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ENTIRE FUNCTIONS AND THEIR FIRST DERIVATIVES SHARING SIMPLE β-POINTS FOR A SMALL FUNCTION β

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Abstract

The following theorem has been proved by A. Schweizer [7]. If a nonconstant entire function f and its derivative f' share their simple zeros and if every simple a-point of f is a (not necessarily simple) a-point of f' for some nonzero constant a, then $f \equiv f'$. In this paper we shall prove that the above result is also true when the nonzero constant a is replaced by a meromorphic small function $\beta(\not\equiv 0, \infty)$.

1. Introduction and results

In this paper, we use the same notation as given in Nevanlinna theory of meromorphic functions ([2, 8]). For a meromorphic function h, let $T(r,h)$ denote the Nevanlinna characteristic of h and let $S(r,h)$ be any quantity satisfying $S(r,h) = o\{T(r,h)\}$ as $r \to \infty$, except possibly on a set of finite linear measure. Then a meromorphic function β is called a small function of h if and only if $T(r,h) = S(r,h)$. Let f and g be two nonconstant entire functions. For $a \in \mathbf{C}$ we say that f and g share the value a CM (counting multiplicities) if the a-points of f and g coincide in locations and multiplicities. If we do not consider the multiplicities, we say that f and g share the value a IM (ignoring multiplicities). Let k be a positive integer and always keeping in mind that the β-points of f are the zeros of $f - \beta$ and their multiplicity is the multiplicity of that zero. We denote by $N_{k)}\left(r, \dfrac{1}{f-\beta}\right)$ the counting function of β-points of f with multiplicity $\leq k$, by $N_{(k+1}\left(r, \dfrac{1}{f-\beta}\right)$ the counting function of β-points of f with multiplicity $> k$ and by $N_{=k}\left(r, \dfrac{1}{f-\beta}\right)$ counting function of β-points of f which have the multiplicity k. In the same way we define $\overline{N}_{k)}\left(r, \dfrac{1}{f-\beta}\right)$,

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$\overline{N}_{(k+1}\left(r, \dfrac{1}{f-\beta}\right)$ and $\overline{N}_{=k}\left(r, \dfrac{1}{f-\beta}\right)$ where in counting the β-points of f we ignore the multiplicities ([8]). Finally we denote by $E(\beta, f)$ the set of β-points of f (counting multiplicities), by $\overline{E}(\beta, f)$ the set of β-points of f (ignoring multiplicities) and by $E_{1)}(\beta, f)$ the set of β-points of f with multiplicity one. Thus we say that f and g share β CM, β IM, β simple if $E(\beta, f) = E(\beta, g)$, $\overline{E}(\beta, f) = \overline{E}(\beta, g)$ and $E_{1)}(\beta, f) = E_{1)}(\beta, g)$ respectively.

On the problems of uniqueness of an entire function and its derivative that share some values. Rubel-Yang ([6]) proved that if the entire function f and f' share two distinct finite values CM then $f \equiv f'$. Mues-Steinmetz ([3]) improved this result to the case when f and f' share two distinct finite values IM. Zheng-Wang ([9]) generalized this result to f and f' which share two small functions CM. In 2000 Qiu ([5]) improved this result to the case when f and f' share two small functions IM. Recently, Schweizer [7] proved the following theorem:

Theorem 1.1. *Let f be a nonconstant entire function and let a be a nonzero constant. If f and f' share their simple zeros and if every simple a-point of f is a (not necessarily simple) a-point of f', then $f \equiv f'$.*

It is natural to ask whether the "nonzero constant a" of Theorem 1.1 can be replaced by "small function $\beta(\not\equiv 0, \infty)$"? In this paper, we will give a positive answer to this question. Indeed, we shall prove the following:

Theorem 1.2. *Let f be a nonconstant entire function and let β be a small meromorphic function of f such that $\beta(z) \not\equiv 0, \infty$. If f and f' share their simple zeros and if every simple β-point of f is a (not necessarily simple) β-point of f', then $f \equiv f'$.*

It is obvious that Theorem 1.2 is a generalization of Theorem 1.1. From Theorem 1.2, we directly conclude the following corollary:

