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PSEUDO ASYMPTOTICALLY PERIODIC SOLUTIONS OF TWO-TERM TIME FRACTIONAL DIFFERENTIAL EQUATIONS WITH DELAY

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Abstract

In this paper, the existence and uniqueness of pseudo $\mathcal{S}$-asymptotically ω-periodic mild solutions of class r for the nonlinear two-term time fractional differential equations with delay are investigated. The working tools are based on the generalization of the semigroup theory and fixed point theory. Finally, we present an application to a fractional partial differential equation with delay.

1. Introduction

The study of the existence of asymptotically ω-periodic solutions is one of the most interesting and important topics in the qualitative theory of differential equations. From an applied perspective, asymptotically ω-periodic systems describe world more realistically and accurately than periodic ones, one can see [7, 12, 13, 22, 23] for more details. The notion of $\mathcal{S}$-asymptotic ω-periodicity, introduced by Henríquez et al. in [10, 11], is related to and more general than that of asymptotic ω-periodicity. Since then, it has attracted the attention of many researchers and the interest in this topic still increases [3, 4, 5, 8, 9, 16, 18]. Recently, in [19], the concept of pseudo $\mathcal{S}$-asymptotic ω-periodicity and pseudo $\mathcal{S}$-asymptotic ω-periodicity of class r, which generalizes the notion of $\mathcal{S}$-asymptotic ω-periodicity, was introduced and the applications to semilinear first-order differential equations with delay in Banach spaces were studied.

In this paper, we study the existence, uniqueness of pseudo $\mathcal{S}$-asymptotically ω-periodic solutions of class r for the nonlinear two-term time fractional differential equations with delay

$$(1.1)\qquad \begin{aligned} &D_t^{\alpha+1}u(t) + \mu D_t^{\beta}u(t) - Au(t) = f(t, u_t), \quad t \in \mathbf{R}^+,\ 0 < \alpha \le \beta \le 1,\ \mu > 0,\\ &u_0 = \varphi_1 \in \mathscr{C}, \quad u_0' = \varphi_2 \in \mathscr{C}, \end{aligned}$$

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where A is an $\tilde{\omega}$-sectorial operator of angle $\beta\pi/2$ with $\tilde{\omega} < 0$, $u_t(\theta) := u(t+\theta)$ for $\theta \in [-r, 0]$, $r \geq 0$ is a fixed constant, $\mathscr{C} := C([-r, 0], \mathbf{R})$ denotes the space of continuous function from $[-r, 0]$ to $\mathbf{R}$ with the supremum norm. The fractional derivative is understood in the Caputo sense.

Note that fractional differential equations arise in many areas of applied science, such as physics, engineering, biology, control theory, among other areas. For this reason, those equations have been of a great interest during the last few decades. Our motivation to study (1.1) come from recent investigation on the subject. For (1.1) without delay, the existence, uniqueness of mild solutions in the special case $\alpha = \beta$ were studied in [15]; the nonlinear two-term time fractional diffusion-wave equation with $A = \frac{d^2}{dx^2}$, $0 < \alpha < \beta - 1$ was studied in [20]; asymptotic behavior of mild solutions was studied in [14]; the $\mathcal{S}$-asymptotic ω-periodicity of mild solutions was studied in [6]; the almost periodicity, compact almost automorphy, almost automorphy and pseudo asymptotic behavior of mild solutions were studied in [1], but in the case of delay, i.e., (1.1), to the best of our knowledge, there is no work reported in literature. Moreover, the pseudo $\mathcal{S}$-asymptotic ω-periodicity of (1.1) is quite new and an untreated topic. This is one of the key motivations of this study.

The paper is organized as follows. In Section 2, some notations and preliminary results are presented. In Section 3, we explore some properties of pseudo $\mathcal{S}$-asymptotically ω-periodic function of class r, and establish the composition theorems. Section 4 is devoted to the existence, uniqueness of pseudo $\mathcal{S}$-asymptotically ω-periodic mild solution of class r for (1.1). In Section 5, we present an application to a fractional partial differential equation with delay.

2. Preliminaries and basic results

Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|_Y)$ be two complex Banach spaces and $\mathbf{N}$, $\mathbf{R}$, $\mathbf{R}^+$, and $\mathbf{C}$ stand for the set of natural numbers, real numbers, nonnegative real numbers, and complex numbers, respectively. $\mathscr{R}(u)$ denotes the range of $u(\cdot)$. In order to facilitate the discussion below, we further introduce the following notations:

- $BC(\mathbf{R}^+, X)$ (resp. $BC(\mathbf{R}^+ \times Y, X)$): the Banach space of bounded continuous functions from $\mathbf{R}^+$ to X (resp. from $\mathbf{R}^+ \times Y$ to X) with the supremum norm.
- $C(\mathbf{R}^+, X)$ (resp. $C(\mathbf{R}^+ \times Y, X)$): the set of continuous functions from $\mathbf{R}^+$ to X (resp. from $\mathbf{R}^+ \times Y$ to X).
- $L(X, Y)$: the Banach space of bounded linear operators from X to Y endowed with the operator topology. In particular, we write $L(X)$ when $X = Y$.
- $L^p(\mathbf{R}^+, X)$: the space of all classes of equivalence (with respect to the equality almost everywhere on $\mathbf{R}^+$) of measurable functions $f : \mathbf{R}^+ \to X$ such that $\|f\| \in L^p(\mathbf{R}^+, \mathbf{R}^+)$.

- $L^p_{loc}(\mathbf{R}^+, X)$: stand for the space of all classes of equivalence of measurable functions $f : \mathbf{R}^+ \to X$ such that the restriction of f to every bounded subinterval of $\mathbf{R}^+$ is in $L^p(\mathbf{R}^+, X)$.

Let $p \in [1, \infty)$. The space $BS^p(\mathbf{R}^+, X)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbf{R}^+ \to X$ such that $f^b \in L^\infty(\mathbf{R}^+, L^p([0,1]; X))$, where f^b is the Bochner transform of f defined by $f^b(t,s) := f(t+s)$, $t \in \mathbf{R}^+$, $s \in [0,1]$. $BS^p(\mathbf{R}^+, X)$ is a Banach space with the norm [17]

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbf{R}^+, L^p)} = \sup_{t \in \mathbf{R}^+} \left(\int_t^{t+1} \|f(\tau)\|^p \, \mathrm{d}\tau \right)^{1/p}.$$

It is obvious that $L^p(\mathbf{R}^+, X) \subset BS^p(\mathbf{R}^+, X) \subset L^p_{loc}(\mathbf{R}^+, X)$ and $BS^p(\mathbf{R}^+, X) \subset BS^q(\mathbf{R}^+, X)$ for $p \geq q \geq 1$.

For each $p \in [1, \infty)$, we denote by $\mathscr{UC}^p(\mathbf{R}^+ \times Y, X)$ the set of all continuous functions $f : \mathbf{R}^+ \times Y \to X$ with the property that there exists a function $L_f \in BS^p(\mathbf{R}^+, \mathbf{R}^+)$ satisfying $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\sup_{\tau \in [t-r, t]} \|f(\tau, u(\tau)) - f(\tau, v(\tau))\| \leq L_f(t)\varepsilon, \quad \text{for all } t \in \mathbf{R}^+ \text{ and}$$
$$u, v \in Y \text{ with } \|u - v\|_Y < \delta.$$

It is easy to see that $\mathscr{UC}^p(\mathbf{R}^+ \times Y, X) \subset \mathscr{UC}^q(\mathbf{R}^+ \times Y, X)$ for $1 \leq q \leq p < \infty$.

Let $\alpha > 0$, $m = \lceil \alpha \rceil$ denote the integer part of α and $u : \mathbf{R}^+ \to X$. The Caputo fraction derivative of $u \in \mathbf{R}^+$ of order α is defined by

$$D_t^\alpha u(t) = \int_0^t g_{m-\alpha}(t-s) u^{(m)}(s) \, ds, \quad t > 0,$$

where $g_\beta(t) := \dfrac{t^{\beta-1}}{\Gamma(\beta)}$, $t > 0$, $\beta > 0$, Γ is the Gamma function, and in case $\beta = 0$, set $g_0(t) := \delta_0$, the Dirac measure concentrated at the origin.

