Y. T. NAI AND D. ZHAO
KODAI MATH. J.
38 (2015), 333–342

ON IDEALS OF RINGS OF FRACTIONS AND RINGS OF POLYNOMIALS

Yuan Ting Nai and Dongsheng Zhao

Abstract

We investigate the links between the lattice $\mathrm{Idl}(R)$ of ideals of a commutative ring R and the lattices $\mathrm{Idl}(R')$ of ideals of various new rings R' constructed from R, in particular, the ring $S^{-1}R$ of fractions and the ring $R[X]$ of polynomials. For any partially ordered set P, we construct another poset $\mathbf{N}(P)$ and show that P satisfies the ascending chain condition if and only if $\mathbf{N}(P)$ satisfies the ascending chain condition. As an application of this result, we give an order version proof for Hilbert's Basis Theorem.

1. Introduction

From a given commutative ring R, one can construct new rings in a number of standard ways, such as quotient rings, rings of fractions and the ring $R[X]$ of polynomials. The set $\mathrm{Idl}(R)$ of all ideals of R is a complete lattice with respect to the inclusion order. Since Emmy Noether and her school's series of work, the lattice $\mathrm{Idl}(R)$ has been regarded as the most important order structure associated to the ring R. For instance, the definitions of both Noetherian rings and Artinian rings can be characterized in terms of the order structures of $(\mathrm{Idl}(R), \subseteq)$; a ring R is arithmetical if and only if $(\mathrm{Idl}(R), \subseteq)$ is a distributive lattice [5] [2]. When we have a new ring R' constructed from a given ring R, a naturally raised question would be: how are the lattices $\mathrm{Idl}(R)$ and $\mathrm{Idl}(R')$ linked?

It is well known that for any ideal I of a commutative ring R, there is a one-to-one correspondence between the ideals of the quotient ring R/I and the ideals J of R containing I, that is, the poset $\mathrm{Idl}(R/I)$ is isomorphic to the principal filter $\uparrow I$ of $\mathrm{Idl}(R)$. Also, for any multiplicatively closed subset S of R (that is, $1_R \in S$ and $s_1, s_2 \in S$ imply $s_1 s_2 \in S$), the lattice $\mathrm{Idl}(S^{-1}R)$ of ideals of the ring of fractions determined by S is isomorphic to a subposet of $\mathrm{Idl}(R)$ through the mapping $h^{-1} : \mathrm{Idl}(S^{-1}R) \to \mathrm{Idl}(R)$, where $h : R \to S^{-1}R$ is the natural

2010 *Mathematics Subject Classification*. 06A06, 13E05, 13E99.

Key words and phrases. ring of fractions, multiplicatively closed subset, full m-closed set, ascending chain condition, Hilbert's Basis Theorem.

Received June 12, 2014; revised October 28, 2014.

homomorphism which sends $x \in R$ to $\dfrac{x}{1_R}$ (see Lemma 5.24 of [7]). Thus $\mathrm{Idl}(S^{-1}R)$ is isomorphic to a subposet of $\mathrm{Idl}(R)$ which is closed under arbitrary meets. It seems, however, that there is still no characterization of subsets of $\mathrm{Idl}(R)$ that correspond to the set of ideals of the ring of fractions $S^{-1}R$ for some multiplicatively closed subset S of R.

On the other hand, there generally does not exist order-embedding of $\mathrm{Idl}(R[X])$ into $\mathrm{Idl}(R)$, simply because $\mathrm{Idl}(R[X])$ usually has much more complicated and richer contents than $\mathrm{Idl}(R)$ does. For example, when K is a field, $\mathrm{Idl}(K)$ only has two elements, while $\mathrm{Idl}(K[X])$ contains infinitely many elements.

In this paper, we first characterize the subposets of $\mathrm{Idl}(R)$ which are isomorphic to $\mathrm{Idl}(S^{-1}R)$ for some multiplicative closed set S of a commutative ring R. Next we construct a partially ordered set $\mathbf{N}(P)$ from any given poset P, and show that there is a weak form of order embedding $G : \mathrm{Idl}(R[X]) \to \mathbf{N}(\mathrm{Idl}(R))$; this G, when restricted to any sub-chain of $\mathrm{Idl}(R[X])$, is an order-embedding. It is then proved that a poset P satisfies the ascending chain condition if and only if $\mathbf{N}(P)$ satisfies the ascending chain condition. As an application, we give an order version proof for Hilbert's Basis Theorem.