Corollary 1.3. *Let f be a nonconstant entire function and let β be a small meromorphic function of f such that $\beta(z) \not\equiv 0, \infty$. If $E_{1)}(0, f) = E_{1)}(0, f')$ and $E_{1)}(\beta, f) = E_{1)}(\beta, f')$, then $f \equiv f'$.*

A. Schweizer [7] also gave the following example:

Example 1. Let

$$f(z) = \frac{a}{2}(\sin(2z) + 1), \quad 0 \neq a \in \mathbf{C} \tag{1.1}$$

then $f'(z) = a\cos(2z)$. All a-points of f and of f' and all zeros of f have multiplicity 2. Thus the condition that f and f' share their simple a-points and that if every simple zero of f is a simple zero of f' does not imply $f \equiv f'$.

Note that from (1.1) we see that

$$L = \frac{\left(\dfrac{f-a}{0-a}\right)'}{\dfrac{f-a}{0-a}} = \frac{f'}{f-a} = \frac{2\cos(2z)}{\sin(2z)-1}.$$

Hence $L' = \dfrac{4}{\sin(2z)-1}$, and so $\overline{N}\left(r, \dfrac{1}{L'}\right) = S(r,f)$. In the present paper, we shall prove the following theorem which includes Example 1 as a special case:

THEOREM 1.4. *Let f be a nonconstant entire function and let β_1 and β_2 be two distinct small meromorphic functions of f. If all zeros of $f-\beta_1$ and $f-\beta_2$ have multiplicities greater than* 1, *then either*

$$f(z) - \beta_1(z) = \frac{-1}{4A}(\beta_2(z)-\beta_1(z))e^{-cz}(1-Ae^{cz})^2 \tag{1.2}$$

and

$$f(z) - \beta_2(z) = \frac{-1}{4A}(\beta_2(z)-\beta_1(z))e^{-cz}(1+Ae^{cz})^2 \tag{1.3}$$

or

$$T(r,f) \le 4\overline{N}\left(r, \frac{1}{L'}\right) + S(r,f), \tag{1.4}$$

where A, c are nonzero constants, $L = \dfrac{F'}{F}$ and $F = \dfrac{f-\beta_1}{\beta_2-\beta_1}$.

From Theorem 1.4, we immediately deduce the following corollary:

COROLLARY 1.5. *Let f be a nonconstant entire function, and let β_1 and β_2 be two distinct small meromorphic functions of f satisfying $\overline{N}\left(r, \dfrac{1}{L'}\right) = S(r,f)$ where $L = \dfrac{F'}{F}$ and $F = \dfrac{f-\beta_1}{\beta_2-\beta_1}$. If all zeros of $f-\beta_1$ and $f-\beta_2$ have multiplicities greater than* 1, *then* (1.2) *and* (1.3) *hold.*

Remarks.

1. If $\beta_1 \equiv a$, $\beta_2 \equiv 0$, $c = 2i$ and $A = -i$, then (1.3) becomes (1.1).

2. From (1.2) or (1.3) it is easy to see that $L'(z) = \dfrac{-2Ac^2e^{cz}}{(1-Ae^{cz})^2}$. Hence $\overline{N}\left(r, \dfrac{1}{L'}\right) = S(r,f)$.

3. The following example shows that the condition $\overline{N}\left(r,\dfrac{1}{L'}\right)=S(r,f)$ in Corollary 1.5 cannot be removed.

Example 2. Let $\alpha(z)$ be any nonconstant entire function, A be any nonzero constant and

$$f(z)-\beta_1(z)=\frac{-1}{4A}(\beta_2(z)-\beta_1(z))e^{-\int_0^z\alpha(t)\,dt}(1-Ae^{\int_0^z\alpha(t)\,dt})^2.$$

Then it is easy to see that all zeros of $f-\beta_1$ and $f-\beta_2$ have multiplicities 2 and

$$L(z)=-2A\alpha\left(\frac{1}{2A}+\frac{e^{\int_0^z\alpha(t)\,dt}}{1-Ae^{\int_0^z\alpha(t)\,dt}}\right).$$

Hence

$$L'(z)=\frac{-\alpha'\left[1+\dfrac{\alpha^4}{\alpha'^2}-\left(Ae^{\int_0^z\alpha(t)\,dt}-\dfrac{\alpha^2}{\alpha'}\right)^2\right]}{(1-Ae^{\int_0^z\alpha(t)\,dt})^2}.$$

From this we deduce that $\overline{N}\left(r,\dfrac{1}{L'}\right)\neq S(r,f)$ but the conclusion of Corollary 1.5 is not valid.