In order to give an operator theoretical approach to (1.1), we have the following definition.

Definition 2.1 [14]. A closed and densely defined linear operator A is said to $\tilde{\omega}$-sectorial of angle θ if there exists $\theta \in [0, \pi/2)$ and $\tilde{\omega} \in \mathbf{R}$ such that its resolvent exists in the sector

$$\tilde{\omega} + S_\theta := \{\tilde{\omega} + \lambda : \lambda \in \mathbf{C}, |arg(\lambda)| < \pi/2 + \theta\} \backslash \{\tilde{\omega}\} \tag{2.1}$$
$$\|(\lambda I - A)^{-1}\| \leq \frac{M}{|\lambda - \tilde{\omega}|}, \quad \lambda \in \tilde{\omega} + S_\theta.$$

In the case $\tilde{\omega} = 0$, we merely say that A is sectorial of angle θ.

We should mention that in the general theory of sectorial operator, it is not require that (2.1) holds in a sector of angle $\pi/2$. Our restriction corresponds to the class of operators used in this paper.

Definition 2.2 [14]. Let $\mu > 0$, $0 \le \alpha, \beta \le 1$ be given. Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X. We call A the generator of an $(\alpha, \beta)_\mu$-regularized family if there exist $\tilde{\omega} \ge 0$ and a strongly continuous function $S_{\alpha,\beta} : \mathbf{R}^+ \to L(X)$ such that $\{\lambda^{\alpha+1} + \mu\lambda^\beta : Re\,\lambda > \tilde{\omega}\} \subset \rho(A)$ and

$$\lambda^\alpha(\lambda^{\alpha+1} + \mu\lambda^\beta - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_{\alpha,\beta}(t)x\,dt, \quad Re\,\lambda > \tilde{\omega},\ x \in X.$$

Because of the uniqueness theorem for the Laplace transform, if $\mu = 0$, $\alpha = 0$, this corresponds to the case of a C_0-semigroup whereas the case $\mu = 0$, $\alpha = 1$ corresponds to the concept of cosine family. For more details on the Laplace transform approach to semigroups and cosine functions, we refer to [2].

Sufficient conditions to existence and the integrability for the generators of an $(\alpha, \beta)_\mu$-regularized family are given in the following results.

Theorem 2.1 [14]. *Let $0 < \alpha \le \beta \le 1$, $\mu > 0$ and $\tilde{\omega} < 0$. Assume that A is an $\tilde{\omega}$-sectorial of angle $\beta\pi/2$, then A generates an $(\alpha, \beta)_\mu$-regularized family $S_{\alpha,\beta}(t)$ satisfying*

$$\|S_{\alpha,\beta}(t)\| \le \frac{C}{1 + |\tilde{\omega}|(t^{\alpha+1} + \mu t^\beta)}, \quad t \ge 0, \tag{2.2}$$

for some constant $C > 0$ depending only on α, β.

Note that

$$\int_0^\infty \frac{1}{1 + |\tilde{\omega}|t^{\alpha+1}}\,dt = \frac{|\tilde{\omega}|^{-1/(\alpha+1)}\pi}{(\alpha+1)\sin(\pi/(\alpha+1))}$$

for $0 < \alpha < 1$, therefore $S_{\alpha,\beta}(t)$ is integrable on $(0, \infty)$.

Theorem 2.2 (Hardy-Littlewood [2]). *Let $f \in L^1_{loc}(\mathbf{R}^+, X)$ and $F(t) := \int_0^t f(s)\,ds$. Assume that $M := \sup_{t \ge \tau} t\|f(t)\| < \infty$ for some $\tau \ge 0$ and $F_\infty \in X$. If $\lim_{\lambda \to 0} \hat{f}(\lambda) = F_\infty$, then $\lim_{t \to \infty} F(t) = F_\infty$, where $\hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)\,dt$.*

3. Pseudo $\mathcal{S}$-asymptotic ω-periodicity of class r

For $\omega > 0$, define

$$C_0(\mathbf{R}^+, X) = \left\{ f \in BC(\mathbf{R}^+, X) : \lim_{t \to \infty} \|f(t)\| = 0 \right\}.$$

$$C_\omega(\mathbf{R}^+, X) = \{ f \in BC(\mathbf{R}^+, X) : f \text{ is } \omega\text{-periodic} \}.$$

Definition 3.1 [21]. A function $f \in BC(\mathbf{R}^+, X)$ is called asymptotically ω-periodic if there exist $g \in C_\omega(\mathbf{R}^+, X)$, $\varphi \in C_0(\mathbf{R}^+, X)$ such that $f = g + \varphi$. The collection of those functions is denoted by $AP_\omega(\mathbf{R}^+, X)$.

Definition 3.2 [10]. A function $f \in BC(\mathbf{R}^+, X)$ is said to be $\mathcal{S}$-asymptotically periodic if there exists $\omega > 0$ such that $\lim_{t\to\infty}(f(t+\omega) - f(t)) = 0$. In this case, we say that f is $\mathcal{S}$-asymptotically ω-periodic. The collection of those functions is denoted by $\mathcal{S}AP_\omega(\mathbf{R}^+, X)$. It is clear that $AP_\omega(\mathbf{R}^+, X) \subset \mathcal{S}AP_\omega(\mathbf{R}^+, X)$.

Definition 3.3 [19]. A function $f \in BC(\mathbf{R}^+, X)$ is called pseudo $\mathcal{S}$-asymptotically periodic if there exists $\omega > 0$ such that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \|f(t+\omega) - f(t)\|\, dt = 0.$$

In this case, we say that f is pseudo $\mathcal{S}$-asymptotically ω-periodic. Denote by $P\mathcal{S}AP_\omega(\mathbf{R}^+, X)$ the set of such functions, $P\mathcal{S}AP_\omega(\mathbf{R}^+, X)$ is a Banach space when endowed with the supremum norm and $\mathcal{S}AP_\omega(\mathbf{R}^+, X) \subset P\mathcal{S}AP_\omega(\mathbf{R}^+, X)$.

Definition 3.4 [19]. A function $f \in BC(\mathbf{R}^+, X)$ is called pseudo $\mathcal{S}$-asymptotically ω-periodic of class r $(r \geq 0)$ if there exists $\omega > 0$ such that

$$\lim_{T\to\infty} \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|f(\tau+\omega) - f(\tau)\|\, dt = 0.$$

Denote by $P\mathcal{S}AP_{\omega,r}(\mathbf{R}^+, X)$ the set of such functions, $P\mathcal{S}AP_{\omega,r}(\mathbf{R}^+, X)$ is a Banach space when endowed with the supremum norm.

Definition 3.5 [19]. A function $F \in BC(\mathbf{R}^+ \times Y, X)$ is called uniformly pseudo $\mathcal{S}$-asymptotically ω-periodic of class r $(r \geq 0)$ if there exists $\omega > 0$ such that

$$\lim_{T\to\infty} \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \sup_{\|z\|_Y \leq R} \|F(\tau+\omega, z) - F(\tau, z)\|\, dt = 0,$$

for all $R > 0$. Denote by $P\mathcal{S}AP_{\omega,r}(\mathbf{R}^+ \times Y, X)$ the set of such functions.

Next, we show some properties of the space $P\mathcal{S}AP_{\omega,r}(\mathbf{R}^+, X)$.

Lemma 3.1. *Let* $r \geq 0$, *then*
(i) $P\mathcal{S}AP_{\omega,r}(\mathbf{R}^+, X) \subseteq P\mathcal{S}AP_\omega(\mathbf{R}^+, X)$.
(ii) $P\mathcal{S}AP_{\omega,r}(\mathbf{R}^+, X)$ *is a closed subspace of* $BC(\mathbf{R}^+, X)$.
(iii) $P\mathcal{S}AP_{\omega,r}(\mathbf{R}^+, X)$ *is a Banach space under the supermum norm.*

Proof. From the estimate

$$\begin{aligned}
&\frac{1}{T}\int_0^T \|f(t+\omega)-f(t)\|\,dt \\
&\quad = \frac{1}{T}\int_0^r \|f(t+\omega)-f(t)\|\,dt + \frac{1}{T}\int_r^T \|f(t+\omega)-f(t)\|\,dt \\
&\quad \le \frac{1}{T}\int_0^r \|f(t+\omega)-f(t)\|\,dt + \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|f(\tau+\omega)-f(\tau)\|\,dt,
\end{aligned}$$

it is easy to see that (i) holds.

