KODAI MATH. J.
38 (2015), 249–269

ESTIMATES OF EIGENVALUES OF A CLAMPED PROBLEM

Tao Zheng

Abstract

In this paper, for the eigenvalue problem of a clamped plate problem on complex projective space with holomorphic sectional curvature $c(>0)$ and $n(\geq 3)$-dimensional noncompact simply connected complete Riemannian manifold with sectional curvature Sec satisfying $-a^2 \leq Sec \leq -b^2$, where $a \geq b \geq 0$ are constants, we obtain universal eigenvalue inequalities. Moreover, we deduce the estimates of the upper bounds of eigenvalues.

1. Introduction

Let M be an n-dimensional complete Riemannian manifold and $\Omega \subset M$ be a bounded domain with smooth boundary (possible empty) in M. A Dirichlet eigenvalue problem of the biharmonic operator or a clamped plate problem, which describes the characteristic vibrations of a clamped plate is given by

$$\begin{cases} \Delta^2 u = \Gamma u, & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ^2 is the biharmonic operator on M and ν denotes the outward normal derivative on $\partial\Omega$. We will denote eigenvalues and the corresponding real eigenfunctions by $\{\Gamma_i\}_{i=1}^{\infty}$ and $\{u_i\}_{i=1}^{\infty}$, respectively. The eigenvalues Γ_i satisfy

$$0 < \Gamma_1 \leq \Gamma_2 \leq \Gamma_3 \leq \cdots \nearrow \infty,$$

where each Γ_i has finite multiplicity which is repeated according to its multiplicity.

2010 *Mathematics Subject Classification*. Primary 35P15, 58C40; Secondary 58J50.

Key words and phrases. biharmonic operator, eigenvalue, Hermitian metric, complex projective space, hyperbolic space.

This work was supported by the fundamental fund of Beijing Institute of Technology No. 20131742009, China Postdoctoral Science Foundation funded project No. 2014M550620, NSFC grant No. 11401023, and partially supported by NSFC grant No. 11471180.

Received June 17, 2014.

For the universal inequalities for eigenvalues of the clamped plate problem in a bounded domain in $\mathbf{R}^n$, in 1956, Payne, Pólya and Weinberger [15, 16] proved

$$\Gamma_{k+1} - \Gamma_k \le \frac{8(n+2)}{n^2 k} \sum_{i=1}^{k} \Gamma_i, \quad k = 1, 2, \ldots. \tag{1.2}$$

In 1984, Hile and Yeh [12] obtained

$$\sum_{i=1}^{k} \frac{\Gamma_i^{1/2}}{\Gamma_{k+1} - \Gamma_i} \ge \frac{n^2 k^{3/2}}{8(n+2)} \left(\sum_{i=1}^{k} \Gamma_i \right)^{-1/2}, \quad k = 1, 2, \ldots. \tag{1.3}$$

In 1990, Hook [13], Chen and Qian [3] independently proved

$$\frac{n^2 k^2}{8(n+2)} \le \left(\sum_{i=1}^{k} \frac{\Gamma_i^{1/2}}{\Gamma_{k+1} - \Gamma_i} \right) \left(\sum_{i=1}^{k} \Gamma_i^{1/2} \right), \quad k = 1, 2, \ldots. \tag{1.4}$$

In [8], Cheng and Yang have given an affirmative answer for a problem on universal inequalities for eigenvalues, proposed by Ashbaugh [2]; that is, they have proved

$$\Gamma_{k+1} - \frac{1}{k} \sum_{i=1}^{k} \Gamma_i \le \left(\frac{8(n+2)}{n^2} \right)^{1/2} \frac{1}{k} \left(\sum_{i=1}^{k} \Gamma_i (\Gamma_{k+1} - \Gamma_i) \right)^{1/2}, \quad k = 1, 2, \ldots. \tag{1.5}$$

For domains in a unit sphere, Wang and Xia [21] gave a universal inequality for the clamped plate problem (1.1). They proved

$$\sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)^2 \le \frac{8(n+2)}{n^2} \sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i) \left(\Gamma_i^{1/2} + \frac{n^2}{2n+4} \right) \left(\Gamma_i^{1/2} + \frac{n^2}{4} \right) \tag{1.6}$$

For an n-dimensional complete manifold M, Cheng, Ichikawa and Mametsuka [6] obtained

$$\sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)^2 \le \frac{8(n+2)}{n^2} \sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i) \left(\Gamma_i^{1/2} + \frac{n^2}{2n+4} \sup_{\Omega} |H|^2 \right) \times \left(\Gamma_i^{1/2} + \frac{n^2}{4} \sup_{\Omega} |H|^2 \right). \tag{1.7}$$

For the real hyperbolic space $\mathbf{H}^n(-1)$, Cheng and Yang [11] proved that

$$\sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)^2 \le 24 \sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i) \left(\Gamma_i^{1/2} - \frac{(n-1)^2}{4} \right) \left(\Gamma_i^{1/2} - \frac{(n-1)^2}{6} \right) \tag{1.8}$$

and Wang and Xia [22] proved

$$(1.9)\qquad \sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \le \left\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)(4\Gamma_i^{1/2}-(n-1)^2)\right\}^{1/2} \times \left\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2(6\Gamma_i^{1/2}-(n-1)^2)\right\}^{1/2}$$

which implies (1.8).