2. Some lemmas

For the proof of our theorems we need the following lemmas:

Lemma 2.1 [4]. *Let f be a nonconstant meromorphic function and $\Psi=a_nf^n+a_{n-1}f^{n-1}+\cdots+a_1f+a_0$, where $a_n\not\equiv 0$, $a_{n-1},\ldots,a_1,a_0$ are meromorphic small functions of f. If $\overline{N}\left(r,\dfrac{1}{\Psi}\right)=S(r,f)$, then three cases are possible*

(i) $\Psi=a_n\left(f+\dfrac{a_{n-1}}{na_n}\right)^n$;

(ii) *There exist a meromorphic small function $\alpha_0\not\equiv 0$ and an integer μ such that $n=2\mu$ and*

$$\Psi=a_n\left(f^2+2\frac{a_{n-1}}{na_n}f+\left(\frac{a_{n-1}}{na_n}\right)^2+\alpha_0\right)^{\mu};$$

(iii) *There exist a meromorphic small function $\alpha_0\not\equiv 0$, positive integers μ_1 and μ_2, and distinct complex numbers λ_1 and λ_2 such that $\mu_1+\mu_2=n$, $\mu_1\lambda_1+\mu_2\lambda_2=0$, and*

$$\Psi=a_n\left(f+\frac{a_{n-1}}{na_n}-\lambda_1\alpha_0\right)^{\mu_1}\left(f+\frac{a_{n-1}}{na_n}-\lambda_2\alpha_0\right)^{\mu_2}.$$

LEMMA 2.2 [2, P. 47]. *Let f be a nonconstant meromorphic function, and a_1, a_2, a_3 be distinct small functions of f. Then*

$$T(r, f) \leq \sum_{j=1}^{3} \overline{N}\left(\frac{1}{f - a_j}\right) + S(r, f).$$

LEMMA 2.3 [5]. *Let f be a nonconstant entire function, α_1 and α_2 be two distinct small functions of f with $\alpha_1 \not\equiv \infty$ and $\alpha_2 \not\equiv \infty$. Set*

$$\Delta(f) = \begin{vmatrix} f - \alpha_1 & \alpha_1 - \alpha_2 \\ f' - \alpha_1' & \alpha_1' - \alpha_2' \end{vmatrix} = \begin{vmatrix} f - \alpha_2 & \alpha_1 - \alpha_2 \\ f' - \alpha_2' & \alpha_1' - \alpha_2' \end{vmatrix}$$

Then

$$\Delta(f) \not\equiv 0 \quad \text{and} \quad m\left(r, \frac{\Delta(f)}{f - \alpha_i}\right) = S(r, f) \quad (i = 1, 2)$$

LEMMA 2.4 [1]. *Let f' be a nonconstant meromorphic function and let*

$$\Gamma = \left(\frac{f''}{f'}\right)^2 - 2\left(\frac{f''}{f'}\right)'.$$

Then

$$T(r, \Gamma) \leq 2\overline{N}\left(r, \frac{1}{f'}\right) + 2\overline{N}_{(2}(r, f) + S(r, f').$$

3. Proof of Theorems

3.1. Proof of Theorem 1.2

The proof here is by contradiction. Assume that $f \not\equiv f'$. Set

$$W = \frac{[(f/\beta)']^2 (f - f')}{f^2 (f - \beta)}. \tag{3.1}$$

From Nevanlinna's fundamental estimate of the logarithmic derivative we obtain

$$\begin{aligned} m(r, W) &\leq m\left(r, \frac{1}{\beta} . \frac{(f/\beta)'}{(f/\beta)}\right) + m\left(r, \frac{1}{\beta} . \frac{(f/\beta)'}{((f/\beta) - 1)}\right) + m\left(r, 1 - \frac{f'}{f}\right) + O(1) \\ &\leq 2m\left(r, \frac{1}{\beta}\right) + S(r, f) = S(r, f). \end{aligned} \tag{3.2}$$