Let $f_n \in PSAP_{\omega,r}(\mathbf{R}^+, X)$ and $f_n \to f$ in $BC(\mathbf{R}^+, X)$, then

$$\begin{aligned}
&\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|f(\tau+\omega)-f(\tau)\|\,dt \\
&\quad = \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|f_n(\tau+\omega)-f_n(\tau)\|\,dt \\
&\qquad + \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|f(\tau+\omega)-f(\tau)-(f_n(\tau+\omega)-f_n(\tau))\|\,dt \\
&\quad \le \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|f_n(\tau+\omega)-f_n(\tau)\|\,dt \\
&\qquad + \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|f(\tau+\omega)-f_n(\tau+\omega)\|\,dt \\
&\qquad + \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|f(\tau)-f_n(\tau)\|\,dt,
\end{aligned}$$

which yields that $f \in PSAP_{\omega,r}(\mathbf{R}^+, X)$, then (ii) holds, therefore (iii) holds. □

LEMMA 3.2. *Let* $r_1 > 0$, $r_2 > 0$, *then*

$$PSAP_{\omega,r_1}(\mathbf{R}^+, X) = PSAP_{\omega,r_2}(\mathbf{R}^+, X).$$

Proof. Let $r > 0$, first we show that

$$PSAP_{\omega,r}(\mathbf{R}^+, X) \subset PSAP_{\omega,2r}(\mathbf{R}^+, X). \tag{3.1}$$

For $f \in PSAP_{\omega,r}(\mathbf{R}^+, X)$, one has

$$\begin{aligned}
&\frac{1}{T}\int_{2r}^T \sup_{\tau\in[t-2r,t]} \|f(\tau+\omega)-f(\tau)\|\,dt \\
&\quad \le \frac{1}{T}\int_{2r}^T \sup_{\tau\in[t-r,t]} \|f(\tau+\omega)-f(\tau)\|\,dt + \frac{1}{T}\int_{2r}^T \sup_{\tau\in[t-2r,t-r]} \|f(\tau+\omega)-f(\tau)\|\,dt
\end{aligned}$$

$$\le \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|f(\tau+\omega)-f(\tau)\|\,dt + \frac{1}{T}\int_r^{T-r} \sup_{\tau\in[u-r,u]} \|f(\tau+\omega)-f(\tau)\|\,du$$

$$\le \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|f(\tau+\omega)-f(\tau)\|\,dt + \frac{1}{T}\int_r^{T} \sup_{\tau\in[u-r,u]} \|f(\tau+\omega)-f(\tau)\|\,du,$$

so

$$\lim_{T\to\infty}\frac{1}{T}\int_{2r}^T \sup_{\tau\in[t-2r,t]} \|f(\tau+\omega)-f(\tau)\|\,dt = 0,$$

thus $f \in PSAP_{\omega,2r}(\mathbf{R}^+,X)$. Hence (3.1) holds.

Now, let $r_1 > r_2 > 0$. If $f \in PSAP_{\omega,r_1}(\mathbf{R}^+,X)$, then

$$\lim_{T\to\infty}\frac{1}{T}\int_{r_1}^T \sup_{\tau\in[t-r_1,t]} \|f(\tau+\omega)-f(\tau)\|\,dt = 0.$$

From

$$\frac{1}{T}\int_{r_2}^T \sup_{\tau\in[t-r_2,t]} \|f(\tau+\omega)-f(\tau)\|\,dt$$

$$= \frac{1}{T}\int_{r_2}^{r_1} \sup_{\tau\in[t-r_2,t]} \|f(\tau+\omega)-f(\tau)\|\,dt + \frac{1}{T}\int_{r_1}^{T} \sup_{\tau\in[t-r_2,t]} \|f(\tau+\omega)-f(\tau)\|\,dt$$

$$\le \frac{1}{T}\int_{r_2}^{r_1} \sup_{\tau\in[t-r_2,t]} \|f(\tau+\omega)-f(\tau)\|\,dt + \frac{1}{T}\int_{r_1}^{T} \sup_{\tau\in[t-r_1,t]} \|f(\tau+\omega)-f(\tau)\|\,dt,$$

then

$$\lim_{T\to\infty}\frac{1}{T}\int_{r_2}^T \sup_{\tau\in[t-r_2,t]} \|f(\tau+\omega)-f(\tau)\|\,dt = 0,$$

so $f \in PSAP_{\omega,r_2}(\mathbf{R}^+,X)$, i.e., one has

$$PSAP_{\omega,r_1}(\mathbf{R}^+,X) \subset PSAP_{\omega,r_2}(\mathbf{R}^+,X). \tag{3.2}$$

On the other hand, since $r_1 > r_2$, there exists $k \in \mathbf{N}$ such that $2^k r_2 > r_1$. By (3.1), (3.2), one has

$$PSAP_{\omega,r_2}(\mathbf{R}^+,X) \subset PSAP_{\omega,2^k r_2}(\mathbf{R}^+,X) \subset PSAP_{\omega,r_1}(\mathbf{R}^+,X).$$

Thus, $PSAP_{\omega,r_1}(\mathbf{R}^+,X) = PSAP_{\omega,r_2}(\mathbf{R}^+,X)$. The proof is complete. □

Remark 3.1. It is interesting that $PSAP_{\omega,r}(\mathbf{R}^+,X) = PSAP_{\omega,1}(\mathbf{R}^+,X)$ for all $r > 0$ by Lemma 3.2, but for $r = 0$, it is not necessarily holds, i.e., $PSAP_{\omega,0}(\mathbf{R}^+,X) = PSAP_{\omega,1}(\mathbf{R}^+,X)$ is not true.

LEMMA 3.3. *Let* $f \in PSAP_{\omega,r}(\mathbf{R}^+, X)$, *then* $f(\cdot + \eta) \in PSAP_{\omega,r}(\mathbf{R}^+, X)$ *for all* $\eta \geq 0$.

Proof. For $\eta \geq 0$,

$$\begin{aligned}
&\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|f(\tau+\eta+\omega) - f(\tau+\eta)\|\,dt \\
&\quad = \frac{1}{T}\int_r^T \sup_{\tau\in[t+\eta-r,t+\eta]} \|f(\tau+\omega) - f(\tau)\|\,dt \\
&\quad = \frac{1}{T}\int_{r+\eta}^{T+\eta} \sup_{\tau\in[t-r,t]} \|f(\tau+\omega) - f(\tau)\|\,dt \\
&\quad = \frac{T+\eta}{T}\cdot\frac{1}{T+\eta}\int_r^{T+\eta} \sup_{\tau\in[t-r,t]} \|f(\tau+\omega) - f(\tau)\|\,dt \\
&\qquad - \frac{1}{T}\int_r^{r+\eta} \sup_{\tau\in[t-r,t]} \|f(\tau+\omega) - f(\tau)\|\,dt,
\end{aligned}$$

then

$$\lim_{T\to\infty}\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|f(\tau+\eta+\omega) - f(\tau+\eta)\|\,dt = 0,$$

which implies that $f(\cdot + \eta) \in PSAP_{\omega,r}(\mathbf{R}^+, X)$ for all $\eta \geq 0$. □

We will establish some composition theorems for pseudo $\mathcal{S}$-asymptotically ω-periodic function of class r.

THEOREM 3.1. *Let* $f \in PSAP_{\omega,r}(\mathbf{R}^+ \times Y, X)$ *and there exists a constant* $L_f > 0$ *such that*

$$\|f(t,u) - f(t,v)\| \leq L_f\|u - v\|_Y, \quad t \in \mathbf{R}^+,\ u, v \in Y,$$

then $h(\cdot) = f(\cdot, u(\cdot)) \in PSAP_{\omega,r}(\mathbf{R}^+, X)$ *if* $u(\cdot) \in PSAP_{\omega,r}(\mathbf{R}^+, Y)$.

