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WEIERSTRASS SEMIGROUPS ON DOUBLE COVERS OF PLANE CURVES OF DEGREE 5

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Abstract

We investigate Weierstrass semigroups of ramification points on double covers of plane curves of degree d. Using the results we determine all the Weierstrass semigroups in the case $d = 5$ when the genus of the covering curve is greater than 17 and the ramification point is on a non-ordinary flex.

1. Introduction

Let C be a smooth irreducible curve of genus g, where a *curve* means a projective curve over an algebraically closed field of characteristic 0. For a point P of C we define the *Weierstrass semigroup* $H(P)$ of P as follows:

$$H(P) = \{n \in \mathbf{N}_0 \mid \text{there is a rational function } f \text{ on } C \text{ such that } (f)_\infty = nP\},$$

where $\mathbf{N}_0$ is the additive monoid of non-negative integers and $(f)_\infty$ means the polar divisor of f. Then $H(P)$ is a *numerical semigroup of genus g*, which means a submonoid of $\mathbf{N}_0$ whose complement is a finite set with cardinality g. The genus of a numerical semigroup H is denoted by $g(H)$. For a numerical semigroup H we denote by $d_2(H)$ the set consisting of the elements h for $2h \in H$, which is a numerical semigroup. For positive integers $a_1, \dots, a_s$ we denote by $\langle a_1, \dots, a_s \rangle$ the additive monoid generated by $a_1, \dots, a_s$.

We will study about the numerical semigroups H which are the Weierstrass semigroups of ramification points on double covers of smooth plane curves of degree d. In this paper such a numerical semigroup H is said to be *of double covering type of a plane curve*, which is abbreviated to *DCP*. In this case, $d_2(H)$ is the Weierstrass semigroup of a point on a smooth plane curve of degree d. If $d_2(H)$ is the Weierstrass semigroup of a point on a smooth plane curve of degree $d \leqq 3$, i.e., $d_2(H) = \mathbf{N}_0$ or $\langle 2, 3 \rangle$, then we can show that H is DCP (for

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example, see [8]). In the case $d = 4$, i.e., $d_2(H) = \langle 3,4\rangle, \langle 3,5,7\rangle$ or $\langle 4,5,6,7\rangle$ the papers [9], [4], [5] and [6] show that every numerical semigroup H with $g(H) \geqq 6$ is DCP except $H = \langle 8,10,12,14,n,n+4\rangle$ with odd $n \geqq 9$, $H = \langle 7 \to 10, 12\rangle$ and $H = \langle 5,7,8\rangle$. The excluded semigroups are not DCP.

Let C be a smooth plane curve and P its point. Let Z be a plane curve. We denote by $C.Z$ the intersection divisor of C with Z. Moreover, let $\mathrm{ord}_P(C.Z)$ be the multiplicity of $C.Z$ at P. We denote by T_P the tangent line at P on C. We note the following:

i) If P is a total flex on a smooth plane curve C of degree 5, i.e., $\mathrm{ord}_P\, C.T_P = 5$, then $H(P) = \langle 4,5\rangle$.

ii) If P is a point with $\mathrm{ord}_P\, C.T_P = 4$ on a smooth plane curve C of degree 5, then $H(P) = \langle 4,7,10,13\rangle$.

The following is the main result of this article:

Main Theorem. *Let H be a numerical semigroup of genus* $\geqq 18$.

i) *If $d_2(H) = \langle 4,5\rangle$, then H is DCP.*

ii) *Assume that $d_2(H) = \langle 4,7,10,13\rangle$. If H is distinct from $2d_2(H) + \langle n,n+4\rangle$ and $2d_2(H) + \langle n,n+12\rangle$ with odd $n \geqq 13$, then it is DCP. The excluded semigroups are not DCP.*

Corollary 2.7 in Section 2 shows i) in the above theorem. Corollary 3.2 in Section 3 and Theorem 4.2 in Section 4 mean ii) in Main Theorem.

2. Ramification points over total flexes

A numerical semigroup H is called an *a-semigroup* if the least positive integer in H is a. For an a-semigroup H we set $S(H) = \{a, s_1, \ldots, s_{a-1}\}$ where $s_i = \min\{h \in H \mid h \equiv i \bmod a\}$, which is called the *standard basis* for H. Let d be an integer which is larger than 2. In this section we set

$$H_d = \langle d-1, d\rangle \quad \text{and} \quad s_i = id \quad \text{for } 1 \leqq i \leqq d-2.$$

Then we have $S(H_d) = \{d-1, s_1, s_2, \ldots, s_{d-2}\}$.

First we will show that eight kinds of numerical semigroups H with $d_2(H) = \langle d-1, d\rangle$ are DCP. We use the following lemma when we calculate the genera $g(H)$ of such numerical semigroups H.

Lemma 2.1. *Let m and l be positive integers with $2 \leqq m \leqq d-1$ and $l \leqq ((d-m)d)/(d-1)$. Let n be an odd number with $n \geqq d(d-2)$. Set*

$$H = 2H_d + \langle n, n + 2s_{d-m} - 2l(d-1)\rangle.$$

Then

$$H = (2H_d + n\mathbf{N}_0) \cup \{n + s_{d-i} - 2j(d-1) \mid 2 \leqq i \leqq m, 1 \leqq j \leqq l\},$$

which implies that $g(H) = (d-1)(d-2) + (n-1)/2 - l(m-1)$.

Proof. By the assumption on n and Remark 2.1 in [7] we have

$$S(2H_d + n\mathbf{N}_0) = \{2(d-1), 2s_1, \dots, 2s_{d-2}, n, n+2s_1, \dots, n+2s_{d-2}\}.$$

Assume that $n + 2s_{d-m-1} - 2(d-1)$ belongs to H. Then the element $s \in S(H)$ with $s \equiv n + 2d(d-m-1) \bmod 2(d-1)$ is written by

$$s = n + 2s_{d-m} - 2l(d-1) + t,$$

where t is the minimum in $H = 2H_d + \langle n, n + 2s_{d-m} - 2l(d-1)\rangle$ with $t \equiv 2d(d-2) \bmod 2(d-1)$. Since

$$2(n + 2s_{d-m} - 2l(d-1)) - 2s_{d-2} = 2(n - s_{d-2}) + 4(s_{d-m} - l(d-1)) \geqq 0$$

by the assumptions $n \geqq d(d-2)$ and $l \leqq ((d-m)d)/(d-1)$, we have $t = 2s_{d-2}$. Hence, we get $n + 2s_{d-m-1} - 2(d-1) \geqq n + 2s_{d-m} - 2l(d-1) + 2s_{d-2}$, which implies that $(l-1)(d-1) \geqq (d-1)d$. Thus, we have $l \geqq d+1$. Then the assumption on l induces $d + 1 \leqq l \leqq ((d-m)d)/(d-1)$, which implies that

$$d^2 - 1 < (d-m)d \leqq (d-2)d = d^2 - 2d.$$

This is a contradiction. Therefore, we obtain $n + 2s_{d-m-1} - 2(d-1) \notin H$.

Moreover, we will show that $n + 2s_{d-2} - 2(l+1)(d-1) \notin H$. Assume that $n + 2s_{d-2} - 2(l+1)(d-1) \in H$. Then the element $s \in S(H)$ with $s \equiv n + 2d(d-2) \bmod 2(d-1)$ is written by

$$s = n + 2s_{d-m} - 2l(d-1) + t,$$

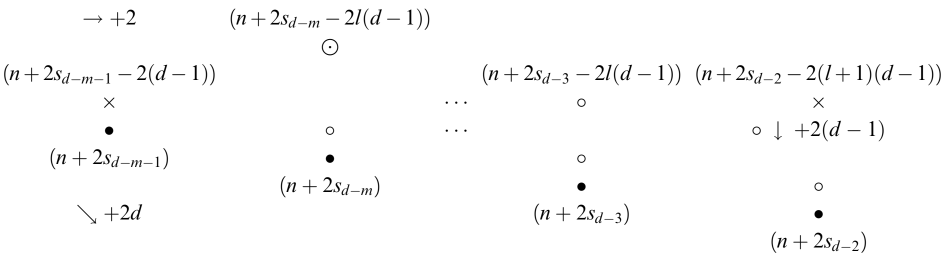
where t is the minimum in $H = 2H_d + \langle n, n + 2s_{d-m} - 2l(d-1)\rangle$ with $t \equiv 2d(m-2) \bmod 2(d-1)$. Since

$$\begin{aligned} &2(n + 2s_{d-m} - 2l(d-1)) - 2s_{m-2} \\ &\quad = 2n - 4l(d-1) + 2d(2d - 3m + 2) \geqq 2d(d-m) \geqq 2d > 0 \end{aligned}$$

by the assumptions $n \geqq d(d-2)$, $l \leqq ((d-m)d)/(d-1)$ and $m \leqq d-1$, we have $t = 2s_{m-2}$. Hence, we get

$$n + 2s_{d-2} - 2(l+1)(d-1) \geqq n + 2s_{d-m} - 2l(d-1) + 2s_{m-2},$$

which implies that $1 \geqq d$. This is a contradiction.