Since f and f' share their simple zeros, every zero of f has multiplicity at least 3 and f' has no simple zeros. That is

$$N\left(r, \frac{1}{f}\right) = N_{(3}\left(r, \frac{1}{f}\right) \quad \text{and} \quad N\left(r, \frac{1}{f'}\right) = N_{(2}\left(r, \frac{1}{f'}\right). \tag{3.3}$$

Further it follows from (3.1) that if z_0 is a zero of f with multiplicity $p \geq 3$ and $\beta(z_0) \neq 0, \infty$, then

$$W(z) = O((z - z_0)^{p-3}), \quad \text{as } z \to z_0. \tag{3.4}$$

Since every simple zero of $f - \beta$ is a zero of $f' - \beta$, we can also conclude from (3.1) that if z_1 is a zero of $f - \beta$ with multiplicity $q \geq 1$ and $\beta(z_1) \neq 0, \infty$,

$$W(z) = \begin{cases} O(1) & \text{if } q = 1 \\ O((z - z_1)^{q-2}) & \text{if } q \geq 2 \end{cases} \tag{3.5}$$

as $z \to z_1$. Thus, from (3.2), (3.4) and (3.5) we get

$$\begin{aligned} T(r, W) &= N(r, W) + m(r, W) = N(r, W) + S(r, f) \\ &\leq 5N\left(r, \frac{1}{\beta}\right) + S(r, f) \\ &\leq 5T(r, \beta) + S(r, f) = S(r, f). \end{aligned} \tag{3.6}$$

We can rewrite (3.1) in the form

$$W\left(\frac{\beta - f}{f}\right) = \left(\frac{(f/\beta)'}{f}\right)^2 \left(\frac{f'}{f} - 1\right) = \frac{1}{\beta^2}\left(\frac{f'}{f} - \frac{\beta'}{\beta}\right)^2 \left(\frac{f'}{f} - 1\right),$$

or

$$\begin{aligned} G &= (g - 1)(g - \lambda)^2 + W\beta^2 \\ &= g^3 - (2\lambda + 1)g^2 + \lambda(\lambda + 2)g + W\beta^2 - \lambda^2, \end{aligned} \tag{3.7}$$

where $G = \dfrac{W\beta^3}{f}$, $g = \dfrac{f'}{f}$ and $\lambda = \dfrac{\beta'}{\beta}$. Note that f is an entire function. So by (3.6), (3.7) and (3.3) we find that

$$\begin{aligned} T(r, f) &= T(r, G) + S(r, f) \leq 3T(r, g) + S(r, f) \leq 3\overline{N}\left(r, \frac{1}{f}\right) + S(r, f) \\ &\leq 3\overline{N}_{(3}\left(r, \frac{1}{f}\right) + S(r, f) \leq N_{(3}\left(r, \frac{1}{f}\right) + S(r, f) \leq T(r, f) + S(r, f), \end{aligned}$$

so that $T(r, f) = 3T(r, g) + S(r, f)$. It follows that every $S(r, f)$ is also an $S(r, g)$ and vice versa. From now on we will write $S(r)$ for the common error term.

Since any zero of $G = \dfrac{W\beta^3}{f}$ can only occur at a zero of W or a zero of β, we have

$$\begin{aligned} \overline{N}\left(r, \frac{1}{G}\right) &\leq \overline{N}\left(r, \frac{1}{W}\right) + \overline{N}\left(r, \frac{1}{\beta}\right) \\ &\leq T(r, W) + T(r, \beta) + O(1) = S(r). \end{aligned} \tag{3.8}$$

By Lemma 2.1, only three cases are possible.

CASE 1. G can be expressed as

$$G=\left(g-\frac{2\lambda+1}{3}\right)^3=g^3-(2\lambda+1)g^2+3\left(\frac{2\lambda+1}{3}\right)^2g-\left(\frac{2\lambda+1}{3}\right)^3.$$

From this and (3.7), equating the coefficients of g and g^0 terms, we obtain

$$3\left(\frac{2\lambda+1}{3}\right)^2\equiv\lambda(\lambda+2)\quad\text{and}\quad W\beta^2-\lambda^2\equiv-\left(\frac{2\lambda+1}{3}\right)^3,$$

or, equivalently,

$$(\lambda-1)^2\equiv 0\quad\text{and}\quad W\beta^2-\lambda^2\equiv-\left(\frac{2\lambda+1}{3}\right)^3.$$

That is $W\beta^2\equiv 0$. Since $\beta\not\equiv 0$, therefore $W\equiv 0$. Hence $f\equiv f'$, a contradiction.