Proof. It is clear that $f(\cdot, u(\cdot)) \in BC(\mathbf{R}^+, X)$. For $\varepsilon > 0$, there exists $L_\varepsilon > 0$ such that

$$\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]}\ \sup_{\|z\|\leq R} \|f(\tau+\omega, z) - f(\tau, z)\|\,dt \leq \varepsilon,$$

$$\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|u(\tau+\omega) - u(\tau)\|_Y\,dt \leq \varepsilon/L_f,$$

for every $T \geq L_\varepsilon$, $R > 0$. For $T \geq L_\varepsilon$,

$$\begin{aligned}
&\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|f(\tau+\omega,u(\tau+\omega)) - f(\tau,u(\tau))\|\,dt\\
&\quad\le \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|f(\tau+\omega,u(\tau+\omega)) - f(\tau,u(\tau+\omega))\|\,dt\\
&\qquad+\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|f(\tau,u(\tau+\omega)) - f(\tau,u(\tau))\|\,dt\\
&\quad\le \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]}\ \sup_{\|z\|\le R} \|f(\tau+\omega,z) - f(\tau,z)\|\,dt\\
&\qquad+\frac{L_f}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|u(\tau+\omega) - u(\tau)\|_Y\,dt\\
&\quad\le 2\varepsilon,
\end{aligned}$$

implies that $h(\cdot)\in PSAP_{\omega,r}(\mathbf{R}^+,X)$. □

In the following, we establish another composition theorem which weakens the assumptions on f.

LEMMA 3.4. *Let $f\in BC(\mathbf{R}^+,X)$, then $f\in PSAP_{\omega,r}(\mathbf{R}^+,X)$ if and only if for any $\varepsilon>0$,*

$$\lim_{T\to\infty}\frac{1}{T}\,\mathrm{mes}(M_{T,\varepsilon}(f)) = 0, \tag{3.3}$$

where mes$(\cdot)$ *denotes the Lebesgue measure and*

$$M_{T,\varepsilon}(f) = \left\{ t\in[r,T] : \sup_{\tau\in[t-r,t]} \|f(\tau+\omega) - f(\tau)\| \ge \varepsilon \right\}.$$

Proof. Sufficiency: From the statement of the Lemma it is clear that $\|f\|<\infty$ and for any $\varepsilon>0$, there exists $T_0>0$ such that for $T>T_0$,

$$\frac{1}{T}\,\mathrm{mes}(M_{T,\varepsilon}(f)) < \frac{\varepsilon}{2\|f\|}.$$

Then,

$$\begin{aligned}
&\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|f(\tau+\omega) - f(\tau)\|\,dt\\
&\quad=\frac{1}{T}\Bigg(\int_{M_{T,\varepsilon}} \sup_{\tau\in[t-r,t]} \|f(\tau+\omega) - f(\tau)\|\,dt\\
&\qquad\qquad+\int_{[r,T]\setminus M_{T,\varepsilon}} \sup_{\tau\in[t-r,t]} \|f(\tau+\omega) - f(\tau)\|\,dt\Bigg)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{2\|f\|}{T}\operatorname{mes}(M_{T,\varepsilon}(f)) + \frac{\varepsilon}{T}\int_{[r,T]\setminus M_{T,\varepsilon}} dt \\
&\leq \frac{2\|f\|}{T}\operatorname{mes}(M_{T,\varepsilon}(f)) + \frac{\varepsilon}{T}\int_0^T dt \\
&\leq 2\varepsilon,
\end{aligned}$$

so

$$\lim_{T\to\infty}\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]}\|f(\tau+\omega)-f(\tau)\|\,dt = 0.$$

That is $f \in PSAP_{\omega,r}(\mathbf{R}^+, X)$.

Necessity: Suppose the contrary, that there exists $\varepsilon_0 > 0$, such that $\frac{1}{T}\operatorname{mes}(M_{T,\varepsilon}(f))$ does not converge to 0 as $T\to\infty$. That is there exists $\delta > 0$, such that for each n,

$$\frac{1}{T_n}\operatorname{mes}(M_{T_n,\varepsilon_0}(f)) \geq \delta \quad \text{for some } T_n > n,$$

then

$$\begin{aligned}
&\frac{1}{T_n}\int_r^{T_n} \sup_{\tau\in[t-r,t]}\|f(\tau+\omega)-f(\tau)\|\,dt \\
&\quad = \frac{1}{T_n}\int_{M_{T_n,\varepsilon_0}} \sup_{\tau\in[t-r,t]}\|f(\tau+\omega)-f(\tau)\|\,dt \\
&\qquad + \int_{[r,T_n]\setminus M_{T_n,\varepsilon_0}} \sup_{\tau\in[t-r,t]}\|f(\tau+\omega)-f(\tau)\|\,dt \\
&\quad \geq \frac{1}{T_n}\int_{M_{T_n,\varepsilon_0}} \sup_{\tau\in[t-r,t]}\|f(\tau+\omega)-f(\tau)\|\,dt \\
&\quad \geq \frac{1}{T_n}\operatorname{mes}(M_{T_n,\varepsilon_0}(f))\varepsilon_0 \\
&\quad \geq \varepsilon_0\delta,
\end{aligned}$$

which contradicts the fact that

$$\lim_{T\to\infty}\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]}\|f(\tau+\omega)-f(\tau)\|\,dt = 0.$$

Thus (3.3) holds. □

THEOREM 3.2. *Assume that* $f \in PSAP_{\omega,r}(\mathbf{R}^+\times Y, X) \cap \mathscr{UC}^p(\mathbf{R}^+\times Y, X)$, $p\in[1,\infty)$, *then* $h(\cdot) = f(\cdot, u(\cdot)) \in PSAP_{\omega,r}(\mathbf{R}^+, X)$ *if* $u \in PSAP_{\omega,r}(\mathbf{R}^+, Y)$.

Proof. It is clear that $f(\cdot, u(\cdot)) \in BC(\mathbf{R}^+, X)$. Moreover, since $f \in PSAP_{\omega,r}(\mathbf{R}^+ \times Y, X) \cap \mathscr{UC}^p(\mathbf{R}^+ \times Y, X)$, for $\forall\varepsilon > 0$, there exists $L_\varepsilon > 0$, $\delta > 0$, $L_f \in BS^p(\mathbf{R}^+, \mathbf{R}^+)$ such that

$$\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \sup_{\|z\|_Y \le R} \|f(\tau+\omega, z) - f(\tau, z)\|\, dt \le \varepsilon, \quad \text{for every } T \ge L_\varepsilon,\ R > 0.$$

$$\|F(t)\| := \sup_{\tau\in[t-r,t]} \|f(\tau, u(\tau+\omega)) - f(\tau, u(\tau))\| \le L_f(t)\varepsilon, \quad \text{if}$$

$$\|z(t)\|_Y := \|u(t+\omega) - u(t)\|_Y < \delta, \quad t \in \mathbf{R}^+.$$

Denote

$$M_{T,\delta}(z) = \left\{ t \in [r, T] : \sup_{\tau\in[t-r,t]} \|u(\tau+\omega) - u(\tau)\|_Y \ge \delta \right\}.$$

$$\tilde{M}_{T,\delta}(z) = \{ t \in [r, T] : \|u(t+\omega) - u(t)\|_Y \ge \delta \}.$$

then $\tilde{M}_{T,\delta}(z) \subset M_{T,\delta}(z)$ and $\lim_{T\to\infty} \dfrac{1}{T}\, \mathrm{mes}(M_{T,\delta}(z)) = 0$ by Lemma 3.4.