The elements of $H = 2H_d + \langle n, n + 2s_{d-m} - 2l(d-1)\rangle$

Let $i \geqq 2$ and $j \geqq l+1$. Then

$$n + 2s_{d-2} - 2(l+1)(d-1) - (n + 2s_{d-i} - 2j(d-1))$$
$$= 2(i-2)d + 2(j-l-1)(d-1) \in 2H_d.$$

Since $n + 2s_{d-2} - 2(l+1)(d-1) \notin H$, we must have $n + 2s_{d-i} - 2j(d-1) \notin H$. Let $i \geqq m+1$ and $j \geqq 1$. Then

$$n + 2s_{d-m-1} - 2(d-1) - (n + 2s_{d-i} - 2j(d-1))$$
$$= 2(i-m-1)d + 2(j-1)(d-1) \in 2H_d.$$

Since $n + 2s_{d-m-1} - 2(d-1) \notin H$, we obtain $n + 2s_{d-i} - 2j(d-1) \notin H$. Hence, the largest odd number n' in the complement of H is $n + 2s_{d-m-1} - 2(d-1)$ or $n + 2s_{d-2} - 2(l+1)(d-1)$ and $g(H + \langle n' \rangle) = g(H) - 1$, which follows from the above figure. Thus, we have

$$H = (2H_d + n\mathbf{N}_0) \cup \{n + 2s_{d-i} - 2j(d-1) \mid 2 \leqq i \leqq m, 1 \leqq j \leqq l\},$$

because $H \backslash 2H_d$ contains no even number. Since we have $g(2H_d + n\mathbf{N}_0) = (d-1)(d-2) + (n-1)/2$, we get our desired result. □

In the rest of this section we are in the following situation: Let C be a smooth plane curve of degree $d \geqq 5$ with a point P satisfying $\mathrm{ord}_P\, C.T_P = d$, i.e., $H(P) = H_d$. We state the following Namba's famous lemma (see Lemma 2.3.2 in [12]), since it plays an important role in the calculation of $\mathrm{ord}_P(C_1.C_2)$, which is the intersection multiplicity of plane curves C_1 and C_2 at P.

Lemma 2.2. *Let C_1, C_2 and C_3 be plane curves. Assume that C_1 is irreducible and is neither a component of C_2 nor of C_3. Let P be a smooth point of C_1. Then*

$$\mathrm{ord}_P(C_2.C_3) \geqq \min\{\mathrm{ord}_P(C_1.C_2), \mathrm{ord}_P(C_1.C_3)\}.$$

The following lemma is useful for determining the Weierstrass semigroup of a ramification point on a double cover of a plane curve.

Lemma 2.3. *Let C_{d-3} be a plane curve of degree $d-3$ such that $\mathrm{ord}_P(C_{d-3}.C) \geqq (d-3-l)d$ with an integer $l \leqq d-4$. Then $C_{d-3} = T_P^{d-3-l} C_l$, where C_l is a plane curve of degree l, which implies that $\mathrm{ord}_P(C_l.C) \geqq \mathrm{ord}_P(C_{d-3}.C) - (d-3-l)d$.*

Proof. We have $\mathrm{ord}_P(T_P^{d-3-l}.C) = (d-3-l)d$. Hence, by the assumption and Lemma 2.2 we get $\mathrm{ord}_P(C_{d-3}.T_P^{d-3-l}) \geqq (d-3-l)d$. Thus, we have $C_{d-3} = T_P C_{d-4}$, where C_{d-4} is a plane curve of degree $d-4$. Moreover, we get

$$\mathrm{ord}_P(C_{d-4}.C) \geqq (d-4-l)d \quad \text{and} \quad \mathrm{ord}_P(T_P^{d-4-l}.C) = (d-4-l)d,$$

which implies that $\mathrm{ord}_P(C_{d-4}.T_P^{d-4-l}) \geqq (d-4-l)d$. Hence, we get $C_{d-3} = T_P^2 C_{d-5}$ if $d-4-l \geqq 1$, where C_{d-5} is a plane curve of degree $d-5$. Using this method successively we get $C_{d-3} = T_P^{d-3-l} C_l$, where C_l is a plane curve of degree l. □

To prove that a numerical semigroup is DCP we use the following theorem many times, which is stated in Theorem 2.2 of [9].

Theorem 2.4. *Let H be a numerical semigroup. Set*

$$n = \min\{h \in H \mid h \text{ is odd}\}.$$

Then

$$g(H) = 2g(d_2(H)) + (n-1)/2 - r$$

with some non-negative integer r (for example, see Lemma 3.1 in [4]). Assume that $d_2(H)$ is Weierstrass. Take a pointed curve (C, P) with $H(P) = d_2(H)$. Let $Q_1, \dots, Q_r$ be points of C different from P with $h^0(Q_1 + \cdots + Q_r) = 1$. Moreover, assume that H has an expression

$$H = 2d_2(H) + \langle n, n+2l_1, \dots, n+2l_s \rangle$$

of generators with positive integers $l_1, \dots, l_s$ such that

$$h^0(l_i P + Q_1 + \cdots + Q_r) = h^0((l_i - 1)P + Q_1 + \cdots + Q_r) + 1$$

for all i. If the divisor $nP - 2Q_1 - \cdots - 2Q_r$ is linearly equivalent to some reduced divisor not containing P, then there is a double covering $\pi : \tilde{C} \to C$ with a ramification point $\tilde{P}$ over P satisfying $H(\tilde{P}) = H$.

We may replace the assumption in Theorem 2.2 of [9] that the complete linear system $|nP - 2Q_1 - \cdots - 2Q_r|$ is base point free by the above assumption that the divisor $nP - 2Q_1 - \cdots - 2Q_r$ is linearly equivalent to some reduced divisor not containing P, because the same proof as in Theorem 2.2 of [9] works well under our assumption.

Theorem 2.5. *Let n be an odd number with $n \geqq d(d-2)$. Let H_d denote $H(P) = \langle d-1, d \rangle$. Let H be a numerical semigroup which is one of the following:*

(i) *$2H_d + \langle n, n+2t_1 \rangle$ with $t_1 = s_{d-2} - l(d-1)$ where l is a positive integer with $l \leqq d-2$ and $n \geqq (d-1)(d-2) + 1 + 2l$.*

(ii) *$2H_d + \langle n, n+2t_1 \rangle$ with $t_1 = s_{d-m} - (d-1)$ where m is an integer with $2 \leqq m \leqq d-1$ and $n \geqq (d-1)(d-2) - 1 + 2m$.*

(iii) *$2H_d + \langle n, n+2t_1 \rangle$ with $t_1 = s_{d-m} - 2(d-1)$ where m is an integer with $2 \leqq m \leqq d-2$ and $n \geqq (d-1)(d-2) - 3 + 4m$.*

(iv) *$2H_d + \langle n, n+2t_1, n+2t_2 \rangle$ with $t_1 = s_{d-2} - 2(d-1)$ and $t_2 = s_{d-m} - (d-1)$ where m is an integer with $3 \leqq m \leqq d-1$ and $n \geqq (d-1)(d-2) + 1 + 2m$.*

(v) *$2H_d + \langle n, n+2t_1, n+2t_2 \rangle$ with $t_1 = s_{d-2} - l(d-1)$ and $t_2 = s_{d-3} - (d-1)$ where l is an integer with $3 \leqq l \leqq d-2$ and $n \geqq (d-1)(d-2) + 3 + 2l$.*

(vi) $2H_d + \langle n, n+2t_1, n+2t_2 \rangle$ *with* $t_1 = s_{d-4} - (d-1)$ *and* $t_2 = s_{d-2} - 3(d-1)$ *where* $n \geqq (d-1)(d-2) + 11$.