All the rings considered in this paper are commutative rings with identity. The identity of R will be denoted by 1_R.

2. Ideals of rings of fractions

Let R be a commutative ring and let $S^{-1}R$ be the ring of fractions determined by a multiplicatively closed subset S (also called an m-closed subset) of R. Let $h : R \to S^{-1}R$ be the natural ring homomorphism, where $h(x) = \dfrac{x}{1_R}$ for each $x \in R$. For any $\mathscr{I} \in \mathrm{Idl}(S^{-1}R)$, $h^{-1}(\mathscr{I}) \in \mathrm{Idl}(R)$. Let $\mathrm{Idl}_S(R) = \{h^{-1}(\mathscr{I}) : \mathscr{I} \in \mathrm{Idl}(S^{-1}R)\}$. Then $\mathrm{Idl}_S(R)$ is a subposet of $\mathrm{Idl}(R)$. The mapping $h^{-1} : \mathrm{Idl}(S^{-1}R) \to \mathrm{Idl}(R)$ is an order embedding [7], thus $\mathrm{Idl}_S(R)$ is order isomorphic to $\mathrm{Idl}(S^{-1}R)$. One natural converse problem is: given a collection $\mathscr{A}$ of ideals of R, under what conditions is there an m-closed set S of R, such that $\mathscr{A} = \mathrm{Idl}_S(R)$?

Proposition 2.1. *Let S be an m-closed subset of the commutative ring R. The following statements are equivalent for an ideal I of R:*

(1) $I \in \mathrm{Idl}_S(R)$.

(2) *For every $x \in R$ and $s \in S$,*

$$sx \in I \quad \textit{implies} \ x \in I.$$

Proof. Let $h : R \to S^{-1}R$ be the natural ring homomorphism. For any $J \in \mathrm{Idl}(R)$, let $J^e = \langle h(J) \rangle$ be the ideal of $S^{-1}R$ generated by $h(J)$, called the extension of J to $S^{-1}R$.

If $I \in \mathrm{Idl}_S(R)$, then there is an ideal $\mathscr{I}$ of $S^{-1}R$ such that $I = h^{-1}(\mathscr{I})$, which implies that $I = h^{-1}(I^e)$. Now if $x \in R$ and $s \in S$ such that $sx \in I$, then $h(x) = \dfrac{x}{1_R} = \dfrac{a}{s} \in I^e$, where $a = sx$. Hence $x \in h^{-1}(I^e) = I$. So (1) implies (2).

Now let I satisfy the condition in (2). We prove that $I = h^{-1}(I^e)$, showing that $I \in \mathrm{Idl}_S(R)$. Suppose $x \in h^{-1}(I^e)$, then $h(x) \in I^e$, i.e. $\dfrac{x}{1_R} = \dfrac{a}{s}$ for some $a \in I$ and $s \in S$. Thus there is an $u \in S$ such that $usx = ua$. Put $s' = us$. Then $s' \in S$ and $s'x = ua \in I$. Hence $x \in I$ by the assumption. Thus $h^{-1}(I^e) \subseteq I$. As the converse inclusion is always true, thus $I = h^{-1}(I^e)$.

It follows that (1) and (2) are equivalent. □

Corollary 2.2. *If* $\{I_k\}_{k \in M} \subseteq \mathrm{Idl}_S(R)$ *then*

$$\bigcap_{k \in M} I_k \in \mathrm{Idl}_S(R).$$

Corollary 2.3. *If* $\mathscr{B}$ *is a directed family of members of* $\mathrm{Idl}_S(R)$ (*for any* $J, K \in \mathscr{B}$ *there is* $H \in \mathscr{B}$ *such that* $J \subseteq H$, $K \subseteq H$), *then*

$$\bigcup_{I \in \mathscr{B}} I \in \mathrm{Idl}_S(R).$$

Corollary 2.4. *If* $I \in \mathrm{Idl}_S(R)$ *then* $\sqrt{I} \in \mathrm{Idl}_S(R)$.