So

$$\begin{aligned}
(3.4)\qquad & \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|f(\tau, u(\tau+\omega)) - f(\tau, u(\tau))\|\, dt \\
&\le \frac{1}{T}\int_{\tilde{M}_{T,\delta}(z)} \sup_{\tau\in[t-r,t]} \|f(\tau, u(\tau+\omega)) - f(\tau, u(\tau))\|\, dt \\
&\quad + \frac{1}{T}\int_{[r,T]\setminus\tilde{M}_{T,\delta}(z)} \sup_{\tau\in[t-r,t]} \|f(\tau, u(\tau+\omega)) - f(\tau, u(\tau))\|\, dt \\
&\le \frac{\|F\|_\infty}{T}\, \mathrm{mes}(\tilde{M}_{T,\delta}(z)) + \frac{\varepsilon}{T}\int_r^T L_f(t)\, dt \\
&\le \frac{\|F\|_\infty}{T}\, \mathrm{mes}(M_{T,\delta}(z)) + \frac{\varepsilon}{T}\sum_{k=0}^{[T]}\int_k^{k+1} L_f(t)\, dt \\
&\le \frac{\|F\|_\infty}{T}\, \mathrm{mes}(M_{T,\delta}(z)) + \frac{\varepsilon}{T}\sum_{k=0}^{[T]}\left(\int_k^{k+1} L_f(t)^p\, dt\right)^{1/p} \\
&\le \frac{\|F\|_\infty}{T}\, \mathrm{mes}(M_{T,\delta}(z)) + \frac{[T]+1}{T}\|L_f\|_{S^p}\varepsilon,
\end{aligned}$$

where $\|F\|_\infty := \sup_{t\in\mathbf{R}^+}\|F(t)\|$.

For $T \ge L_\varepsilon$, by (3.4), one has

$$
\begin{aligned}
&\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|f(\tau+\omega,u(\tau+\omega))-f(\tau,u(\tau))\|\,dt \\
&\quad\le \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|f(\tau+\omega,u(\tau+\omega))-f(\tau,u(\tau+\omega))\|\,dt \\
&\qquad+\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|f(\tau,u(\tau+\omega))-f(\tau,u(\tau))\|\,dt \\
&\quad\le \varepsilon+\frac{\|F\|_\infty}{T}\ \mathrm{mes}(M_{T,\delta}(z))+\frac{[T]+1}{T}\|L_f\|_{S^p}\varepsilon,
\end{aligned}
$$

Due to the arbitrariness of ε, one has

$$
\lim_{T\to\infty}\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|f(\tau+\omega,u(\tau+\omega))-f(\tau,u(\tau))\|\,dt=0,
$$

implies that $h(\cdot)\in PSAP_{\omega,r}(\mathbf{R}^+,X)$. □

LEMMA 3.5. *Let* $u\in BC([-r,\infty),X)$ *and assume that* $u|_{[0,\infty)}\in PSAP_{\omega,r}(\mathbf{R}^+,X)$, *then* $u_t\in PSAP_{\omega,r}(\mathbf{R}^+,\mathscr{C})$.

Proof. Note that

$$
\begin{aligned}
&\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \|u_{\tau+\omega}-u_\tau\|_{\mathscr{C}}\,dt \\
&\quad\le \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]}\left(\sup_{\theta\in[-r,0]}\|u(\tau+\omega+\theta)-u(\tau+\theta)\|\right)dt \\
&\quad\le \frac{1}{T}\int_r^T \sup_{\tau\in[t-2r,t]}\|u(\tau+\omega)-u(\tau)\|\,dt \\
&\quad\le \frac{1}{T}\int_r^T \sup_{\tau\in[t-2r,t-r]}\|u(\tau+\omega)-u(\tau)\|\,dt+\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]}\|u(\tau+\omega)-u(\tau)\|\,dt \\
&\quad= \frac{1}{T}\int_0^{T-r} \sup_{\tau\in[t-r,t]}\|u(\tau+\omega)-u(\tau)\|\,dt+\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]}\|u(\tau+\omega)-u(\tau)\|\,dt \\
&\quad= \frac{1}{T}\int_0^{r} \sup_{\tau\in[t-r,t]}\|u(\tau+\omega)-u(\tau)\|\,dt+\frac{1}{T}\int_r^{T-r} \sup_{\tau\in[t-r,t]}\|u(\tau+\omega)-u(\tau)\|\,dt \\
&\qquad+\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]}\|u(\tau+\omega)-u(\tau)\|\,dt \\
&\quad\le \frac{1}{T}\int_0^{r} \sup_{\tau\in[t-r,t]}\|u(\tau+\omega)-u(\tau)\|\,dt+\frac{2}{T}\int_r^T \sup_{\tau\in[t-r,t]}\|u(\tau+\omega)-u(\tau)\|\,dt,
\end{aligned}
$$

By $u|_{[0,\infty)} \in PSAP_{\omega,r}(\mathbf{R}^+, X)$, it is easy to see that

$$\lim_{T\to\infty}\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]}\|u_{\tau+\omega}-u_\tau\|_{\mathscr{C}}\,dt = 0.$$

Hence $u_t \in PSAP_{\omega,r}(\mathbf{R}^+, \mathscr{C})$. The proof is complete. □

Lemma 3.6. *Let $\{S(t)\}_{t\geq 0} \subset L(X)$ be a strongly continuous family of bounded and linear operators such that $\|S(t)\| \leq \phi(t)$, $t \in \mathbf{R}^+$, where $\phi \in L^1(\mathbf{R}^+)$ is nonincreasing. If $f \in PSAP_{\omega,r}(\mathbf{R}^+, X)$, then*

$$(\Lambda f)(t) := \int_0^t S(t-s)f(s)\,ds \in PSAP_{\omega,r}(\mathbf{R}^+, X), \quad t \in \mathbf{R}^+.$$

Proof. Note that

$$\begin{aligned}(\Lambda f)(\tau+\omega)-(\Lambda f)(\tau) &= \int_0^\omega S(\tau+\omega-s)f(s)\,ds + \int_0^\tau S(\tau-s)[f(s+\omega)-f(s)]\,ds\\ &:= I(\tau)+J(\tau),\end{aligned}$$

where

$$I(\tau) = \int_0^\omega S(\tau+\omega-s)f(s)\,ds, \quad J(\tau) = \int_0^\tau S(\tau-s)[f(s+\omega)-f(s)]\,ds.$$

Since

$$\begin{aligned}\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]}\|I(\tau)\|\,dt &\leq \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]}\left(\int_0^\omega \phi(\tau+\omega-s)\|f(s)\|\,ds\right)dt\\ &\leq \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]}\phi(\tau)\left(\int_0^\omega \|f(s)\|\,ds\right)dt\\ &\leq \frac{1}{T}\int_r^T \phi(t-r)\left(\int_0^\omega \|f(s)\|\,ds\right)dt\\ &= \frac{1}{T}\int_0^{T-r}\phi(t)\,dt\cdot\left(\int_0^\omega \|f(s)\|\,ds\right)\\ &\leq \frac{\|\phi\|_{L^1}}{T}\left(\int_0^\omega \|f(s)\|\,ds\right) \to 0, \quad T\to\infty,\end{aligned}$$

then

$$\lim_{T\to\infty}\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]}\|I(\tau)\|\,dt = 0. \tag{3.5}$$

Next, we will prove that

$$\lim_{T\to\infty}\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]}\|J(\tau)\|\,dt = 0. \tag{3.6}$$

Since $f \in PSAP_{\omega,r}(\mathbf{R}^+, X)$, $f \in PSAP_{\omega}(\mathbf{R}^+, X)$ by Lemma 3.1, then

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \|f(t+\omega)-f(t)\|\,dt = 0.$$

$$\lim_{T\to\infty}\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]}\|f(\tau+\omega)-f(\tau)\|\,dt = 0.$$

Define

$$\varphi_T(s) := \frac{1}{T}\int_0^{T-s}\|f(t+\omega)-f(t)\|\,dt, \quad s\in\mathbf{R}^+,$$

then $\varphi_T(s)$ is decreasing on $\mathbf{R}^+$ and

$$\lim_{T\to\infty}\varphi_T(0) = \lim_{T\to\infty}\frac{1}{T}\int_0^T\|f(t+\omega)-f(t)\|\,dt = 0.$$