CASE 2. There exists a nonzero small function α_0 and an integer μ such that $2\mu=3$ and

$$G=\left(g^2-2\left(\frac{2\lambda+1}{3}\right)g+\left(\frac{2\lambda+1}{3}\right)^2+\alpha_0\right)^\mu.$$

In this case it is impossible to find an integer number μ such that $2\mu=3$.

CASE 3. There exists a nonzero small function α_0, positive integers μ_1 and μ_2, and distinct complex numbers κ_1 and κ_2 such that $\mu_1+\mu_2=3$, $\mu_1\kappa_1+\mu_2\kappa_2=0$ and

$$G=\left(g-\frac{1}{3}(2\lambda+1)-\kappa_1\alpha_0\right)^{\mu_1}\left(g-\frac{1}{3}(2\lambda+1)-\kappa_2\alpha_0\right)^{\mu_2}. \tag{3.9}$$

Without loss of generality we suppose $\mu_1=1$ and $\mu_2=2$. Thus $\kappa_1+2\kappa_2=0$ and (3.9) becomes

$$G=(g-v_1)(g-v_2)^2, \tag{3.10}$$

where

$$v_1=\frac{1}{3}(2\lambda+1)+\kappa_1\alpha_0\quad\text{and}\quad v_2=\frac{1}{3}(2\lambda+1)-\frac{1}{2}\kappa_1\alpha_0.$$

Since $\kappa_1\neq 0$ and $\alpha_0\not\equiv 0$, therefore $v_1\not\equiv v_2$. If $v_1\equiv 0$, then $v_2=\frac{1}{2}(2\lambda+1)$. From this, (3.10) and (3.7), equating the coefficient of g and g^0 terms, we get

$$\lambda\equiv\frac{1}{4}\quad\text{and}\quad W\beta^2\equiv\frac{1}{16}.$$

Then (3.10) reads

$$\frac{\beta}{16f} = g\left(g - \frac{3}{4}\right)^2. \tag{3.11}$$

Differentiating (3.11) and then using $\lambda \equiv \frac{1}{4}$, we find that

$$\frac{\beta}{16f}\left(g - \frac{1}{4}\right) = -3g'\left(g - \frac{3}{4}\right)\left(g - \frac{1}{4}\right),$$

and eliminating $\dfrac{\beta}{16f}$ between this and (3.11) leads to

$$g\left(g - \frac{3}{4}\right)^2\left(g - \frac{1}{4}\right) = -3g'\left(g - \frac{3}{4}\right)\left(g - \frac{1}{4}\right). \tag{3.12}$$

If $\left(g - \frac{3}{4}\right)\left(g - \frac{1}{4}\right) \equiv 0$, then it is easy to see that g is a constant. From this and (3.11), it follows that $T(r, f) = S(r, f)$, a contradiction. Therefore $\left(g - \frac{3}{4}\right)\left(g - \frac{1}{4}\right) \not\equiv 0$ and (3.12) becomes $g' = -\frac{1}{3}g\left(g - \frac{3}{4}\right)$. We can write this as $\dfrac{g'}{g} - \dfrac{g'}{g - \frac{3}{4}} = \dfrac{1}{4}$ and integrating both sides, we have $g(z) = \dfrac{3/4}{1 - ce^{-(1/4)z}}$. Integrating again, we obtain

$$f(z) = b(e^{(1/4)z} - c)^3, \tag{3.13}$$

where b and c are nonzero constants. But, since $\dfrac{1}{4} = \lambda = \dfrac{\beta'}{\beta}$, so $\beta(z) = de^{(1/4)z}$, where d is a nonzero constant. Substituting this into (3.13), we conclude that $T(r, f) = S(r, f)$. Again this is a contradiction.