Proof. Suppose $sx \in \sqrt{I}$ with $s \in S$ and $x \in R$. Then there is an $n \in \mathbf{Z}^+$ such that $(sx)^n \in I$, so $s^n x^n \in I$. As $s^n \in S$ and $I \in \mathrm{Idl}_S(R)$, by Proposition 2.1, we have $x^n \in I$, i.e. $x \in \sqrt{I}$. All these show, again by Proposition 2.1, that $\sqrt{I} \in \mathrm{Idl}_S(R)$. □

From the above discussion, it is seen that for any m-closed set S of R, the family $\mathscr{A} = \mathrm{Idl}_S(R)$ satisfies the following conditions:

(1) $R \in \mathscr{A}$;
(2) $\mathscr{A}$ is closed under arbitrary intersections;
(3) $\mathscr{A}$ is closed under unions of directed subfamilies;
(4) $\mathscr{A}$ is closed under taking radicals of ideals.

Now given a family $\mathscr{A}$ of ideals of R satisfying the conditions (1) to (4) above, must there be an m-closed set S of R such that $\mathscr{A} = \mathrm{Idl}_S(R)$? In the next section, we will answer this question.

3. More characterizations of families $\mathrm{Idl}_S(R)$

In this section, we prove a different characterization of $\mathrm{Idl}_S(R)$ for m-closed sets S of a commutative ring R. A more concrete characterization for the ring of integers (actually for every principal ideal domain) will be derived. A negative answer is given to the problem posed at the end of last section.

Let E be a subset of a commutative ring R. Define

$$\Psi(E) = \{I \in \mathrm{Idl}(R) : \text{for any } x \in R \text{ and } t \in E,\ tx \in I \text{ implies } x \in I\}.$$

Also for any collection $\mathscr{A}$ of ideals of R, define

$$\Phi(\mathscr{A}) = \{s \in R : \text{for any } I \in \mathscr{A} \text{ and } x \in R,\ sx \in I \text{ implies } x \in I\}.$$

Remark 3.1. (1) For every $\mathscr{A} \subseteq \mathrm{Idl}(R)$, $S = \Phi(\mathscr{A})$ is closed under multiplication and contains the identity 1_R.

(2) Let $s_1, s_2 \in R$ be any two elements in R such that $s_1 s_2 \in S = \Phi(\mathscr{A})$. For any $I \in \mathscr{A}$ and $x \in R$, if $s_2 x \in I$, then $s_1 s_2 x \in I$ because I is an ideal, which then implies $x \in I$, therefore $s_2 \in S$. Similarly, $s_1 \in S$. It follows that $S = \Phi(\mathscr{A})$ satisfies the following property:

For any $s, t \in R$, $st \in S$ if and only if $s \in S$ and $t \in S$.

In particular, if $s^n \in S$ for some positive integer n, then $s \in S$.

(3) For any $E \subseteq R$, $R \in \Psi(E)$ and $\Psi(E)$ is closed under arbitrary intersections and unions of directed subfamilies.

(4) For every $E \subseteq R$ and $\mathscr{A} \subseteq \mathrm{Idl}(R)$,

$$E \subseteq \Phi(\Psi(E)), \quad \mathscr{A} \subseteq \Psi(\Phi(\mathscr{A})).$$

If $E \subseteq D$, then $\Psi(D) \subseteq \Psi(E)$. Similarly, if $\mathscr{A} \subseteq \mathscr{B}$ then $\Phi(\mathscr{B}) \subseteq \Phi(\mathscr{A})$.

Thus Ψ and Φ define a Galois connection between the power sets of R and that of $\mathrm{Idl}(R)$. It follows from the general properties of a Galois connection, that for any subset E of R and any subset $\mathscr{A}$ of $\mathrm{Idl}(R)$, $\Psi(\Phi(\Psi(E))) = \Psi(E)$ and $\Phi(\Psi(\Phi(\mathscr{A}))) = \Phi(\mathscr{A})$.

Proposition 3.2. (1) *Let $\mathscr{A} \subseteq \mathrm{Idl}(R)$. Then $\mathscr{A} = \mathrm{Idl}_S(R)$ for some m-closed set S if and only if $\Psi(\Phi(\mathscr{A})) = \mathscr{A}$.*

(2) *Let S be an m-closed set of R. Then there is an $\mathscr{A} \subseteq \mathrm{Idl}(R)$ such that $S = \Phi(\mathscr{A})$ if and only if $S = \Phi(\Psi(S))$.*

Proof. As the proofs of the two parts are similar, we just prove (1).