In this paper, motivated by [7, 20], for the eigenvalue problem of a clamped plate problem on complex projective space with holomorphic sectional curvature $c>0$ and $n(\ge 3)$-dimensional noncompact simply connected complete Riemannian manifold with sectional curvature Sec satisfying $-a^2 \le Sec \le -b^2$, where $a \ge b \ge 0$ are constants, we obtain the eigenvalue inequalities in the form of (1.9).

Theorem 1.1. *Let $M=\mathbf{C}P^n(c)$ be n-dimensional complex projective space with holomorphic sectional curvature $c>0$ and $\Omega\subset M$ be a bounded domain with smooth boundary. Then for the eigenvalues Γ_i's of a clamped plate problem* (1.1), *we have*

$$(1.10)\qquad \sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \le \frac{1}{n}\left\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)(2\Gamma_i^{1/2}+cn(n+1))\right\}^{1/2} \times \left\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2(2(n+1)\Gamma_i^{1/2}+cn(n+1))\right\}^{1/2}.$$

Remark 1.1. If $c=4$, C. Xia [23] obtained an eigenvalue inequality for a clamped plate problem.

Corollary 1.2. *Under the same assumption of Theorem* 1.1, *for the eigenvalues Γ_i's of a clamped plate problem* (1.1), *we have*

(1)

$$(1.11)\qquad \sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \le \frac{1}{n^2}\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)(2\Gamma_i^{1/2}+cn(n+1))(2(n+1)\Gamma_i^{1/2}+cn(n+1));$$

(2)

$$(1.12)\qquad \Gamma_{k+1} \le S_{k+1}+\sqrt{S_{k+1}^2-T_{k+1}},$$

where

$$S_{k+1} = \frac{1}{k}\sum_{i=1}^{k}\Gamma_i + \frac{1}{2n^2k}\sum_{i=1}^{k}(2\Gamma_i^{1/2} + cn(n+1))(2(n+1)\Gamma_i^{1/2} + cn(n+1))$$

and

$$T_{k+1} = \frac{1}{k}\sum_{i=1}^{k}\Gamma_i^2 + \frac{1}{n^2k}\sum_{i=1}^{k}\Gamma_i(2\Gamma_i^{1/2} + cn(n+1))(2(n+1)\Gamma_i^{1/2} + cn(n+1)).$$

(3) *For any positive real number* ε,

$$\Gamma_{k+1} + \frac{c^2n^2(n+1)^2(n^2+4n+4+4\varepsilon)}{16\varepsilon(n+1+\varepsilon)} \leq \left(1 + \frac{4(n+1+\varepsilon)}{n^2}\right)k^{(2(n+1+\varepsilon))/n^2} \times \left(\Gamma_1 + \frac{c^2n^2(n+1)^2(n^2+4n+4+4\varepsilon)}{16\varepsilon(n+1+\varepsilon)}\right). \tag{1.13}$$

THEOREM 1.3. *Let* $M = \mathbf{C}P^{n+m}(c)$ *be* $(n+m)$*-dimensional complex projective space with holomorphic sectional curvature* $c > 0$ *and* $\Omega \subset M$ *be a bounded complex submanifold. Then for the eigenvalues of a clamped plate problem* (1.1), *we have*

$$\sum_{i=1}^{k}(\Gamma_{k+1} - \Gamma_i)^2 \leq \frac{1}{n}\left\{\sum_{i=1}^{k}(\Gamma_{k+1} - \Gamma_i)(2\Gamma_i^{1/2} + cn(n+1))\right\}^{1/2} \times \left\{\sum_{i=1}^{k}(\Gamma_{k+1} - \Gamma_i)^2(2(n+1)\Gamma_i^{1/2} + cn(n+1))\right\}^{1/2}. \tag{1.14}$$

COROLLARY 1.4. *Under the same assumption as Theorem* 1.3, *for the eigenvalues* Γ_i*'s of a clamped plate problem* (1.1), *we have*

(1)

$$\sum_{i=1}^{k}(\Gamma_{k+1} - \Gamma_i)^2 \leq \frac{1}{n^2}\sum_{i=1}^{k}(\Gamma_{k+1} - \Gamma_i)(2\Gamma_i^{1/2} + cn(n+1))(2(n+1)\Gamma_i^{1/2} + cn(n+1)); \tag{1.15}$$