(vii) $2H_d + \langle n, n+2t_1, n+2t_2 \rangle$ *with* $t_1 = s_{d-3} - 2(d-1)$ *and* $t_2 = s_{d-2} - 3(d-1)$ *where* $n \geqq (d-1)(d-2) + 11$.

(viii) $2H_d + \langle n, n+2t_1, n+2t_2 \rangle$ *with* $t_1 = s_{d-4} - (d-1)$ *and* $t_2 = s_{d-3} - 2(d-1)$ *where* $n \geqq (d-1)(d-2) + 11$.

Then H is DCP.

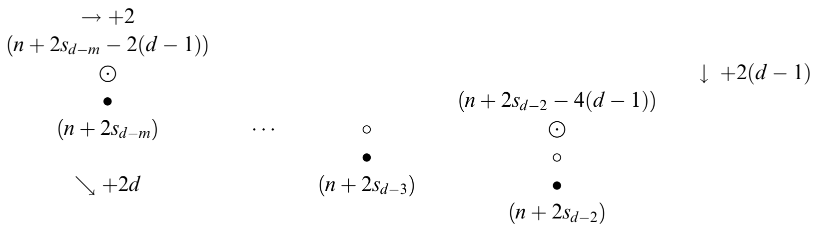
Proof. To prove that H is DCP we use Theorem 2.2 in [9]. We show step by step that H satisfies the conditions of the theorem in [9].

STEP 1. By Lemma 2.1 we have

$$g(H) = (d-1)(d-2) + \frac{n-1}{2} - r$$

where (i) $r = l$, (ii) $r = m-1$ and (iii) $r = 2(m-1)$.

(iv) We note that $n + 2s_{d-2} - 4(d-1)$ is the largest number in the complement of the semigroup $2\langle d-1, d \rangle + \langle n, n+2s_{d-m} - 2(d-1) \rangle$ in $\mathbf{N}_0$ (see the figure below).



The elements of $H = 2\langle d-1, d \rangle + \langle n, n+2s_{d-2} - 4(d-1), n+2s_{d-m} - 2(d-1) \rangle$

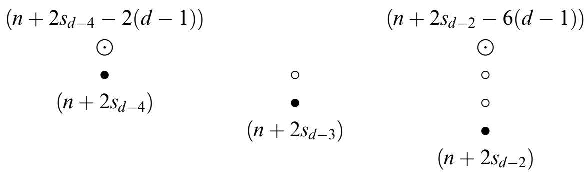
Using Lemma 2.1 we get $r = m$.

(v) We have $r = l+1$. In fact, we note that

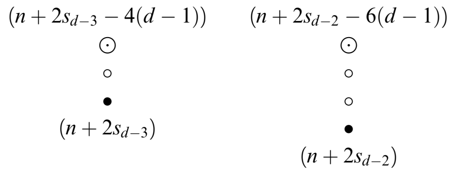
$$n + 2s_{d-3} - 2(d-1) - (n + 2s_{d-2} - 2(l+1)(d-1)) = 2((l-1)(d-1) - 1) > 0,$$

which implies that $n + 2s_{d-3} - 2(d-1)$ is the largest number in the complement of the semigroup $2\langle d-1, d \rangle + \langle n, n+2s_{d-2} - 2l(d-1) \rangle$ in $\mathbf{N}_0$. Using Lemma 2.1 we get $r = l+1$.

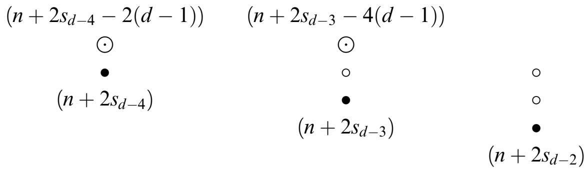
(vi) We have $r = 5$. In fact, we have the following figure:



(vii) We have $r = 5$. In fact, we have the following figure:



(viii) We have $r = 5$. In fact, we have the following figure:



Step 2. Let E be a divisor of degree $n - 2r$ on a smooth plane curve of degree d. By the assumption on n in each case we have $\deg E \geqq (d-2)(d-1) + 1$, which implies that E is very ample.

Step 3. In each case we choose r points $Q_1, \dots, Q_r$ of C with $Q_i \neq P$ for all i satisfying the equality

$$h^0(Q_1 + \cdots + Q_r) = 1. \tag{1}$$

To choose the points $Q_1, \dots, Q_r$ satisfying the equality (1) we use the following: Let $P_1, \dots, P_k$ $(k \leqq d)$ be points on a smooth plane curve of degree $d \geqq 4$. Then $h^0(P_1 + \cdots + P_k) = 1$ unless $k \geqq d - 1$ and at least $d - 1$ points of $P_1, \dots, P_k$ are collinear. This follows from the fact that a smooth plane curve of degree $d \geqq 4$ is $(d-1)$-gonal and has a unique g_d^2, which is cut out by lines. The latter fact is called Namba's Theorem (see [12]).

(i) Let us take a line L with $L \not\ni P$. We set $L.C = Q_1 + \cdots + Q_d$ with $Q_i \neq P$ for all i. Since C is $(d-1)$-gonal (see [1]), we have the equality (1) because $r = l \leqq d - 2$.

(ii) Let L be a line through P distinct from the tangent line T_P. We set $L.C = P + Q_1 + \cdots + Q_{d-1}$ with $Q_i \neq P$, all i. Then we have the equality (1) because $r = m - 1 \leqq d - 2$.

(iii) Take two distinct lines L_1 and L_2 through P different from T_P. We set

$$L_1.C = P + R_1 + \cdots + R_{d-1} \quad \text{and} \quad L_2.C = P + S_1 + \cdots + S_{d-1}.$$

Then

$$h^0(R_1 + \cdots + R_{m-1} + S_1 + \cdots + S_{m-1}) = 1. \tag{2}$$

Indeed, we have

$$|R_1 + \cdots + R_{m-1} + S_1 + \cdots + S_{m-1}| = |L_1 + L_2 - E|,$$

where E is an effective divisor of degree $\geqq 6$. It is known that a complete linear system of degree at most $2d-5$ on a smooth plane curve of degree $d \geqq 4$ is zero-dimensional or empty unless its free part is a g^1_{d-1} or a g^2_d (and hence it contains at least $d-1$ collinear points). This fact follows from Theorem 3.1 in [3]. Hence we get the equality (2). We set $Q_i = R_i$ for $i = 1, \dots, m-1$ and $Q_{m-1+i} = S_i$ for $i = 1, \dots, m-1$. Hence we get the equality (1).

(iv) Let L be a line through P, distinct from T_P. We set $L.C = P + R_1 + \cdots + R_{d-1}$. For $i = 1, \dots, m-1$ we set $Q_i = R_i$ and take a point $Q_m \in C$ which is not in L. If $r = m \leqq d-2$, we get the equality (1). Hence, we may assume that $m = d-1$. Since $Q_1, \dots, Q_{d-1}$ are not collinear, we get the equality (1).

(v) Let L be a line not passing through P with $L.C = R_1 + \cdots + R_d$. We set $Q_i = R_i$ for $i = 1, \dots, l$. Choose $Q_{l+1} \in C$ with $Q_{l+1} \neq P$ and $Q_{l+1} \notin L$. By the same way as in (iv) we get the equality (1).