If $\Psi(\Phi(\mathscr{A})) = \mathscr{A}$, then $S = \Phi(\mathscr{A})$ is an m-closed set of R and $\mathscr{A} = \mathrm{Idl}_S(R)$ by Proposition 2.1.

Now assume that $\mathscr{A} = \mathrm{Idl}_S(R)$ for some m-closed set S of R. Then $\mathscr{A} = \Psi(S)$, by Proposition 2.1. By Remark 3.1(4), $\Psi(\Phi(\mathscr{A})) = \Psi(\Phi(\Psi(S))) = \Psi(S) = \mathscr{A}$. □

An m-closed set S of R is called a *full m-closed set* if for any $x, y \in R$, $xy \in S$ if and only if $x \in S$ and $y \in S$. By Remark 3.1(2), $S = \Phi(\mathscr{A})$ is a full m-closed set of R for any $\mathscr{A} \subseteq \mathrm{Idl}(R)$.

For any m-closed set S of R, there is a smallest full m-closed set containing S, which is the intersection of all full m-closed sets containing S.

Let $\hat{S} = \{x \in R : \text{there exists } y \in R \text{ such that } xy \in S\}$. Then one can easily show that $\hat{S}$ is the smallest full m-closed set containing S.

The following proposition shows that every ring of fractions is defined by a full m-closed set.

Proposition 3.3. *For any m-closed set S of R, $S^{-1}R$ is isomorphic to $\hat{S}^{-1}R$.*

Proof. Define $f : S^{-1}R \to \hat{S}^{-1}R$ by $f\left(\dfrac{x}{s}\right) = \dfrac{x}{s}$. Clearly f is a ring homomorphism. If $f\left(\dfrac{x}{s}\right) = f\left(\dfrac{y}{t}\right)$, then there exists $r \in \hat{S}$ such that $rtx = rsy$ holds in R. Now there exists $u \in R$ such that $ur \in S$, so $urtx = ursy$, implying $\dfrac{x}{s} = \dfrac{y}{t}$ as $ur \in S$. Thus f is injective. For any $\dfrac{z}{v} \in \hat{S}^{-1}R$, there exists $u \in R$ such that $uv \in S$. Then $u \in \hat{S}$ and $\dfrac{z}{v} = \dfrac{uz}{uv} = f\left(\dfrac{uz}{uv}\right)$, showing that f is surjective. Therefore f is a ring isomorphism. $\square$

Using Proposition 2.1, we can derive a more concrete characterization of $\mathrm{Idl}_S(\mathbf{Z})$, where $\mathbf{Z}$ is the ring of integers.

Proposition 3.4. *Let $\mathscr{A} \subseteq \mathrm{Idl}(\mathbf{Z})$ be a non-empty set with $\mathbf{Z} \in \mathscr{A}$. Then the following statements are equivalent:*

(1) *$\mathscr{A} = \mathrm{Idl}_S(\mathbf{Z})$ for some m-closed set S of $\mathbf{Z}$.*

(2) *For any $I, J \in \mathrm{Idl}(\mathbf{Z})$, $IJ \in \mathscr{A}$ if and only if $I \in \mathscr{A}$ and $J \in \mathscr{A}$.*

Proof. (1) implies (2): Since (1) holds, $\mathscr{A} = \Psi(S)$ by Proposition 2.1. Note that for any $\langle a \rangle \in \mathrm{Idl}(\mathbf{Z})$, $\langle a \rangle \in \Psi(S)$ if and only if $\gcd(a, s) = 1$ for all $s \in S$.

For any $I = \langle a \rangle$ and $J = \langle b \rangle$, $IJ \in \mathscr{A} = \Psi(S)$ if and only if $\gcd(ab, s) = 1$ for all $s \in S$, if and only if $\gcd(a, s) = 1$ and $\gcd(b, s) = 1$ for all $s \in S$. This is equivalent to $I = \langle a \rangle \in \mathscr{A}$ and $J = \langle b \rangle \in \mathscr{A}$.