Moreover,

$$\begin{aligned}
\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]}\|J(\tau)\|\,dt &\le \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]}\left(\int_0^\tau \|S(\tau-s)\|\,\|f(s+\omega)-f(s)\|\,ds\right)dt\\
&\le \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]}\left(\int_0^\tau \phi(\tau-s)\|f(s+\omega)-f(s)\|\,ds\right)dt\\
&= \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]}\left(\int_0^{t-r} \phi(\tau-s)\|f(s+\omega)-f(s)\|\,ds\right)dt\\
&\quad + \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]}\left(\int_{t-r}^{\tau} \phi(\tau-s)\|f(s+\omega)-f(s)\|\,ds\right)dt\\
&:= J_1(T)+J_2(T),
\end{aligned}$$

where

$$J_1(T) = \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]}\left(\int_0^{t-r} \phi(\tau-s)\|f(s+\omega)-f(s)\|\,ds\right)dt,$$

$$J_2(T) = \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]}\left(\int_{t-r}^{\tau} \phi(\tau-s)\|f(s+\omega)-f(s)\|\,ds\right)dt.$$

From Fubini's theorem and ϕ is nonincreasing, one has

$$\begin{aligned}
J_1(T) &= \frac{1}{T}\int_r^T \left(\int_0^{t-r} \sup_{\tau\in[t-r,t]} \phi(\tau-s)\|f(s+\omega)-f(s)\|\,ds\right)dt \\
&\le \frac{1}{T}\int_r^T \left(\int_0^{t-r} \phi(t-r-s)\|f(s+\omega)-f(s)\|\,ds\right)dt \\
&= \frac{1}{T}\int_r^T \left(\int_0^{t-r} \phi(s)\|f(t-r-s+\omega)-f(t-r-s)\|\,ds\right)dt \\
&= \frac{1}{T}\int_r^T \left(\int_r^{t} \phi(s-r)\|f(t-s+\omega)-f(t-s)\|\,ds\right)dt \\
&= \frac{1}{T}\int_r^T \phi(s-r)\left(\int_s^{T} \|f(t-s+\omega)-f(t-s)\|\,dt\right)ds \\
&= \int_r^T \phi(s-r)\left(\frac{1}{T}\int_0^{T-s} \|f(t+\omega)-f(t)\|\,dt\right)ds \\
&= \int_r^T \phi(s-r)\varphi_T(s)\,ds \\
&\le \int_r^T \phi(s-r)\varphi_T(0)\,ds \\
&\le \varphi_T(0)\|\varphi\|_{L^1} \to 0, \quad T\to\infty.
\end{aligned}$$

By $f \in PSAP_{\omega,r}(\mathbf{R}^+, X)$, one has

$$\begin{aligned}
J_2(T) &= \frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]} \left(\int_{t-r}^{\tau} \phi(\tau-s)\|f(s+\omega)-f(s)\|\,ds\right)dt \\
&\le \frac{1}{T}\int_r^T \phi(0) \sup_{\tau\in[t-r,t]} \left(\int_{t-r}^{\tau} \|f(s+\omega)-f(s)\|\,ds\right)dt \\
&\le \frac{\phi(0)}{T}\int_r^T \left(\int_{t-r}^{t} \|f(s+\omega)-f(s)\|\,ds\right)dt \\
&\le \frac{\phi(0)}{T}\int_r^T \left(\int_{t-r}^{t} \sup_{s\in[t-r,t]} \|f(s+\omega)-f(s)\|\,ds\right)dt \\
&\le \frac{r\phi(0)}{T}\int_r^T \sup_{s\in[t-r,t]} \|f(s+\omega)-f(s)\|\,dt \to 0, \quad T\to\infty.
\end{aligned}$$

So (3.6) holds.

By (3.5), (3.6), one has

$$\lim_{T\to\infty}\frac{1}{T}\int_r^T \sup_{\tau\in[t-r,t]}\|(\Lambda f)(\tau+\omega)-(\Lambda f)(\tau)\|\,dt=0,$$

which means that $\Lambda f \in PSAP_{\omega,r}(\mathbf{R}^+, X)$. The proof is complete. □

4. Fractional differential equations with delay

In this section, we establish some sufficient criteria for the existence and uniqueness of $PSAP_{\omega,r}$ solutions for (1.1).

We adopt the following concept of mild solution.

DEFINITION 4.1 [14]. Suppose $0 < \alpha \le \beta \le 1$, $\mu > 0$. A function $u \in C([-r,\infty), X)$ is said to be a mild solution of (1.1) if $u_0 = \varphi_1$, $u_0' = \varphi_2$ and for $t \in \mathbf{R}^+$,

$$u(t) = S_{\alpha,\beta}(t)\varphi_1(0) + (g_1 * S_{\alpha,\beta})(t)\varphi_2(0) + \mu(g_{1+\alpha-\beta} * S_{\alpha,\beta})(t)\varphi_1(0) + \int_0^t S_{\alpha,\beta}(t-s)f(s,u_s)\,ds. \tag{4.1}$$

To establish our results, we introduce the following conditions:

(H_1) A is an $\tilde{\omega}$-sectorial operator of angle $\beta\pi/2$ with $\tilde{\omega} < 0$.

(H_2) $f \in PSAP_{\omega,r}(\mathbf{R}^+ \times \mathscr{C}, X)$.

(H_{31}) f satisfies the Lipschitz condition

$$\|f(t,\phi) - f(t,\psi)\| \le L_f\|\phi-\psi\|_{\mathscr{C}}, \quad \phi,\psi \in \mathscr{C},\ t \in \mathbf{R}^+.$$

where $L_f > 0$ is a constant.

(H_{32}) f satisfies the Lipschitz condition

$$\|f(t,\phi) - f(t,\psi)\| \le L_f(t)\|\phi-\psi\|_{\mathscr{C}}, \quad \phi,\psi \in \mathscr{C},\ t \in \mathbf{R}^+,$$

where $L_f \in BS^p(\mathbf{R}^+,\mathbf{R}^+)$, $p \ge 1$.

(H_{33}) f satisfies the Lipschitz condition

$$\|f(t,\phi) - f(t,\psi)\| \le L_f(t)\|\phi-\psi\|_{\mathscr{C}}, \quad \phi,\psi \in \mathscr{C},\ t \in \mathbf{R}^+,$$

where $L_f \in BS^p(\mathbf{R}^+,\mathbf{R}^+) \cap L^1(\mathbf{R}^+,\mathbf{R}^+)$, $p \ge 1$.

THEOREM 4.1. *Assume that* (H_1), (H_2), (H_{31}) *hold, then* (1.1) *has a unique mild solution* $u(t) \in PSAP_{\omega,r}(\mathbf{R}^+, X)$ *if* $CL_f < 1$, *where* C *is the constant defined in Theorem* 2.1.

Proof. By Theorem 2.1, A generates a uniformly integrable $(\alpha,\beta)_\mu$-regularized family $S_{\alpha,\beta}(t)$ on Banach space X. Let $\mathfrak{B} = \{u : [-r,\infty) \to X | u_0$

$= \varphi_1, u_0' = \varphi_2, u|_{[0,\infty)} \in PSAP_{\omega,r}(\mathbf{R}^+, X)\}$ endowed with the metric $d(u,v) = \|u - v\|_{C(\mathbf{R}^+,X)}$ and let $\mathscr{F} : \mathfrak{B} \to \mathfrak{B}$ be the map defined by $(\mathscr{F}u)_0 = \varphi_1$, $(\mathscr{F}u)_0' = \varphi_2$ and

$$(\mathscr{F}u)(t) = S_{\alpha,\beta}(t)\varphi_1(0) + (g_1 * S_{\alpha,\beta})(t)\varphi_2(0) + \mu(g_{1+\alpha-\beta} * S_{\alpha,\beta})(t)\varphi_1(0) + \int_0^t S_{\alpha,\beta}(t-s) f(s, u_s)\, ds, \quad t \in \mathbf{R}^+. \tag{4.2}$$

Let $u \in \mathfrak{B}$ be given,

(i) $\lim_{t\to\infty}\|S_{\alpha,\beta}(t)\varphi_1(0)\| = 0$ by (2.2), so $S_{\alpha,\beta}(t)\varphi_1(0) \in C_0(\mathbf{R}^+, X) \subset PSAP_{\omega,r}(\mathbf{R}^+, X)$.