(vi) Let L be a line through P with $L \neq T_P$ and $L.C = P + R_1 + \cdots + R_{d-1}$. We set $Q_1 = R_1$, $Q_2 = R_2$ and $Q_3 = R_3$. Take two distinct points Q_4 and Q_5 of C which do not belong to the line L such that the line L_{Q_4, Q_5} through Q_4 and Q_5 does not contain P. It suffices to show the equality in the case $d = 5, 6$. Let $d = 5$. Take a curve C_2 of degree 2 with $C_2.C \geqq Q_1 + \cdots + Q_5$. Since we have $L.C \geqq Q_1 + Q_2 + Q_3$, we get $C_2.L \geqq Q_1 + Q_2 + Q_3$. Hence, we have $C_2 = LL_1$ where L_1 is a line. Since the line L contains neither Q_4 nor Q_5, a line L_1 must contain Q_4 and Q_5. Hence, L_1 is uniquely determined. Thus, C_2 is uniquely determined. Therefore, we get $h^0(Q_1 + \cdots + Q_5) = 1$. Let $d = 6$. The points $Q_1, \dots, Q_5$ are not collinear. Hence we obtain the equality (1).

(vii) Let $Q_1, \dots, Q_5$ be general points. Then we get the equality (1).

(viii) Let L_1 be a line through P with $L_1 \neq T_P$ and $L_1.C = P + R_1 + \cdots + R_{d-1}$. We set $Q_1 = R_1$ and $Q_2 = R_2$. Take a point Q_3 of C which does not belong to the line L_1. Let L_2 be the line through Q_3 and P. We set $L_2.C = P + Q_3 + S_1 + \cdots + S_{d-2}$. Let $Q_4 = S_1$ and $Q_5 = S_2$. Then we have $Q_4, Q_5 \notin \{Q_1, Q_2\}$ and $L_2.C \ngeqq Q_i$ for $i = 1, 2$. It suffices to show the equality in the case $d = 5, 6$. Let $d = 5$. Let C_2 be a conic with $C_2.C \geqq Q_1 + \cdots + Q_5$. Since $L_2.C \geqq Q_3 + Q_4 + Q_5$, we obtain $C_2 = L_2 L$ where L is a line. Now we have $Q_1 + \cdots + Q_5 \leqq L_2.C + L.C$, which implies that $L.C \geqq Q_1 + Q_2$. Hence, we get $L = L_1$. Thus, a conic C_2 is uniquely determined. Let $d = 6$. The points $Q_1 \dots, Q_5$ are not collinear. Thus, we have the equality (1).

Step 4. We set $D_r = Q_1 + \cdots + Q_r$. In this step C_i means a plane curve of degree i. We will show that

$$h^0(K - t_i P - D_r) = h^0(K - (t_i - 1)P - D_r)$$

for $i = 1, 2$ where K is a canonical divisor on C. Let $C_{d-3}.C \geqq (t_i - 1)P + D_r$. It suffices to show that $C_{d-3}.C \geqq t_i P + D_r$ because of the fact that $H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(d-3)) \simeq H^0(C, \mathcal{O}_C(K))$.

(i) By Lemma 2.3 we obtain $C_{d-3} = T_P^{d-2-l} C_{l-1}$. Hence, we have

$$T_P^{d-2-l}.C + C_{l-1}.C \geqq (t_1 - 1)P + D_r = (d-2-l)dP + (l-1)P + D_r,$$

which implies that $C_{l-1}.C \geqq (l-1)P + D_r$. Thus, we get $L.C_{l-1} \geqq D_r$ where L is as in Step 3. In view of $r = l$ we get $C_{l-1} = LC_{l-2}$, which implies that $C_{d-3} = T_P^{d-2-l} LC_{l-2}$. Moreover, we obtain

$$(d-2-l)dP + Q_1 + \cdots + Q_d + C_{l-2}.C \geqq ((d-2-l)d + l - 1)P + D_r.$$

Hence, we get $C_{l-2}.C \geqq (l-1)P$. Since $T_P.C = dP$ and $d - 3 \geqq l - 1$, we have $C_{l-2}.T_P \geqq (l-1)P$, which implies that $C_{l-2} = T_P C_{l-3}$. Thus, we get $C_{d-3} = T_P^{d-1-l} LC_{l-3}$. Therefore, we have

$$\begin{aligned} C_{d-3}.C &= T_P^{d-1-l}.C + L.C + C_{l-3}.C \\ &\geqq (d-1-l)dP + D_r > ((d-2-l)d + l)P + D_r. \end{aligned}$$

(ii) By Lemma 2.3 we get $C_{d-3} = T_P^{d-m-1} C_{m-2}$. Hence, we have

$$(d-m-1)dP + C_{m-2}.C = (d-m-1)T_P.C + C_{m-2}.C \geqq (d-m-1)dP + D_r,$$

which implies that $C_{m-2}.C \geqq D_r$. Since $L.C \geqq D_r$, we have $L.C_{m-2} \geqq D_r$, which implies that $C_{m-2} = LC_{m-3}$. Thus, we obtain $C_{d-3} = T_P^{d-m-1} LC_{m-3}$. Hence, we have

$$C_{d-3}.C = (d-m-1)dP + L.C + C_{m-3}.C \geqq (d-m-1)dP + P + D_r = t_1 P + D_r.$$

(iii) By Lemma 2.3 we obtain $C_{d-3} = T_P^{d-2-m} C_{m-1}$, which implies that $C_{m-1}.C \geqq P + D_r$. On the other hand, we have

$$L_1.C \geqq P + Q_1 + \cdots + Q_{m-1} \quad \text{and} \quad L_2.C \geqq P + Q_m + \cdots + Q_{2(m-1)},$$

from which we get

$$C_{m-1}.L_1 \geqq P + Q_1 + \cdots + Q_{m-1} \quad \text{and} \quad C_{m-1}.L_2 \geqq P + Q_m + \cdots + Q_{2(m-1)}.$$

Hence, we obtain $C_{m-1} = L_1 L_2 C_{m-3}$. Thus, we get

$$\begin{aligned} C_{d-3}.C = T_P^{d-2-m} L_1 L_2 C_{m-3}.C &\geqq T_P^{d-2-m}.C + L_1.C + L_2.C \\ &\geqq ((d-2-m)d + 2)P + D_r = t_1 P + D_r. \end{aligned}$$

(iv) Let $t_i = t_2$. By Lemma 2.3 we obtain $C_{d-3} = T_P^{d-m-1} C_{m-2}$, which implies that $C_{m-2}.C \geqq D_r$. Hence, we get $C_{m-2}.L \geqq Q_1 + \cdots + Q_{m-1}$, which implies that $C_{m-2} = LC_{m-3}$. Thus, we have $C_{d-3} = T_P^{d-m-1} LC_{m-3}$. Hence, we obtain

$$C_{d-3}.C = T_P^{d-m-1}.C + L.C + C_{m-3}.C \geqq ((d-m-1)d + 1)P,$$

which implies that $C_{d-3}.C \geqq ((d-m-1)d + 1)P + D_r = t_2 P + D_r$.

We have

$$h^0(K - (t_1 - 1)P - Q_1) = h^0((d-1)P - Q_1) = 1.$$

Hence, there is a unique effective divisor E which is linearly equivalent to $(d-1)P - Q_1$. Then E should be $Q_2 + \cdots + Q_{m-1} + R_m + \cdots + R_{d-1}$, because we have

$$dP = T_P.C \sim L.C = P + Q_1 + Q_2 + \cdots + Q_{m-1} + R_m + \cdots + R_{d-1}.$$

Since Q_m is different from $Q_1, \dots, Q_{m-1}$, $R_m, \dots, R_{d-1}$, we get

$$h^0((d-1)P - Q_1 - Q_m - Q_2 - \cdots - Q_{m-1}) = 0.$$

Thus, it follows that $0 = h^0(K - (t_1 - 1)P - D_r) = h^0(K - t_1P - D_r)$.