(2) implies (1): By Proposition 3.2, it is enough to show $\Psi(\Phi(\mathscr{A})) = \mathscr{A}$. If $\mathscr{A} = \{\mathbf{Z}\}$, then $\Psi(\Phi(\mathscr{A})) = \mathscr{A}$ is clearly true. If $\{0\} \in \mathscr{A}$, then by (2), we can deduce that $\mathscr{A} = \mathrm{Idl}(\mathbf{Z})$ and so $\Psi(\Phi(\mathscr{A})) = \mathscr{A}$ also holds. Now we consider the case where $\mathscr{A} \neq \{\mathbf{Z}\}$ and $\{0\} \notin \mathscr{A}$. To show $\Psi(\Phi(\mathscr{A})) = \mathscr{A}$, we only need to check $\Psi(\Phi(\mathscr{A})) \subseteq \mathscr{A}$ since $\mathscr{A} \subseteq \Psi(\Phi(\mathscr{A}))$ always holds. Since (2) holds, $\langle a \rangle \in \mathscr{A}$ if and only if $\langle a \rangle = \mathbf{Z}$ or $\langle p \rangle \in \mathscr{A}$ for any prime factor p of a. It follows that there is a set E of positive prime integers such that

$$\mathscr{A} = \{\langle p_1^{t_1} p_2^{t_2} \cdots p_n^{t_n} \rangle : p_i \in E,\ t_i \in \mathbf{Z} \text{ and } t_i \geq 0,\ n \in \mathbf{Z}^+\}.$$

(i) We claim that $a \in \Phi(\mathscr{A})$ iff $\langle a \rangle = \mathbf{Z}$ or a has no prime factor in E. As a matter of fact, for any $a \in \Phi(\mathscr{A})$, suppose $\langle a \rangle \neq \mathbf{Z}$ and a has a prime factor

p in E. So $a = a_1 p$ for some $a_1 \in \mathbf{Z}$. Then $\langle p \rangle \in \mathscr{A}$, and for any $s \in \mathbf{Z}$, $sa = sa_1 p \in \langle p \rangle$ which implies $s \in \langle p \rangle$ by the definition of $\Phi(\mathscr{A})$. Therefore, $\mathbf{Z} \subseteq \langle p \rangle$, which is not possible. Conversely, if $\langle a \rangle = \mathbf{Z}$ or a has no prime factor in E, then clearly $a \in \Phi(\mathscr{A})$.

(ii) Let $\langle m \rangle \in \Psi(\Phi(\mathscr{A}))$ and $\langle m \rangle \neq \mathbf{Z}$. We show that all the prime factors of m are in E, therefore $\langle m \rangle \in \mathscr{A}$. Suppose $m = pm_1$, with p a positive prime integer such that $p \notin E$. Then $p \in \Phi(\mathscr{A})$ by (i). Now $pm_1 \in \langle m \rangle$. So $m_1 \in \langle m \rangle$ since $\langle m \rangle \in \Psi(\Phi(\mathscr{A}))$. Hence there is a $t \in \mathbf{Z}$ such that $m_1 = tm = tpm_1$ which implies $p = 1$, a contradiction. All these show that $\langle m \rangle \in \mathscr{A}$. Hence $\Psi(\Phi(\mathscr{A})) \subseteq \mathscr{A}$. □

Remark 3.5. From the above proof, we can see that the above proposition also holds for every principal ideal domain (where the prime numbers are replaced by prime elements).

Example 3.6. Consider $\mathscr{A} = \{\langle 0 \rangle, \langle 4 \rangle, \langle 2 \rangle, \langle 1 \rangle\}$. Then $\mathscr{A}$ is a family of ideals of $\mathbf{Z}$ and satisfies the four properties listed at the end of Section 2. Now $\langle 4 \rangle \in \mathscr{A}$ but $\langle 4 \rangle^2 = \langle 16 \rangle \notin \mathscr{A}$. By Proposition 3.4, $\mathscr{A} \neq \mathrm{Idl}_S(\mathbf{Z})$ for any m-closed set S.

Proposition 3.7. *An m-closed subset S of $\mathbf{Z}$ is full if and only if $S = \Phi(\Psi(S))$.*

Proof. If $S = \Phi(\Psi(S))$, then by Remark 3.1 (1) and (2), S is full.

Now let S be full. Then $a \in S$ implies that each prime factor of a is in S. It follows that there is a set E of positive prime integers such that

$$S = \{\pm p_1^{t_1} p_2^{t_2} \cdots p_n^{t_n} : p_i \in E,\ t_i \in \mathbf{Z} \text{ and } t_i \geq 0,\ n \in \mathbf{Z}^+\}.$$

One can then deduce that $\langle m \rangle \in \Psi(S)$ iff $\langle m \rangle = \mathbf{Z}$ or m has no prime factor in E.