(ii) By (2.2), one has $\sup_{t>\tau}\|tS_{\alpha,\beta}(t)\| < \infty$ for each $\tau > 0$. Since A is an $\tilde{\omega}$-sectorial of angle $\beta\pi/2$, then $\|\hat{S}_{\alpha,\beta}(\lambda)\| \to 0$ as $\lambda \to 0$. Thus, by the Hardy-Littlewood theorem (Theorem 2.2), we conclude that $\lim_{t\to\infty}\|(g_1 * S_{\alpha,\beta})(t)\| = 0$. Hence $(g_1 * S_{\alpha,\beta})(t)\varphi_2(0) \in C_0(\mathbf{R}^+, X) \subset PSAP_{\omega,r}(\mathbf{R}^+, X)$.

(iii) Let $0 < \varepsilon < \beta - \alpha$, then

$$\begin{aligned}
&\|g_{1+\alpha-\beta} * S_{\alpha,\beta}(t)\| \\
&\quad = \left\| \int_0^t g_{1+\alpha-\beta}(t-\tau) S_{\alpha,\beta}(\tau)\, d\tau \right\| \\
&\quad = \left\| \Gamma(\beta-\alpha-\varepsilon) \int_0^t g_{1+\alpha-\beta}(t-\tau) g_{\beta-\alpha-\varepsilon}(\tau) \tau^{\alpha-\beta+\varepsilon+1} S_{\alpha,\beta}(\tau)\, d\tau \right\| \\
&\quad \le \Gamma(\beta-\alpha-\varepsilon) \int_0^t g_{1+\alpha-\beta}(t-\tau) g_{\beta-\alpha-\varepsilon}(\tau) \tau^{\alpha-\beta+\varepsilon+1} \|S_{\alpha,\beta}(\tau)\|\, d\tau,
\end{aligned}$$

By (2.2), one has

$$\Gamma(\beta-\alpha-\varepsilon)\tau^{\alpha-\beta+\varepsilon+1}\|S_{\alpha,\beta}(\tau)\| \le \frac{\tilde{M}\tau^{\alpha-\beta+\varepsilon+1}}{1 + |\tilde{\omega}|\tau^{\alpha+1}} = \frac{\tilde{M}\tau^{-\beta+\varepsilon}}{\frac{1}{\tau^{\alpha+1}} + |\tilde{\omega}|}, \quad \tau > 0,$$

where $\tilde{M}$ is a constant. Since $\varepsilon < \beta$, there exists a constant $\tilde{C} > 0$ such that

$$\Gamma(\beta-\alpha-\varepsilon)\tau^{\alpha-\beta+\varepsilon+1}\|S_{\alpha,\beta}(\tau)\| \le \tilde{C}.$$

Therefore,

$$\begin{aligned}
\|g_{1+\alpha-\beta} * S_{\alpha,\beta}(t)\| &\le \tilde{C} \int_0^t g_{1+\alpha-\beta}(t-\tau) g_{\beta-\alpha-\varepsilon}(\tau)\, d\tau \\
&= \tilde{C} g_{1-\varepsilon}(t) = \frac{\tilde{C} t^{-\varepsilon}}{\Gamma(1-\varepsilon)},
\end{aligned}$$

which shows that $\|g_{1+\alpha-\beta} * S_{\alpha,\beta}(t)\| \to 0$ as $t \to \infty$. Therefore, $g_{1+\alpha-\beta} * S_{\alpha,\beta} \in PSAP_{\omega,r}(\mathbf{R}^+, X)$.

(iv) For $u \in PSAP_{\omega,r}(\mathbf{R}^+, X)$, $u_s \in PSAP_{\omega,r}(\mathbf{R}^+, \mathscr{C})$ by Lemma 3.5. It is not difficult to see that $f(s, u_s) \in PSAP_{\omega,r}(\mathbf{R}^+, X)$ by Theorem 3.1. By Theorem 2.1, Lemma 3.6, one has

$$\int_0^t S_{\alpha,\beta}(t-s) f(s, u_s)\, ds \in PSAP_{\omega,r}(\mathbf{R}^+, X).$$

By (i)–(iv), $\mathscr{F} u \in \mathfrak{B}$, so $\mathscr{F}$ is well defined.

Moreover, let $u, v \in \mathfrak{B}$, one has

$$\begin{aligned}
\|(\mathscr{F}u)(t) - (\mathscr{F}v)(t)\| &\le \int_0^t \|S_{\alpha,\beta}(t-s)\| \, \|f(s,u_s) - f(s,v_s)\|\, ds \\
&\le L_f \|u - v\| \cdot \int_0^t \frac{C}{1 + |\tilde{\omega}|[(t-s)^{\alpha+1} + \mu(t-s)^{\beta}]}\, ds \\
&\le C L_f \|u - v\|,
\end{aligned}$$

which implies that

$$d(\mathscr{F}u, \mathscr{F}v) \le C L_f d(u, v).$$

By the Banach contraction mapping principle, $\mathscr{F}$ has a unique fixed point in $\mathfrak{B}$, which is the unique $PSAP_{\omega,r}$ mild solution of (1.1). □

THEOREM 4.2. *Assume that* (H_1), (H_2), (H_{32}) *hold and*

$$C\left(1 + \frac{|\tilde{\omega}|^{-1/(\alpha+1)}\pi}{(\alpha+1)\sin(\pi/(\alpha+1))}\right)\|L_f\|_{S^p} < 1,$$

then (1.1) *has a unique mild solution* $u(t) \in PSAP_{\omega,r}(\mathbf{R}^+, X)$.

Proof. Define the operator $\mathscr{F}$ as in (4.2), it is easy to see that $f(s, u_s) \in PSAP_{\omega,r}(\mathbf{R}^+, X)$ by Theorem 3.2 and $\mathscr{F}$ is well defined similarly as the proof of Theorem 4.1.

For $u, v \in \mathfrak{B}$, one has

$$\begin{aligned}
\|(\mathscr{F}u)(t) - (\mathscr{F}v)(t)\| &\le \int_0^t \|S_{\alpha,\beta}(t-s)\| \, \|f(s,u_s) - f(s,v_s)\|\, ds \\
&\le \|u - v\| \cdot \int_0^t \frac{C L_f(s)}{1 + |\tilde{\omega}|[(t-s)^{\alpha+1} + \mu(t-s)^{\beta}]}\, ds \\
&\le \|u - v\| \cdot \int_0^t \frac{C L_f(s)}{1 + |\tilde{\omega}|(t-s)^{\alpha+1}}\, ds,
\end{aligned}$$

- If $t = m \in \mathbf{N}$, in this case

$$\begin{aligned}
(4.3)\qquad & \int_0^t \frac{1}{1+|\tilde{\omega}|(t-s)^{\alpha+1}} L_f(s)\,ds \\
&= \int_0^m \frac{1}{1+|\tilde{\omega}|(m-s)^{\alpha+1}} L_f(s)\,ds \\
&= \sum_{k=0}^{m-1} \int_k^{k+1} \frac{1}{1+|\tilde{\omega}|(m-s)^{\alpha+1}} L_f(s)\,ds \\
&\le \sum_{k=0}^{m-1} \frac{1}{1+|\tilde{\omega}|(m-k-1)^{\alpha+1}} \int_k^{k+1} L_f(s)\,ds \\
&\le \sum_{k=0}^{m-1} \frac{1}{1+|\tilde{\omega}|(m-k-1)^{\alpha+1}} \left(\int_k^{k+1} L_f(s)^p\,ds\right)^{1/p} \\
&\le \left[1+\left(\int_0^1+\int_1^2+\cdots+\int_{m-2}^{m-1}\right)\frac{1}{1+|\tilde{\omega}|t^{\alpha+1}}\,dt\right]\|L_f\|_{S^p} \\
&\le \left(1+\int_0^\infty \frac{1}{1+|\tilde{\omega}|t^{\alpha+1}}\,dt\right)\|L_f\|_{S^p} \\
&\le \left(1+\frac{|\tilde{\omega}|^{-1/(\alpha+1)}\pi}{(\alpha+1)\sin(\pi/(\alpha+1))}\right)\|L_f\|_{S^p},
\end{aligned}$$