If $v_2 \equiv 0$, then similarly as the above discussion, we will arrive at the same contradiction. In the following we assume $v_1 \not\equiv 0$ and $v_2 \not\equiv 0$. By Lemma 2.2, (3.10) and (3.8) we see that

$$\begin{aligned} T(r, g) &\le \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g - v_1}\right) + \overline{N}\left(r, \frac{1}{g - v_2}\right) + S(r, g) \\ &\le \overline{N}\left(r, \frac{1}{g}\right) + 2\overline{N}\left(r, \frac{1}{G}\right) + S(r) \le \overline{N}\left(r, \frac{1}{g}\right) + S(r). \end{aligned} \tag{3.14}$$

We know from $g = \dfrac{f'}{f}$ that the zeros of g can only occur at the zeros of f' which are not zero of f. Consequently, from (3.3),

$$\begin{aligned} T(r, g) &\le \overline{N}\left(r, \frac{1}{g}\right) + S(r) \le \overline{N}_{(2}\left(r, \frac{1}{g}\right) + S(r) \\ &\le \frac{1}{2}N\left(r, \frac{1}{g}\right) + S(r) \le \frac{1}{2}T(r, g) + S(r) \end{aligned}$$

a contradiction, and the proof of Theorem 1.2 is complete. □

3.2. Proof of Theorem 1.4

Consider the auxiliary function

$$h = \frac{\Delta^2(f)}{(f - \beta_1)(f - \beta_2)}. \tag{3.15}$$

From Lemma 2.3 we know that $\Delta(f) \not\equiv 0$ and

$$m(r,h) \le m\left(r, \frac{\Delta(f)}{f - \beta_1}\right) + m\left(r, \frac{\Delta(f)}{f - \beta_2}\right) = S(r,f). \tag{3.16}$$

From the hypotheses of Theorem 1.4 and Lemma 2.2 we deduce that

$$\begin{aligned} T(r,f) &\le \sum_{j=1}^{2} \overline{N}\left(r, \frac{1}{f - \beta_j}\right) + \overline{N}(r,f) + S(r,f) \\ &= \sum_{j=1}^{2} \overline{N}_{(2}\left(r, \frac{1}{f - \beta_j}\right) + S(r,f) \\ &\le \sum_{j=1}^{2} \frac{1}{2} N_{(2}\left(r, \frac{1}{f - \beta_j}\right) + S(r,f) \le T(r,f) + S(r,f). \end{aligned}$$

Therefore

$$T(r,f) = \sum_{j=1}^{2} \overline{N}_{(2}\left(r, \frac{1}{f - \beta_j}\right) + S(r,f) = \frac{1}{2}\sum_{j=1}^{2} N_{(2}\left(r, \frac{1}{f - \beta_j}\right) + S(r,f).$$

This implies that

$$T(r,f) = 2\overline{N}_{=2}\left(r, \frac{1}{f - \beta_1}\right) + S(r,f) = 2\overline{N}_{=2}\left(r, \frac{1}{f - \beta_2}\right) + S(r,f). \tag{3.17}$$

From Lemma 2.3, it is easy to see that

$$\Delta(f) = \left(\frac{f - \beta_1}{\beta_2 - \beta_1}\right)'(\beta_2 - \beta_1)^2 = \left(\frac{f - \beta_2}{\beta_2 - \beta_1}\right)'(\beta_2 - \beta_1)^2. \tag{3.18}$$

By (3.17) and (3.18), if any zero of $f - \beta_j$ $(j = 1, 2)$ has multiplicity two, then it must be a zero of $\Delta^2(f)$ with multiplicity two. Thus, from (3.15) we get $N(r,h) = S(r,f)$. Together with (3.16) we have

$$T(r,h) = S(r,f). \tag{3.19}$$

We consider the transformation

$$F = \frac{f - \beta_1}{\beta_2 - \beta_1}. \tag{3.20}$$

Substitution of expressions (3.20) and (3.18) into (3.15) now tells us that

$$\frac{h}{(\beta_2-\beta_1)^2}=\frac{F'^2}{F(F-1)}=\left(\frac{F'}{F}\right)^2\left(\frac{F}{F-1}\right)$$

or

$$H=L^2+\delta, \tag{3.21}$$

where $\delta=\dfrac{-h}{(\beta_2-\beta_1)^2}$, $H=\dfrac{\delta}{F}$ and $L=\dfrac{F'}{F}$. Differentiating (3.21) once we obtain