(v) Let $t_i = t_1$. By Lemma 2.3 we obtain $C_{d-3} = T_P^{d-2-l}C_{l-1}$. Hence, we get $C_{l-1}.C \geqq (l-1)P + D_r$. Moreover, we have $L.C = Q_1 + \cdots + Q_{r-1} + R_{l+1} + \cdot + R_d$. Thus, we obtain $C_{l-1}.L \geqq Q_1 + \cdots + Q_{r-1}$, which implies that $C_{l-1} = LC_{l-2}$. Hence, we get $C_{d-3} = T_P^{d-2-l}LC_{l-2}$. Moreover, we have

$$Q_1 + \cdots + Q_{r-1} + R_{l+1} + \cdot + R_d + C_{l-2}.C = L.C + C_{l-2}.C \geqq (l-1)P + D_r,$$

which implies that $C_{l-2}.C \geqq (l-1)P + Q_r$. Hence, we get $C_{l-2}.T_P \geqq (l-1)P$. Thus, we have $C_{l-2} = T_PC_{l-3}$. Hence, we get $C_{d-3} = T_P^{d-2-l}LT_PC_{l-3}$. Therefore, we obtain

$$C_{d-3}.C \geqq (d-2-l)dP + Q_1 + \cdots + Q_l + dP + Q_{l+1} > t_1P + D_r.$$

We have $K - (t_2 - 1)P \sim (d^2 - 3d - d^2 + 4d)P = dP$ where $t_2 = (d-4)d + 1$. Since C is $(d-1)$-gonal, we get $h^0(dP - Q_1 - Q_2) = 1$. We note that Q_{l+1} is general. Hence, we get $h^0(dP - Q_1 - Q_2 - Q_{l+1}) = 0$. Thus, we get

$$0 = h^0(K - (t_2 - 1)P - D_r) = h^0(K - t_2P - D_r).$$

(vi) Let $t_i = t_1$. By Lemma 2.3 we have $C_{d-3} = T_P^{d-5}C_2$. Hence, we get $C.C_2 \geqq Q_1 + \cdots + Q_5$, which implies that $C_2.L \geqq Q_1 + Q_2 + Q_3$. Therefore, we obtain $C_2 = LL_{Q_4,Q_5}$. Hence, we have

$$\begin{aligned} C_{d-3}.C &= T_P^{d-5}.C + L.C + L_{Q_4,Q_5}.C \\ &= d(d-5)P + P + Q_1 + Q_2 + Q_3 + R_4 + \cdots + R_{d-1} + L_{Q_4,Q_5}.C, \end{aligned}$$

which implies that $C_{d-3}.C \geqq t_1P + D_r$.

Suppose that there exists a curve C_{d-3} such that $C_{d-3}.C \geqq (t_2 - 1)P + D_r$, where $t_2 = (d-5)d + 3$. By Lemma 2.3 and the above method we have $C_{d-3} = T_P^{d-5}LL_{Q_4,Q_5}$. Then we obtain

$$(d-5)d + 2 \leqq \mathrm{ord}_P\, C_{d-3}.C = (d-5)d + 1,$$

which is a contradiction. Hence, we get

$$0 = h^0(K - (t_2 - 1)P - D_r) = h^0(K - t_2P - D_r).$$

(vii) We have

$$h^0(K - (t_1 - 1)P - D_r) = h^0((2d-1)P - (Q_1 + \cdots + Q_5)) = 5 - 5 = 0,$$

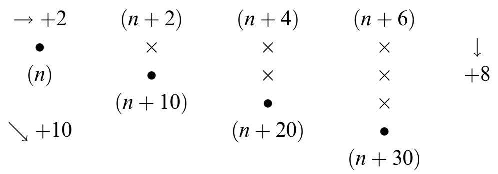
because $Q_1, \ldots, Q_5$ are general points. It is enough to show that $h^0(K - (t_2 - 1)P - D_r) = 0$, which is clear since $t_2 = t_1 + 1$.

(viii) Let $t_i = t_1$. We get $C_{d-3} = T_P^{d-5} C_2$. Hence, we have $C.C_2 \geqq Q_1 + \cdots + Q_5$. Since $C.L_2 \geqq Q_3 + Q_4 + Q_5$, we have $C_2 = L_1 L_2$, which implies that

$$1 = h^0(K - (t_1 - 1)P - D_r) = h^0(K - t_1 P - D_r) = h^0(K - t_2 P - D_r).$$

Step 5. By Step 2 the divisor $nP - 2Q_1 - \cdots - 2Q_r$ is very ample. It follows from Step 4 and Theorem 2.2 in [9] that H is DCP. □

In the rest of this section we denote by H a numerical semigoroup with $d_2(H) = \langle 4, 5 \rangle$. Using Theorem 2.5 we will prove that the numerical semigroup H is DCP. Let n be the least odd number in H. In the following figure a cross $\times$ is one of the candidates of the elements $\mathbf{N}_0 \backslash H$ which are odd numbers larger than n.



The candidates of odd gaps $> n$

In fact, for any odd $n \geqq 13$ we have

$$S(2\langle 4, 5 \rangle + n\mathbf{N}_0) = \{8, 10, 20, 30, n, n + 10, n + 20, n + 30\}$$

(see [7] if $n \geqq 15$). We note that $H \supseteqq 2\langle 4, 5 \rangle + n\mathbf{N}_0$.

Lemma 2.6. *A numerical semigroup H with $d_2(H) = \langle 4, 5 \rangle$ and the least odd number $n \geqq 13$ in H is one of the following*:

(a) $H_n = 2\langle 4, 5 \rangle + n\mathbf{N}_0$, (b) $H_n + \langle n + 2t \rangle$, $t = 1, 2, 3, 6, 7, 11$,

(c) $H_n + \langle n + 2t, n + 14 \rangle$, $t = 1, 6$, (d) $H_n + \langle n + 6, n + 12 \rangle$, (e) $H_n + \langle n + 4, n + 6 \rangle$,

(f) $H_n + \langle n + 2, n + 4, n + 6 \rangle$, (g) $H_n + \langle n + 2, n + 6 \rangle$, (h) $H_n + \langle n + 2, n + 4 \rangle$.

Proof. By the figure "The candidates of odd gaps $> n$" we get the classification. □

Applying Theorem 2.5 to the cases of Lemma 2.6 we get the following:

Corollary 2.7. *Let H be a numerical semigroup of genus $\geqq 18$ with $d_2(H) = \langle 4, 5 \rangle$. Then H is DCP.*

Proof. We use the classification in Lemma 2.6, because the least odd number $n \leqq 11$ in H implies $g(H) \leqq g(2\langle 4, 5 \rangle + n\mathbf{N}_0) \leqq 2 * 6 + (11 - 1)/2 = 17$.

In the case (a) by Proposition 2.3 in [7] we get the result if $g(H) \geqq 18$.

In the case (b) we can apply Theorem 2.5 (i) to the cases $t = 3, 7, 11$ if $g(H) \geqq 18$. We can apply Theorem 2.5 (ii) to the cases $t = 1, 6$ if $g(H) \geqq 18$. Theorem 2.5 (iii) is applied to the case $t = 2$ if $g(H) \geqq 18$. We can apply Theorem 2.5 (iv), (v), (vii), (vi) and (viii) to the cases (c), (d), (e), (g) and (h) respectively if $g(H) \geqq 18$.

By Lemma 2.3 and Proposition 2.4 in [9] we get the result (f) if $g(H) \geqq 18$. □

3. Non-DCP numerical semigroups

By [9] we know that any numerical semigroup H of genus $g \geqq 9$ with $d_2(H) = \langle 3, 5, 7 \rangle$ is DCP. We note that a point P on a smooth plane curve C of degree 4 with $H(P) = \langle 3, 5, 7 \rangle$ satisfies $\mathrm{ord}_P(C.T_P) = 3$. But the following theorem shows that for any $d \geqq 5$ there is a numerical semigroup H whose $d_2(H)$ is the Weierstrass semigroup of a point P on a plane curve C of degree d with $\mathrm{ord}_P\, C.T_P = d - 1$ such that H is not DCP. In this case we have

$$d_2(H) = \langle d-1, d-1+d-2, 2(d-1)+d-3, \dots, (d-2)(d-1)+1 \rangle \text{ (see [2])},$$

which is denoted by H'_d. In this section we assume that $d \geqq 5$.