Let $x \in \Phi(\Psi(S))$. If x has a positive prime factor, say p, such that $p \notin E$, then $\langle p \rangle \in \Psi(S)$. For any $s \in E$, $xs \in \langle p \rangle$, thus $s \in \langle p \rangle$ since $x \in \Phi(\Psi(S))$, which implies $s = p$, a contradiction. All these show that $x \in S$. So $\Phi(\Psi(S)) \subseteq S$, implying $S = \Phi(\Psi(S))$ since $S \subseteq \Phi(\Psi(S))$ always holds. □

We still do not know the answer of the following problem.

Problem. Is it true that for any full m-closed set S of a commutative ring R, $S = \Phi(\Psi(S))$?

4. Posets satisfying the ascending chain condition

In the following, by a poset (an abbreviation for partially ordered set) we mean a non-empty set P equipped with a partial order "$\leq$" that is reflexive, transitive and antisymmetric.

For a subset B of P, the supremum (infimum) of B in P, denoted by *sup* B (*inf* B) or $\bigvee B$ ($\bigwedge B$) if it exists, is the least upper bound (greatest lower bound) of B. For more about posets and lattices, see [1] [3] [4].

An ascending chain in P is a sequence $\{a_i : i \in \mathbf{Z}^+\}$ of elements such that

$$a_1 \leq a_2 \leq \cdots \leq a_i \leq \cdots.$$

A descending chain in P is a sequence of elements $\{a_i : i \in \mathbf{Z}^+\}$ such that

$$a_1 \geq a_2 \geq \cdots \geq a_i \geq \cdots.$$

A poset P is said to satisfy the ascending chain condition (or ACC for short) if for any ascending chain $\{a_i : i \in \mathbf{Z}^+\}$ in P, there is i_0 such that $a_k = a_{i_0}$ for all $k \geq i_0$. The dual notion is the descending chain condition (DCC).

Example 4.1. (1) If R is a commutative ring and $\mathrm{Idl}(R)$ is the poset of all ideals ordered by inclusion, then R is Noetherian if and only if $\mathrm{Idl}(R)$ satisfies ACC. Dually, R is Artinian if and only if $\mathrm{Idl}(R)$ satisfies DCC.

(2) Let V be a vector space and $(\mathrm{Sub}(V), \subseteq)$ be the poset of all subspaces of V ordered by inclusion. Then V is finite dimensional if and only if $\mathrm{Sub}(V)$ satisfies ACC.

(3) Let G be a group and $(\mathrm{Sub}(G), \subseteq)$ be the poset of all subgroups of G ordered by inclusion. Then every subgroup of G is finitely generated if and only if $\mathrm{Sub}(G)$ satisfies ACC.

(4) Let M be an R-module with R a commutative ring and $(\mathrm{Sub}(M), \subseteq)$ be the poset of all submodules of M ordered by inclusion. Then M is Noetherian if and only if $\mathrm{Sub}(M)$ satisfies ACC.

An ideal I of a poset P is a nonempty subset of P satisfying the following:

(i) For any $x \in P$ and $y \in I$, if $x \leq y$, then $x \in I$.

(ii) For any two elements x, y in I, there is a $z \in I$ such that $x \leq z$ and $y \leq z$.

Let $\mathrm{Idl}(P)$ denote the poset of all ideals of P ordered by inclusion. The following results can be easily verified.

Proposition 4.2. (1) *If P satisfies ACC (DCC resp.) and P_1 is a subposet of P, then P_1 satisfies ACC (DCC resp.).*

(2) *If P and Q satisfy ACC (DCC resp.) then the cartesian product $P \times Q$ also satisfies ACC (DCC resp.).*

(3) *A poset P satisfies ACC if and only if* $(\mathrm{Idl}(P), \subseteq)$ *satisfies ACC.*

A monotone map $f : P \to Q$ between posets is a mapping such that $f(a) \leq f(b)$ holds in Q for any $a \leq b$ in P. A mapping $f : P \to Q$ is an order embedding if for any $a, b \in P$, $f(a) \leq f(b)$ if and only if $a \leq b$.