$$H'=2LL'+\delta'. \tag{3.22}$$

On the other hand differentiating $H=\dfrac{\delta}{F}$ we find that

$$H'=\frac{\delta}{F}\left(\frac{\delta'}{\delta}-L\right); \tag{3.23}$$

and eliminating H' between (3.22) and (3.23) leads to

$$2LL'=-L^3+\frac{\delta'}{\delta}L^2-\delta L. \tag{3.24}$$

If $L\equiv 0$, then F is a constant and so $T(r,f)=S(r,f)$ a contradiction. Therefore $L\not\equiv 0$, and (3.24) becomes

$$2L'=-L^2+\frac{\delta'}{\delta}L-\delta. \tag{3.25}$$

Let z_1 be a zero of $f-\beta_1$ with multiplicity two and $\beta_i(z_1)\neq 0,\infty$ $(i=1,2)$, $(\beta_2-\beta_1)(z_1)\neq 0,\infty$. Then the Taylor expansion of F about z_1 is

$$F(z)=a_2(z-z_1)^2+a_3(z-z_1)^3+a_4(z-z_1)^4+\cdots,\quad a_2\neq 0.$$

Hence

$$L=\frac{F'}{F}=2(z-z_1)^{-1}+\frac{a_3}{a_2}+\left[2\frac{a_4}{a_2}-\left(\frac{a_3}{a_2}\right)^2\right](z-z_1)+O((z-z_1)^2), \tag{3.26}$$

$$L'=-2(z-z_1)^{-2}+2\frac{a_4}{a_2}-\left(\frac{a_3}{a_2}\right)^2+O((z-z_1)) \tag{3.27}$$

and

$$L^2=4(z-z_1)^{-2}+4\frac{a_3}{a_2}(z-z_1)^{-1}+8\frac{a_4}{a_2}-3\left(\frac{a_3}{a_2}\right)^2+O((z-z_1)). \tag{3.28}$$

Substituting (3.26), (3.27) and (3.28) into (3.25) gives

$$\frac{a_3}{a_2}=\frac{\delta'}{2\delta}\quad\text{and}\quad\frac{a_4}{a_2}=\frac{7}{48}\left(\frac{\delta'}{\delta}\right)^2-\frac{1}{12}\delta. \tag{3.29}$$

From (3.27) we find that

$$\frac{L''}{L'} = -2(z-z_1)^{-1} - \left[2\frac{a_4}{a_2} - \left(\frac{a_3}{a_2}\right)^2\right](z-z_1) + O((z-z_1)^2).$$

Hence

$$\left(\frac{L''}{L'}\right)' = 2(z-z_1)^{-2} - \left[2\frac{a_4}{a_2} - \left(\frac{a_3}{a_2}\right)^2\right] + O((z-z_1)) \tag{3.30}$$

and

$$\left(\frac{L''}{L'}\right)^2 = 4(z-z_1)^{-2} + 4\left[2\frac{a_4}{a_2} - \left(\frac{a_3}{a_2}\right)^2\right] + O((z-z_1)). \tag{3.31}$$

We set

$$\eta = \left(\frac{L''}{L'}\right)^2 - 2\left(\frac{L''}{L'}\right)'. \tag{3.32}$$

From (3.25) it is easy to see that $N(r,L') = 2N(r,L) + S(r,f)$. This implies that $N(r,L) = \overline{N}(r,L) + S(r,f)$. From this we conclude that

$$N_{(2}(r,L) = S(r,f). \tag{3.33}$$

Substituting (3.30) and (3.31) into (3.32) and then using (3.29) we arrive at

$$\eta = \frac{1}{4}\left(\frac{\delta'}{\delta}\right)^2 - \delta + O((z-z_1)).$$

If $\eta \not\equiv \frac{1}{4}\left(\frac{\delta'}{\delta}\right)^2 - \delta$, then

$$\begin{aligned}\overline{N}_{=2}\left(r,\frac{1}{f-\beta_1}\right) &\le N\left(r,\frac{1}{\eta - \frac{1}{4}\left(\frac{\delta'}{\delta}\right)^2 + \delta}\right) + S(r,f).\\ &\le T\left(r,\eta - \frac{1}{4}\left(\frac{\delta'}{\delta}\right)^2 + \delta\right) + S(r,f) \le T(r,\eta) + S(r,f).\end{aligned}$$

