \{\mathfrak{m}\}$ and so $\operatorname{Att}_R \mathfrak{F}_{\mathfrak{a}}^i(M)/N \subseteq \{\mathfrak{m}\}$. Now [3, 7.2.12] implies that $\mathfrak{F}_{\mathfrak{a}}^i(M)/N$ is finitely generated. Since N is finitely generated we conclude that $\mathfrak{F}_{\mathfrak{a}}^i(M)$ is finitely generated for all $i < n$, as required. □

LEMMA 2.9. *Let* $(R, \mathfrak{m})$ *be a local ring and* M *be an* R*-module such that* $\operatorname{Coass}_R(M) \subseteq \{\mathfrak{m}\}$. *Then* $\operatorname{Hom}_R(R_x, M) = 0$ *for all* $x \in \mathfrak{m}$.

Proof. Since $\operatorname{Coass}_R(M) \subseteq \{\mathfrak{m}\}$, there is an integer $t \geq 1$ such that $\mathfrak{m}^t M$ is finitely generated. Now if $f \in \operatorname{Hom}_R(R_x, M)$ then $f(1/x^n) = x^k x^t f(1/x^{t+k+n}) \in x^k \mathfrak{m}^t M$ for all $k, n \in \mathbf{N}$. Thus $f\left(\frac{1}{x^n}\right) \in \bigcap_k x^k \mathfrak{m}^t M = 0$ for all $n \in \mathbf{N}$, by Krull Theorem. Therefore $f = 0$. □

THEOREM 2.10. *Let* $\mathfrak{a}$ *be an ideal of a local ring* $(R, \mathfrak{m})$ *and* M *a finitely generated* R*-module, and let* $n \in \mathbf{N}$. *Then the following statements are equivalent*:

i) $\mathfrak{F}_{\mathfrak{a}}^i(M)$ *is finitely generated for all* $i > n$.

ii) $\operatorname{Coass}_R(\mathfrak{F}_{\mathfrak{a}}^i(M)) \subseteq \{\mathfrak{m}\}$ *for all* $i > n$.

iii) $\mathfrak{F}_{\mathfrak{a}}^i(M) = 0$ *for all* $i > n$.

Proof. i) $\Rightarrow$ ii): By [8, Theorem 2.10].

ii) $\Rightarrow$ iii): Let $x \in \mathfrak{m} \backslash \mathfrak{a}$. Then by Lemma 2.9, $\operatorname{Hom}_R(R_x, \mathfrak{F}_{\mathfrak{a}}^i(M)) = 0$ for all $i > n$. But by [7, Theorem 3.15], there exists a long exact sequence

$$\cdots \to \operatorname{Hom}_R(R_x, \mathfrak{F}_{\mathfrak{a}}^i(M)) \to \mathfrak{F}_{\mathfrak{a}}^i(M) \to \mathfrak{F}_{\langle \mathfrak{a}, x \rangle}^i(M) \to \operatorname{Hom}_R(R_x, \mathfrak{F}_{\mathfrak{a}}^{i+1}(M)) \to \cdots$$

Hence we have $\mathfrak{F}^i_{\mathfrak{a}}(M) \simeq \mathfrak{F}^i_{\langle \mathfrak{a}, x\rangle}(M)$ for all $i > n$. Continuing in this way, we get $\mathfrak{F}^i_{\mathfrak{a}}(M) \simeq \mathfrak{F}^i_{\mathfrak{m}}(M)$ for all $i > n$. Since $\mathfrak{F}^i_{\mathfrak{m}}(M) = 0$ for all $i \geq 0$, we get $\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$ for all $i > n$ and the proof is complete.

iii) $\Rightarrow$ i): It is clear. □

The next result is a generalization of [1, Theorem 2.6].

Corollary 2.11. *Let* $\mathfrak{a}$ *be an ideal of a local ring* $(R, \mathfrak{m})$ *and* M *a finitely generated R-module. Let* $l := \dim(M/\mathfrak{a}M)$. *Then* $\operatorname{Coass}_R \mathfrak{F}^l_{\mathfrak{a}}(M) \nsubseteq \{\mathfrak{m}\}$.

Proof. By [7, Theorem 4.5], $\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$ for all $i > l$ and $\mathfrak{F}^l_{\mathfrak{a}}(M) \neq 0$. If $\operatorname{Coass}_R \mathfrak{F}^l_{\mathfrak{a}}(M) \subseteq \{\mathfrak{m}\}$ then by the above theorem $\mathfrak{F}^l_{\mathfrak{a}}(M) = 0$ which is a contradiction. Therefore the proof is complete.

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