Example 4.3. (1) Let $f : R \to A$ be a surjective ring homomorphism. Then

$$f^{-1} : \mathrm{Idl}(A) \to \mathrm{Idl}(R)$$

is an order embedding, where f^{-1} sends $I \in \mathrm{Idl}(A)$ to $f^{-1}(I)$.

(2) Let S be a multiplicatively closed subset of a commutative ring R and $S^{-1}R$ be the ring of fractions of R. Let $h : R \to S^{-1}R$ be the natural ring homomorphism that sends $r \in R$ to $\dfrac{r}{1_R}$. Then $h^{-1} : \mathrm{Idl}(S^{-1}R) \to \mathrm{Idl}(R)$ is an order embedding (see [7], Lemma 5.24).

Definition 4.4. A monotone map $f : P \to Q$ between two posets is called a *pre-order embedding* if for any $a, b \in P$, $a \leq b$ and $f(a) = f(b)$ imply $a = b$.

Example 4.5. (1) Every order embedding is a pre-order embedding.

(2) Let $\mathscr{I}(\mathbf{R})$ be the set of all open intervals of $\mathbf{R}$ ordered by inclusion. Define $f : \mathscr{I}(\mathbf{R}) \to \mathbf{R}$ by $f((a,b)) = b - a$. Then f is a pre-order embedding that is not injective.

(3) Let $L = \{(x,y) : x, y \in \mathbf{R}, x \geq 0, y \geq 0\}$ with the pointwise order. Define $g : L \to \mathbf{R}$ by $g(a,b) = \sqrt{a^2 + b^2}$. Then g is a pre-order embedding.

(4) Let $(\mathrm{Sub}_{\mathrm{fin}}(V), \subseteq)$ be the poset of all the subspaces of finite dimensions of a vector space V. Define $g : \mathrm{Sub}_{\mathrm{fin}}(V) \to E$ by $g(W) = \dim(W)$ for each $W \in \mathrm{Sub}_{\mathrm{fin}}(V)$, where $E = \{0, 1, 2, \ldots\}$ is equipped with the ordinary order of numbers. Then g is a pre-order embedding.

The following lemma follows directly from the respective definitions.

Lemma 4.6. *If $g : P \to Q$ is a pre-order embedding and Q satisfies ACC (DCC resp.), then P also satisfies ACC (DCC resp.).*

5. The monotone extension $\mathbf{N}(P)$ of P

Let $\mathbf{N}$ be the set of non-negative integers with the ordinary order of numbers.

In this section, we construct a poset $\mathbf{N}(P)$ from any given poset P, called the monotone extension of P, and show that P satisfies ACC if and only if $\mathbf{N}(P)$ satisfies ACC.

Definition 5.1. For any poset P, let $\mathbf{N}(P)$ be the poset of all monotone mappings $f : \mathbf{N} \to P$ with the order $\leq$ defined by $f \leq g$ iff $f(n) \leq g(n)$ for every $n \in \mathbf{N}$.

In the following, we shall use $\mathbf{N}(P)$ to denote the poset $(\mathbf{N}(P), \leq)$.

Remark 5.2. It can be verified that if P is a meet (join) semilattice, then $\mathbf{N}(P)$ is also a meet (join) semilattice, where $f \wedge g$ and $f \vee g$ are defined by

$(f \wedge g)(n) = f(n) \wedge g(n)$ and $(f \vee g)(n) = f(n) \vee g(n)$ for every $n \in \mathbf{N}$ respectively. Furthermore, if P is a distributive (modular) lattice, then $\mathbf{N}(P)$ is also a distributive (modular) lattice.

For any poset P, define $F_P : P \to \mathbf{N}(P)$ by

$$F_P(a) : \mathbf{N} \to P, \quad \text{for all } a \in P,$$

where $(F_P(a))(n) \equiv a$ is the constant mapping with value a.

Define $G_P : \mathbf{N}(P) \to P$ by $G_P(f) = f(0)$ for each $f \in \mathbf{N}(P)$.

Then both F_P and G_P are monotone, and for any $a \in P$ and $f \in \mathbf{N}(P)$,

$$G_P(F_P(a)) = a \quad \text{and} \quad F_P(G_P(f)) \leq f.$$

Obviously G_P is surjective and F_P is an order embedding.