- If $t=m-h$, where $0<h<1$. In this general case,

$$\begin{aligned}
\int_0^t \frac{1}{1+|\tilde{\omega}|(t-s)^{\alpha+1}} L_f(s)\,ds &= \int_0^{m-h} \frac{1}{1+|\tilde{\omega}|(m-h-s)^{\alpha+1}} L_f(s)\,ds \\
&= \int_h^m \frac{1}{1+|\tilde{\omega}|(m-s)^{\alpha+1}} L_f(s-h)\,ds \\
&= \int_0^m \frac{1}{1+|\tilde{\omega}|(m-s)^{\alpha+1}} \tilde{L}_f(s)\,ds \\
&\le \left(1+\frac{|\tilde{\omega}|^{-1/(\alpha+1)}\pi}{(\alpha+1)\sin(\pi/(\alpha+1))}\right)\|\tilde{L}_f\|_{S^p},
\end{aligned}$$

where $\tilde{L}_f$ is defined by

$$\tilde{L}_f(s) = \begin{cases} 0, & 0 \le s < h, \\ L_f(s-h), & s \ge h, \end{cases}$$

then $\|\tilde{L}_f\|_{S^p} = \|L_f\|_{S^p}$. So we infer that

$$(4.4)\qquad \int_0^t \frac{1}{1+|\tilde{\omega}|(t-s)^{\alpha+1}} L_f(s)\,ds \le \left(1+\frac{|\tilde{\omega}|^{-1/(\alpha+1)}\pi}{(\alpha+1)\sin(\pi/(\alpha+1))}\right)\|L_f\|_{S^p}$$

By (4.3), (4.4), one has

$$\|(\mathscr{F}u)(t) - (\mathscr{F}v)(t)\| \le C\left(1 + \frac{|\tilde{\omega}|^{-1/(\alpha+1)}\pi}{(\alpha+1)\sin(\pi/(\alpha+1))}\right)\|L_f\|_{S^p} \cdot \|u - v\|,$$

so

$$d(\mathscr{F}u, \mathscr{F}v) \le C\left(1 + \frac{|\tilde{\omega}|^{-1/(\alpha+1)}\pi}{(\alpha+1)\sin(\pi/(\alpha+1))}\right)\|L_f\|_{S^p} \cdot d(u, v).$$

By the Banach contraction mapping principle, $\mathscr{F}$ has a unique fixed point in $\mathfrak{B}$, which is the unique $PSAP_{\omega,r}$ mild solution of (1.1). □

THEOREM 4.3. *Assume that* (H_1), (H_2), (H_{33}) *hold, then* (1.1) *has a unique mild solution* $u(t) \in PSAP_{\omega,r}(\mathbf{R}^+, X)$.

Proof. Define the operator $\mathscr{F}$ as in (4.2). Let $u, v \in \mathfrak{B}$, one has

$$\begin{aligned}
\|(\mathscr{F}u)(t) - (\mathscr{F}v)(t)\| &\le \int_0^t \|S_{\alpha,\beta}(t-s)\| \, \|f(s, u_s) - f(s, v_s)\| \, ds \\
&\le \|u - v\| \cdot \int_0^t \frac{CL_f(s)}{1 + |\tilde{\omega}|[(t-s)^{\alpha+1} + \mu(t-s)^{\beta}]} \, ds \\
&\le C\int_0^t L_f(s) \, ds \cdot \|u - v\| \\
&\le C\|L_f\|_{L^1} \cdot \|u - v\|.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|(\mathscr{F}^2u)(t) - (\mathscr{F}^2v)(t)\| &\le \int_0^t \frac{CL_f(s)}{1 + |\tilde{\omega}|[(t-s)^{\alpha+1} + \mu(t-s)^{\beta}]} \|(\mathscr{F}u)(s) - (\mathscr{F}v)(s)\| \, ds \\
&\le C\int_0^t L_f(s)\|(\mathscr{F}u)(s) - (\mathscr{F}v)(s)\| \, ds \\
&\le C^2\|u - v\| \int_0^t L_f(s)\left(\int_0^s L_f(\tau) \, d\tau\right) ds \\
&= C^2\|u - v\| \int_0^t \left(\int_0^s L_f(\tau) \, d\tau\right) d\left(\int_0^s L_f(\tau) \, d\tau\right) \\
&\le \frac{C^2}{2!}\|u - v\| \left(\int_0^t L_f(\tau) \, d\tau\right)^2 \\
&\le \frac{(C\|L_f\|_{L^1})^2}{2!}\|u - v\|.
\end{aligned}$$

By the method of mathematical induction, we have

$$\|(\mathscr{F}^n u)(t) - (\mathscr{F}^n v)(t)\| \le \frac{C^n}{n!}\|u - v\|\left(\int_0^t L_f(\tau)\, d\tau\right)^n.$$

Moreover, since $L_f(t) \in L^1(\mathbf{R}^+, \mathbf{R}^+)$,

$$\|(\mathscr{F}^n u)(t) - (\mathscr{F}^n v)(t)\| \le \frac{(C\|L_f\|_{L^1})^n}{n!}\|u - v\|,$$

which implies that

$$d(\mathscr{F}^n u, \mathscr{F}^n v) \le \frac{(C\|L_f\|_{L^1})^n}{n!} d(u, v).$$

For sufficiently large n, we have $(C\|L_f\|_{L^1})^n/n! < 1$, by the Banach contraction mapping principle, $\mathscr{F}$ has a unique fixed point in $\mathfrak{B}$, which is the unique $PSAP_{\omega,r}$ mild solution of (1.1). □

5. Example

Consider the following fractional partial differential equation with delay

$$D_t^{\alpha+1}u(t,x) + \mu D_t^{\beta}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) - \delta u(t,x) + a(t)\int_{-1}^{0} b(s)\sin[u(t+s,x)]\, ds, \tag{5.1}$$

where $t \in \mathbf{R}^+$, $x \in [0,1]$, $0 < \alpha \le \beta \le 1$, $\mu > 0$, $\delta > 0$, $a(t) \in PSAP_{\omega,1}(\mathbf{R}^+, \mathbf{R})$, with initial and zero boundary conditions.

Let $X = (L^2([0,1], \mathbf{R}), \|\cdot\|_{L^2})$ and define the operator A on X by

$$Au = \frac{\partial^2}{\partial x^2}u - \delta u$$

with

$$D(A) = \{u \in L^2([0,1], \mathbf{R}) : u'' \in L^2[0,1], u(0) = u(1) = 0\},$$

and

$$f(t,\varphi)(x) = a(t)\int_{-1}^{0} b(s)\sin[\varphi(s)(x)]\, ds, \quad t \in \mathbf{R}^+,\ \varphi \in C([-1,0], X),\ x \in [0,1].$$

It is well know that A is a $\tilde{\omega}$-sectorial operator with $\tilde{\omega} = -\delta < 0$ and angle $\pi/2$ (and hence of angle $\beta\pi/2$ with $\beta \le 1$) [14]. (5.1) can be rewritten as an abstract system of the form (1.1), where $u(t) = u(t,\cdot)$. In addition, since $a(t) \in PSAP_{\omega,1}(\mathbf{R}^+, \mathbf{R})$ and

$$\|f(t,\phi) - f(t,\psi)\| \le \|a\|\left(\int_{-1}^{0} |b(s)|^2\, ds\right)^{1/2}\|\phi - \psi\|_{\mathscr{C}}, \quad \text{for all } \phi, \psi \in C([-1,0], X),$$

so (H_{31}) holds with $L_f \equiv \|a\|(\int_{-1}^{0} |b(s)|^2\, ds)^{1/2}$. By Theorem 4.1, we conclude that (5.1) has a unique solution $u \in PSAP_{\omega,1}$ if $CL_f < 1$.

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