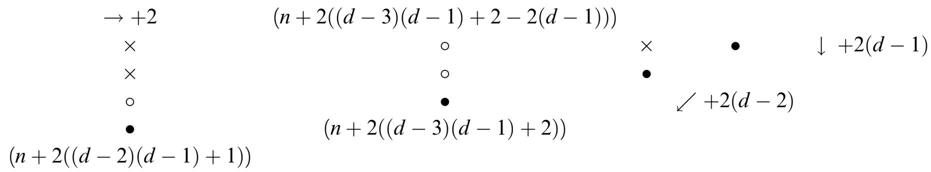
Theorem 3.1. *Let n be an odd number with $n \geqq (d-2)(d-1)+1$. Assume that H is $2H'_d + \langle n, n+2t \rangle$ with $t = (d-3)(d-1)+2-l(d-1)$ for $l = 1$ or 2. Then the semigroup H is not DCP.*

Proof. We have the standard basis $S(H'_d) = \{d-1, s_1, \dots, s_{d-2}\}$ for H'_d, where $s_i = (d-1-i)(d-1)+i$ for all i. It follows from the condition $n \geqq (d-2)(d-1)+1$ and Remark 2.1 in [7] that

$$S(2H'_d + n\mathbf{N}_0) = \{2(d-1), 2s_1, \dots, 2s_{d-2}\} \cup \{n, n+2s_1, \dots, n+2s_{d-2}\}.$$

By Remark 2.1 in [7] we have $g(2H'_d + n\mathbf{N}_0) = 2g(H'_d) + (n-1)/2$.

Step 1. We obtain $g(H) = 2g(H'_d) + (n-1)/2 - r$, where $r = 1$ and 3 for $l = 1$ and $l = 2$ respectively. Indeed, if $l = 1$, then the set $H \backslash (2H'_d + \langle n \rangle)$ consists of one element $n + 2((d-3)(d-1)+2-(d-1))$, which implies that $r = 1$. The semigroup H with $l = 2$ contains the following three elements ∘ in the figure below:



The elements of $H = 2H'_d + n\mathbf{N}_0 + (n + 2((d-3)(d-1)+2-2(d-1)))\mathbf{N}_0$

Assume that there is a double covering $\pi : \tilde{C} \to C$ with a ramification point $\tilde{P}$ over a point P with $H(\tilde{P}) = H$.

STEP 2. There are r points $Q_1, \ldots, Q_r$ distinct from P such that $2D$ is linearly equivalent to a reduced divisor containing P, where $D = \frac{n+1}{2}P - D_r$ with $D_r = Q_1 + \cdots + Q_r$.

STEP 3. We show that the equality $h^0(K - tP - D_r) = h^0(K - (t-1)P - D_r)$ induces a contradiction. Let $T_P.C = (d-1)P + Q$ with $Q \neq P$.

First, let $l = 1$. We consider the case $Q_1 = Q$. Let $C_{d-3} = T_P^{d-4}L$ with a line L passing through P with $L \neq T_P$. Then in view of $d \geqq 5$ we have

$$\begin{aligned} C_{d-3}.C &= (d-4)(d-1)P + (d-4)Q_1 + L.C \\ &\geqq (d-4)(d-1)P + Q_1 + P = ((d-4)(d-1)+1)P + Q_1 \end{aligned}$$

and $C_{d-3}.C \ngeqq ((d-4)(d-1)+2)P$. This is a contradiction.

We consider the case with $Q_1 \neq Q$. We set $C_{d-3} = T_P^{d-4}L$ with the line L passing through P and Q_1 which is a reducible curve of degree $d-3$. In view of $Q_1 \neq Q$ we note that $L \neq T_P$. Then

$$\begin{aligned} C_{d-3}.C &= T_P^{d-4}.C + L.C = (d-4)(d-1)P + (d-4)Q + L.C \\ &\geqq (d-4)(d-1)P + (d-4)Q + P + Q_1 \geqq ((d-4)(d-1)+1)P + Q_1. \end{aligned}$$

But $C_{d-3}.C \ngeqq ((d-4)(d-1)+2)P$. This is a contradiction.

Next, let $l = 2$. We consider the case $Q_1 = Q_2 = Q_3 = Q$.

Let $d = 5$. Assume that $h^0(K - P - 3Q) = h^0(K - 2P - 3Q)$. Let C_2 be a conic such that $C_2.C \geqq 2P + 3Q$. Then $C_2.T_P \geqq 2P + Q$. Hence, we get $C_2 = T_P L$ where L is a line. Moreover, we have

$$2P + 3Q \leqq C_2.C = T_P L.C = T_P.C + L.C = 4P + Q + L.C,$$

which implies that $L = T_Q$. Hence, we get $C_2 = T_P T_Q$. Thus, we obtain

$$1 = h^0(K - 2P - 3Q) = h^0(K - P - 3Q) = 6 + 1 - 6 + h^0(P + 3Q) \geqq 2,$$

which is a contradiction.

Let $d \geqq 6$. Let L_1 be a line through P which is distinct from T_P. We set $L_0 = T_Q$. Then in view of $d \geqq 6$ we have

$$\begin{aligned} T_P^{d-5}L_1L_0.C &\geqq (d-5)(d-1)P + (d-5)Q + P + 2Q \\ &= ((d-5)(d-1)+1)P + (d-3)Q \geqq ((d-5)(d-1)+1)P + 3Q. \end{aligned}$$

But we get $T_P^{d-5}L_1L_0.C \ngeqq ((d-5)(d-1)+2)P$. This is a contradiction.

We consider the case $Q_1 \neq Q$ when we renumber Q_1, Q_2 and Q_3. Let L_0 be the line such that $L_0.C \geqq Q_2 + Q_3$. If $L_0 \ni P$, then we take L_1 as a line through

Q_1 and not containing P. If $L_0 \not\ni P$, then we take L_1 as the line through Q_1 and P. Then we get

$$L_0 L_1 T_p^{d-5}.C \geqq ((d-5)(d-1)+1)P + Q_1 + Q_2 + Q_3$$

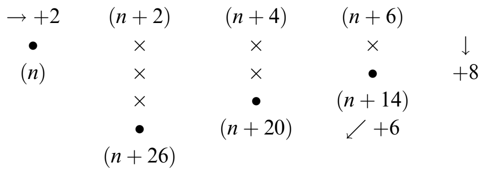
and $L_0 L_1 T_p^{d-5}.C \not\geqq ((d-5)(d-1)+2)P$. This is a contradiction. □

In the case $d = 5$ we get the following by Theorem 3.1:

Corollary 3.2. *Set $H(n) = 2\langle 4, 7, 10, 13\rangle + n\mathbf{N}_0$, where n is an odd number with $n \geqq 13$. Then neither $H(n) + \langle n+4\rangle$ nor $H(n) + \langle n+12\rangle$ is DCP.*

4. Double coverings of plane curves of degree 5

In this section H denotes a numerical semigroup with $d_2(H) = \langle 4, 7, 10, 13\rangle$. Let n be the least odd number in H. Then we note that $g(H) \leqq 12 + (n-1)/2$ (for example, see Lemma 3.1 in [4]). Assume that $n \geqq 13$. In the following figure a cross $\times$ is one of the candidates of the odd numbers in $\mathbf{N}_0 \backslash H$ which are larger than n.



The candidates of odd gaps $> n$

We get $6 + (n-1)/2 \leqq g(H) \leqq 12 + (n-1)/2$ by Lemma 2.2 in [7]. Hence, we set $g(H) = 12 + (n-1)/2 - r$ with $0 \leqq r \leqq 6$. By the above figure "The candidates of odd gaps $> n$" the numerical semigroups H are determined as follows:

Lemma 4.1. *Set $H(n) = 2\langle 4, 7, 10, 13\rangle + n\mathbf{N}_0$. Then H is one of the following*:

(i) *If $g(H) = 12 + (n-1)/2$, then $H = H(n)$.*

(ii) *If $g(H) = 11 + (n-1)/2$, then H is either* 1) $H(n) + \langle n+6\rangle$ *or* 2) $H(n) + \langle n+12\rangle$ *or* 3) $H(n) + \langle n+18\rangle$.