Combining this with (3.33) and Lemma 2.4 yields

$$\overline{N}_{=2}\left(r,\frac{1}{f-\beta_1}\right) \le 2\overline{N}\left(r,\frac{1}{L'}\right) + S(r,f).$$

Together with (3.17) we have $T(r,f) \le 4\overline{N}\left(r,\frac{1}{L'}\right) + S(r,f)$. This is (1.4). In the following, we assume that

$$\eta \equiv \frac{1}{4}\left(\frac{\delta'}{\delta}\right)^2 - \delta. \tag{3.34}$$

On the other hand, differentiating (3.25) twice we obtain

$$2L'' = L\left[-2L' + \left(\frac{\delta'}{\delta}\right)'\right] + \frac{\delta'}{\delta}L' - \delta' \tag{3.35}$$

and

$$2L''' = -2L'^2 - L\left[2L'' - \left(\frac{\delta'}{\delta}\right)''\right] + 2\left(\frac{\delta'}{\delta}\right)'L' + \frac{\delta'}{\delta}L'' - \delta''. \tag{3.36}$$

Suppose that z_2 is a zero of $f - \beta_2$ with multiplicity two and $\beta_i(z_2) \neq 0, \infty$ $(i = 1, 2)$, $(\beta_2 - \beta_1)(z_2) \neq 0, \infty$. Then $F(z) = 1 + O((z - z_2)^2)$ and so $L(z_2) = 0$. Thus we deduce from (3.25), (3.35), (3.36) and (3.32) that

$$\eta = 3\left(\frac{L''}{L'}\right)^2 - 2\frac{L'''}{L'} = \frac{29}{4}\left(\frac{\delta'}{\delta}\right)^2 - 4\frac{\delta''}{\delta} - \delta + O((z - z_2)).$$

As in the above discussion, we find that either (1.4) holds or

$$\eta \equiv \frac{29}{4}\left(\frac{\delta'}{\delta}\right)^2 - 4\frac{\delta''}{\delta} - \delta. \tag{3.37}$$

If we now eliminate η between (3.37) and (3.34) we arrive at

$$7\frac{\delta'^2}{\delta} = 4\delta''. \tag{3.38}$$

If $\delta' \not\equiv 0$, then from (3.38) we have $7\frac{\delta'}{\delta} = 4\frac{\delta''}{\delta'}$. By integrating once,

$$\delta^7 = c\delta'^4. \tag{3.39}$$

where c is a nonzero constant. From this we conclude that

$$3m(r, \delta) = m\left(r, c\left(\frac{\delta'}{\delta}\right)^4\right) = 4m\left(r, \frac{\delta'}{\delta}\right) + O(1) = 4S(r, \delta) + O(1) = S(r, \delta). \tag{3.40}$$

It follows from (3.39) that if z_∞ is a pole of δ with multiplicity p (≥ 1), then $O((z - z_\infty)^{-7p}) = O((z - z_\infty)^{-4(p+1)})$. Hence $p = \frac{4}{3}$, which contradicts with p being an integer. Therefore $N(r, \delta) \equiv 0$. Together with (3.40) we get $T(r, \delta) = S(r, \delta)$ and this gives us δ is a nonzero constant. That is $\delta' \equiv 0$. This and (3.25) imply that $\frac{L'}{L - c} - \frac{L'}{L + c} = -c$, where $c = i\sqrt{\delta}$. By integration, we obtain $L = \frac{-ce^{cz}}{B - e^{cz}} + \frac{cBe^{-cz}}{1 - Be^{-cz}}$. Since $L = \frac{F'}{F}$, it follows from the integration that $F(z) = Ae^{-cz}(1 - Be^{cz})^2$, where A and B are nonzero constants. From this and (3.20) we have

$$f(z) - \beta_1(z) = A(\beta_2(z) - \beta_1(z))e^{-cz}(1 - Be^{cz})^2. \tag{3.41}$$

We rewrite this in the form

$$f(z) - \beta_2(z) = A(\beta_2(z) - \beta_1(z))e^{-cz}\left[1 - \left(2B + \frac{1}{A}\right)e^{cz} + B^2 e^{2cz}\right] \tag{3.42}$$

which, in view of (3.17), leads to $A = \dfrac{-1}{4B}$. Substituting this into (3.41) and (3.42) we arrive at (1.2) and (1.3) respectively. This completes the proof of Theorem 1.4. □

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