THEOREM 5.3. *For any poset P, P satisfies ACC if and only if $\mathbf{N}(P)$ satisfies ACC.*

Proof. Since F_P is an order-embedding, the sufficiency follows from Lemma 4.6.

Now suppose that P satisfies ACC. Let

$$f_1 \leq f_2 \leq \cdots \leq f_n \leq \cdots$$

be an ascending chain in $\mathbf{N}(P)$. Note that if $n \leq m$ then $f_n(n) \leq f_n(m) \leq f_m(m)$, so the following is an ascending chain in P:

$$f_1(1) \leq f_2(2) \leq \cdots \leq f_n(n) \leq \cdots.$$

By the assumption on P, there is an m such that $f_k(k) = f_m(m)$ whenever $k \geq m$. Now for each $i \leq m$,

$$f_1(i) \leq f_2(i) \leq \cdots \leq f_k(i) \leq \cdots$$

is an ascending chain in P, so there is a t_i such that $f_k(i) = f_{t_i}(i)$ for all $k \geq t_i$. Let $t = \max\{t_1, t_2, \ldots, t_m, m\}$.

We claim that $f_k = f_t$ for all $k \geq t$.

As a matter of fact, for any $l \in \mathbf{N}$,

(i) if $l \leq m$ then, as $t_l \leq t \leq k$, $f_k(l) = f_{t_l}(l) = f_t(l)$;

(ii) if $m < l$, then $f_m(m) \leq f_m(l) \leq f_t(l) \leq f_k(l) \leq f_s(s) = f_m(m)$, where $s = \max\{l, k\}$. The last equation holds because $m \leq s$. This again shows that $f_k(l) = f_m(m) = f_t(l)$.

All these show that $f_k = f_t$.

Thus $\mathbf{N}(P)$ satisfies ACC. □

LEMMA 5.4. *Let R be a commutative ring. Then there is a pre-order embedding*

$$G : \mathrm{Idl}(R[X]) \to \mathbf{N}(\mathrm{Idl}(R)).$$

Proof. Define the mapping $G : \mathrm{Idl}(R[X]) \to \mathbf{N}(\mathrm{Idl}(R))$ as follows: for any ideal $\mathscr{I}$ of $R[X]$ and $i \in \mathbf{N}$,

$$G(\mathscr{I})(i) = \{a \in R : \text{there exists } a_0 + a_1 X + \cdots + a_{i-1} X^{i-1} + a X^i \in \mathscr{I}\}.$$

Then Lemma 8.6 of [7] says exactly that G is a pre-order embedding. $\square$

COROLLARY 5.5 (Hilbert's Basis Theorem). *Let R be a commutative Noetherian ring and X be an indeterminant. Then the ring $R[X]$ of polynomials is again a Noetherian ring.*

REFERENCES

[1] B. A. DAVEY AND H. A. PRIESTLEY, Introduction to lattices and order, Cambridge University Press, Cambridge, United Kingdom, 2002.
[2] L. FUCHS, Über die Ideale arithmetischer Ringe, Comment. Math. Helv. **23** (1949), 334–341.
[3] G. GIERZ ET AL., Continuous lattices and domains, Encyclopedia of mathematics and its applications **93**, Cambridge University Press, Cambridge, United Kingdom, 2003.
[4] G. GRÄTZER, General lattice theory, Birkhäuser Verlag, CH-4010 Basel, Switzerland, 2003.
[5] C. U. JENSEN, Arithmetical rings, Acta. Math. Acad. Sci. Hung. **17** (1966), 115–123.
[6] G. C. ROTA, The many lives of lattice theory, preprint.
[7] R. Y. SHARP, Steps in commutative algebra, London Math. Society student texts **19**, Cambridge University Press, Cambridge, United Kingdom, 1990.

Yuan Ting Nai
MATHEMATICS AND MATHEMATICS EDUCATION
NATIONAL INSTITUTE OF EDUCATION
NANYANG TECHNOLOGICAL UNIVERSITY
1 NANYANG WALK, 637616
SINGAPORE
E-mail: 9naiyt@gmail.com

Dongsheng Zhao
MATHEMATICS AND MATHEMATICS EDUCATION
NATIONAL INSTITUTE OF EDUCATION
NANYANG TECHNOLOGICAL UNIVERSITY
1 NANYANG WALK, 637616
SINGAPORE
E-mail: dongsheng.zhao@nie.edu.sg