(iii) *If $g(H) = 10 + (n-1)/2$, then H is either* 1) $H(n) + \langle n+6, n+12\rangle$ *or* 2) $H(n) + \langle n+6, n+18\rangle$ *or* 3) $H(n) + \langle n+10\rangle$ *or* 4) $H(n) + \langle n+12, n+18\rangle$.

(iv) *If $g(H) = 9 + (n-1)/2$, then H is either* 1) $H(n) + \langle n+2\rangle$ *or* 2) $H(n) + \langle n+4\rangle$ *or* 3) $H(n) + \langle n+6, n+10\rangle$ *or* 4) $H(n) + \langle n+6, n+12, n+18\rangle$ *or* 5) $H(n) + \langle n+10, n+12\rangle$.

(v) *If $g(H) = 8 + (n-1)/2$, then H is either* 1) $H(n) + \langle n+2, n+6\rangle$ *or* 2) $H(n) + \langle n+2, n+12\rangle$ *or* 3) $H(n) + \langle n+4, n+6\rangle$ *or* 4) $H(n) + \langle n+4, n+10\rangle$ *or* 5) $H(n) + \langle n+6, n+10, n+12\rangle$.

(vi) *If* $g(H) = 7 + (n-1)/2$, *then* H *is either*
1) $H(n) + \langle n+2, n+4 \rangle$ *or* 2) $H(n) + \langle n+2, n+6, n+12 \rangle$ *or* 3) $H(n) + \langle n+4, n+6, n+10 \rangle$.
(vii) *If* $g(H) = 6 + (n-1)/2$, *then* $H = H(n) + \langle n+2, n+4, n+6 \rangle$.

Theorem 4.2. *If* $g = g(H) \geqq 18$, *then the numerical semigroup* H *except for* (ii) 2) *and* (iv) 2) *is DCP.*

Proof. We give the proofs according to the cases given in Lemma 4.1. Let (C, P) be a pointed plane curve with $H(P) = \langle 4, 7, 10, 13 \rangle$. Then we have $T_P(C).C = 4P + R$ with some point $R \neq P$, which implies that $K \sim 8P + 2R$. To show that H is DCP we use Theorem 2.2 in [9]. So, we need to choose r points $Q_1, \dots, Q_r$ of C satisfying the assumptions of the theorem in [9]. We set $D = \dfrac{n+1}{2}P - Q_1 - \cdots - Q_r$. Then we note that

$$\deg(2D - P) = n - 2r = 2g - 23 \geqq 36 - 23 = 13$$

because $g(H) \geqq 18$. Hence, the divisor $2D - P$ is very ample.

In the case (i) it follows from Proposition 2.3 in [7] that H is DCP.

We consider the case (ii) 1). Let $Q_1 = R$. Since C is not trigonal, we get $h^0(2P + R) = 1$. It is clear that $h^0(3P + R) = 2$ since $|4P + R|$ is a net without base points. Thus, we get the result. Theorem 3.1 implies that H is not DCP in the case (ii) 2). In the case (ii) 3) it follows from Proposition 2.4 in [7] that H is DCP.

Let H be the semigroup in the case (iii) 1). We set $Q_1 = Q_2 = R$. We have

$$h^0(2P + 2R) = 4 + 1 - 6 + h^0(6P) = 1 \quad \text{and}$$

$$h^0(3P + 2R) = 5 + 1 - 6 + h^0(5P) = 2.$$

Moreover, we get

$$h^0(5P + 2R) = 7 + 1 - 6 + h^0(3P) = 3 \quad \text{and}$$

$$h^0(6P + 2R) = 8 + 1 - 6 + h^0(2P) = 4.$$

In the case (iii) 2) we take a general point Q. Let $Q_1 = R$ and $Q_2 = Q$. Then we have $h^0(9P + R + Q) = 6$ and $h^0(8P + R + Q) = 5$, because of $8P + R + Q \not\sim 8P + 2R \sim K$. Moreover, we get $h^0(2P + R + Q) = -1 + h^0(6P + R - Q)$. Now we have

$$h^0(6P + R) = 2 + h^0(2P + R) = 3,$$

because C is 4-gonal. Hence, we get $h^0(2P + R + Q) = -1 + 2 = 1$, because Q is general. We see that $h^0(3P + Q + R) = 2$ since $|4P + R|$ is a net and $h^0(2P + R + Q) = 1$.

In the case (iii) 3) we have

$$h^0(K - 5P) = h^0(5P) = 2 \quad \text{and} \quad h^0(K - 6P) = -1 + h^0(6P) = 1.$$

Let Q_1 be a general point. Since $h^0(K - 5P - Q_1) = 1$, there exists a unique effective divisor $E = S_1 + S_2 + S_3 + S_4$ of degree 4 with $E \sim K - 5P - Q_1$. The effective divisor E does not contain P, because $h^0(K - 6P) = 1$. Moreover, we have $E \neq 4R$. Indeed, assume that $E = 4R$. Then we get

$$4R \sim K - 5P - Q_1 \sim 3P + 2R - Q_1,$$

which implies that $2R + Q_1 \sim 3P$. This contradicts $h^0(3P) = 1$. We may assume that $S_4 \neq R$ and $S_4 \neq P$. We set $Q_2 = S_4$. Then we have

$$h^0(K - 5P - Q_1 - Q_2) = h^0(S_1 + S_2 + S_3) = 1.$$

Hence, there exists a unique conic C_2 with $C.C_2 \geqq 5P + Q_1 + Q_2$. Take a conic C_2' with $C.C_2' \geqq 4P + Q_1 + Q_2$. Since Q_1 and Q_2 are different from R, we must have $C_2' = T_P L_{Q_1,Q_2}$, where L_{Q_1,Q_2} is the line through Q_1 and Q_2. Hence, we obtain $h^0(K - 4P - Q_1 - Q_2) = 1$.

In the case (iii) 4) let Q_1 and Q_2 be general points. Then we have

$$h^0(9P + Q_1 + Q_2) = 6 \quad \text{and} \quad h^0(8P + Q_1 + Q_2) = 5,$$

because $8P + Q_1 + Q_2 \not\sim K$. Since $h^0(3P + 2R) = h^0(5P) = 2$, we obtain

$$h^0(5P + Q_1 + Q_2) = 2 + h^0(3P + 2R - Q_1 - Q_2) = 2 \quad \text{and}$$
$$h^0(6P + Q_1 + Q_2) = 3.$$

Let H be the semigroup in the case (iv) 1). We take a line L_P through P distinct from T_P. Then we have $L_P.C = P + S_1 + S_2 + S_3 + S_4$. We set $Q_i = S_i$ for all $i = 1, 2, 3$. It is clear that $h^0(4P + R) = 3$ and $h^0(P + Q_1 + Q_2 + Q_3) = 2$ by the choice of R and Q_i's.

In the case (iv) 2) H is not DCP by Theorem 3.1.

We consider the case (iv) 3). Let L_P be a line as in the case (iv) 1). We set $Q_1 = R$, $Q_2 = S_3$ and $Q_3 = S_4$. Then we have

$$\begin{aligned} &h^0(K - 5P - Q_1 - Q_2 - Q_3) \\ &\quad = h^0(4P + R + P + S_1 + S_2 + Q_2 + Q_3 - 5P - Q_1 - Q_2 - Q_3) \\ &\quad = h^0(S_1 + S_2) = 1. \end{aligned}$$

Moreover, it is enough to show that $h^0(K - 2P - Q_1 - Q_2 - Q_3) = 1$, which is clear by the choice of Q_i's.

Let H be the semigroup in the case (iv) 4). We set $Q_1 = R$. Take two general points Q_2 and Q_3. We have

$$h^0(9P + Q_1 + Q_2 + Q_3) = 7 = h^0(8P + Q_1 + Q_2 + Q_3) + 1.$$

Moreover, we have

$$h^0(6P + Q_1 + Q_2 + Q_3) = 4 + h^0(2P + R - Q_2 - Q_3) = 4$$

and

$$h^0(5P + Q_1 + Q_2 + Q_3) = 3 + h^0(3P + R - Q_2 - Q_3) = 3,$$

because Q_2 and Q_3 are general. Let C_2 be a conic with $C_2.C \geqq 2P + Q_1 + Q_2 + Q_3$. Then C_2 is uniquely determined. Hence, we get

$$h^0(2P + Q_1 + Q_2 + Q_3) = 1 \quad \text{and} \quad h^0(3P + Q_1 + Q_2 + Q_3) = 2.$$

We are in the case (iv) 5). Let Q_1, Q_2 and Q_3 be general points of C. We have

$$h^0(6P + Q_1 + Q_2 + Q_3) = 4 + h^0(2P + 2R - Q_1 - Q_2 - Q_3) = 4.$$

In view of $h^0(3P + 2R) = h^0(5P) = 2$ we have $h^0(5P + Q_1 + Q_2 + Q_3) = 3$. Moreover, we get $h^0(4P + 2R) = 1 + h^0(4P) = 3$, which implies that

$$h^0(4P + Q_1 + Q_2 + Q_3) = 2.$$

We consider the case (v). We note that by Namba's Theorem we have $h^0(Q_1 + Q_2 + Q_3 + Q_4) = 1$ if four points Q_1, Q_2, Q_3 and Q_4 of C do not lie on a line.

In the case (v) 1) let L_P be a line through P with $L_P \neq T_P$. We set $L_P.C = P + Q_1 + Q_2 + Q_3 + S$ and $Q_4 = R$. Let C_2 be a conic with $C_2.C \geqq P + Q_1 + Q_2 + Q_3 + Q_4$. Then we get $C_2.L_P \geqq P + Q_1 + Q_2 + Q_3$. Hence, we have $C_2 = L_P L$ where L is any line through Q_4, which implies that $h^0(K - P - Q_1 - Q_2 - Q_3 - Q_4) = 2$. Let C_2' be a conic with $C_2'.C \geqq 2P + Q_1 + Q_2 + Q_3 + Q_4$. Then we have $C_2'.L_P \geqq Q_1 + Q_2 + Q_3 + P$. In view of $Q_4 = R$ we get $C_2' = L_P T_P$, which implies that $h^0(K - 2P - Q_1 - Q_2 - Q_3 - Q_4) = 1$. It is clear that $h^0(K - 3P - Q_1 - Q_2 - Q_3 - Q_4) = 1$ since $C_2'.C = L_P T_P.C \geqq 5P + C_1 + C_2 + C_3 + C_4$.

We are in the case (v) 2). Let L_P be a line through P with $L_P \neq T_P$. We set $L_P.C = P + Q_1 + Q_2 + Q_3 + S$. Let Q_4 be a point of C not on the line L_P with $Q_4 \neq R$. Then we have

$$\begin{aligned}
&h^0(5P + Q_1 + Q_2 + Q_3 + Q_4)\\
&\quad = 4 + h^0(K - 5P - Q_1 - Q_2 - Q_3 - Q_4)\\
&\quad = 4 + h^0(5P + R + Q_1 + Q_2 + Q_3 + S - 5P - Q_1 - Q_2 - Q_3 - Q_4)\\
&\quad = 4 + h^0(R + S - Q_4) = 4.
\end{aligned}$$

Moreover, we get $h^0(6P + Q_1 + Q_2 + Q_3 + Q_4) = 5$. It is clear that $h^0(P + Q_1 + Q_2 + Q_3 + Q_4) = 2$ since the four points P, Q_1, Q_2, Q_3 lie on the line L_P and $Q_4 \notin L_P$.

Let H be the semigroup in the case (v) 3). We take a line L containing neither P nor R. We set $L.C = Q_1 + Q_2 + Q_3 + S + T$ and $Q_4 = R$. Let C_2 be a conic with $C_2.C \geqq P + Q_1 + Q_2 + Q_3 + Q_4$. Then $C_2.L \geqq Q_1 + Q_2 + Q_3$. Hence, we get $C_2 = LT_P$. We note that $C.C_2 \geqq 4P + Q_1 + Q_2 + Q_3 + Q_4$.

We consider the case (v) 4). Let L_1 be a line through P with $L_1 \neq T_P$ such that $L_1.C = P + Q_1 + Q_2 + S_1 + T_1$. Let L_2 be a line through P different from

T_P and L_1 such that $L_2.C = P + Q_3 + Q_4 + S_2 + T_2$. Then $h^0(K - 4P - Q_1 - Q_2 - Q_3 - Q_4) = 0$ since L_1L_2 is the only conic passing through P and all Q_i's. Hence, we get

$$h^0(5P + Q_1 + Q_2 + Q_3 + Q_4) = 4 \quad \text{and} \quad h^0(4P + Q_1 + Q_2 + Q_3 + Q_4) = 3.$$

On the other hand, let C_2' be a conic with $C_2'.C \geqq P + Q_1 + Q_2 + Q_3 + Q_4$. Then we have $C_2'.L_1 \geqq P + Q_1 + Q_2$. Hence, we obtain $C_2' = L_1L'$ where L' is the line with $L'.C \geqq Q_3 + Q_4$. The line L' must be L_2. Thus, C_2' is uniquely determined. Moreover, we get $C_2'.C \geqq 2P + Q_1 + Q_2 + Q_3 + Q_4$.

Let H be the semigroup in the case (v) 5). We set $Q_1 = Q_2 = R$. Let Q_3 and Q_4 be general points of C. Then we have

$$h^0(K - 2P - Q_1 - Q_2 - Q_3 - Q_4) = h^0(6P - Q_3 - Q_4) = 0.$$

We consider the case (vi) 1). Let Q_1, Q_2 and Q_3 be general points of C. Then we have $h^0(K - 2P - Q_1 - Q_2 - Q_3) = 1$. Hence there is a unique conic C_2 with $C_2.C \geqq 2P + Q_1 + Q_2 + Q_3$, which is irreducible, because T_P does not contain any Q_i and no three of the four points P, Q_1, Q_2 and Q_3 are collinear. Let $C_2.C = 2P + Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + S_1 + S_2 + S_3$. Here, we have $Q_i \neq P$ for all i and $S_j \neq P$ for all j, because C_2 is irreducible. Then we get

$$h^0(K - 2P - Q_1 - Q_2 - Q_3 - Q_4 - Q_5) = 1.$$

Moreover, let C_2' be a conic with $C_2'.C \geqq Q_1 + Q_2 + Q_3 + Q_4 + Q_5$. Then $C_2.C_2' \geqq Q_1 + Q_2 + Q_3 + Q_4 + Q_5$. Since C_2 is irreducible, we must have $C_2' = C_2$. Hence, we get $1 = h^0(K - Q_1 - Q_2 - Q_3 - Q_4 - Q_5)$.

Let H be the semigroup in the case (vi) 2). We take general points Q_1, Q_2, Q_3 and Q_4 of C. We have $h^0(K - 2P - Q_1 - Q_2 - Q_3 - Q_4) = 0$, because Q_1, Q_2, Q_3 and Q_4 are general. Since $h^0(K - P - Q_1 - Q_2 - Q_3 - Q_4) = 1$, there is a unique effective divisor E which is linearly equivalent to $K - P - Q_1 - Q_2 - Q_3 - Q_4$. We have $E \neq 5P$, because $h^0(2P + 2R) = 1$. We take a point Q_5 with $Q_5 \neq P$ such that $E \geqq Q_5$. Then we get

$$h^0(K - P - Q_1 - Q_2 - Q_3 - Q_4 - Q_5) = h^0(E - Q_5) = 1.$$

Since no four points of Q_1, Q_2, Q_3, Q_4 and Q_5 are collinear, there exists a unique conic passing through all Q_i's. Thus, we get $h^0(K - Q_1 - Q_2 - Q_3 - Q_4 - Q_5) = 1$.

In the case (vi) 3) let Q_1, Q_2, Q_3, Q_4 and Q_5 be general points of C. Then we have

$$h^0(K - P - Q_1 - Q_2 - Q_3 - Q_4 - Q_5) = 0.$$

In the case (vii) we get the result by Corollary 2.8 in [7]. □

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