(2)

$$\Gamma_{k+1} \leq S_{k+1} + \sqrt{S_{k+1}^2 - T_{k+1}}, \tag{1.16}$$

where

$$S_{k+1} = \frac{1}{k}\sum_{i=1}^{k}\Gamma_i + \frac{1}{2n^2k}\sum_{i=1}^{k}(2\Gamma_i^{1/2} + cn(n+1))(2(n+1)\Gamma_i^{1/2} + cn(n+1))$$

and

$$T_{k+1} = \frac{1}{k}\sum_{i=1}^{k}\Gamma_i^2 + \frac{1}{n^2k}\sum_{i=1}^{k}\Gamma_i(2\Gamma_i^{1/2} + cn(n+1))(2(n+1)\Gamma_i^{1/2} + cn(n+1));$$

(3)

$$\Gamma_{k+1} + \frac{c^2n^2(n+1)^2(n^2+4n+4+4\varepsilon)}{16\varepsilon(n+1+\varepsilon)} \leq \left(1 + \frac{4(n+1+\varepsilon)}{n^2}\right)k^{(2(n+1+\varepsilon))/n^2} \times \left(\Gamma_1 + \frac{c^2n^2(n+1)^2(n^2+4n+4+4\varepsilon)}{16\varepsilon(n+1+\varepsilon)}\right), \tag{1.17}$$

where ε is any positive real number.

Remark 1.2. Let Ω be bounded domain with smooth boundary in Euclidean space. Then, for the eigenvalue problem (1.1), Agmon [1] and Pleijel [17] proved the following Weyl's asymptotic formula

$$\Gamma_k \sim \frac{16\pi^4}{(\omega_n \operatorname{vol} \Omega)^{4/n}}k^{4/n}, \quad k \to \infty, \tag{1.18}$$

where ω_n is the volume of the unit ball in $\mathbf{R}^n$. The Weyl's asymptotic formula (1.18) also holds in general Riemannian manifolds. In the case of n-dimensional complex manifold, since the real dimension is $2n$, the Weyl's asymptotic formula (1.18) can be rewritten as

$$\Gamma_k \sim \frac{16\pi^4}{(\omega_{2n} \operatorname{vol} \Omega)^{2/n}}k^{2/n}, \quad k \to \infty. \tag{1.19}$$

Therefore, we can conjecture that there exists a constant $C_{n,\Omega}$ depending only on the dimension n and Ω such that

$$\Gamma_{k+1} \leq C_{n,\Omega}k^{2/n}\Gamma_1. \tag{1.20}$$

Theorem 1.5. *Let M be $n(\geq 3)$-dimensional noncompact simply connected complete Riemannian manifold with sectional curvature Sec satisfying $-a^2 \leq Sec \leq -b^2$, where $a \geq b \geq 0$ are constants. Assume that $\Omega \subset M$ is a bounded domain with smooth boundary. For the eigenvalue problem* (1.1), *we have*

$$\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \tag{1.21}$$
$$\leq \left\{4\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left(\Gamma_i^{1/2}-\frac{(n-1)^2}{4}b^2+\frac{a^2-b^2}{4}\right)\right\}^{1/2}$$
$$\times\left\{6\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\left(\Gamma_i^{1/2}-\frac{(n-1)^2}{6}b^2+\frac{a^2-b^2}{6}\right)\right\}^{1/2}.$$

Remark 1.3. If $a=b=1$, that is, M is the hyperbolic space, the inequality (1.21) is the one of Wang and Xia [22] (see (1.9)).

COROLLARY 1.6. *Under the same assumption as in Theorem* 1.5, *for the eigenvalues* Γ_i*'s of a clamped plate problem* (1.1), *we have*

$$\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \leq 24\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left(\Gamma_i^{1/2}-\frac{(n-1)^2}{4}b^2+\frac{a^2-b^2}{4}\right) \tag{1.22}$$
$$\times\left(\Gamma_i^{1/2}-\frac{(n-1)^2}{6}b^2+\frac{a^2-b^2}{6}\right).$$

Remark 1.4. The inequality (1.22) is better than the one in [4]. If $a=b=1$, the inequality (1.22) is the one of Cheng and Yang [11] (see (1.8)).

2. Preliminaries

Let M be n-dimensional Hermitian manifold and h be its Hermitian metric. Then the real part of h is a Riemannian metric g on M and (M,g) is a $2n$-dimensional Riemannian manifold. Assume that $z=(z^1,\dots,z^n)=(x^1+\sqrt{-1}x^{n+1},\dots,x^n+\sqrt{-1}x^{2n})$ is the local coordinate system on M. we have (see [14])

$$\begin{aligned} g &= \sum_{\alpha,\beta=1}^{2n} g_{\alpha\beta}\,\mathrm{d}x^\alpha\otimes\mathrm{d}x^\beta,\\ h &= \sum_{i,j=1}^{n} h_{i\bar{j}}\,\mathrm{d}z^i\otimes\mathrm{d}\overline{z^j}\\ &= \sum_{i,j=1}^{n}(g_{i,j}+\sqrt{-1}g_{i,n+j})\,\mathrm{d}z^i\otimes\mathrm{d}\overline{z^j}\\ &= \sum_{\alpha,\beta=1}^{2n} g_{\alpha\beta}\,\mathrm{d}x^\alpha\otimes\mathrm{d}x^\beta-\frac{\sqrt{-1}}{2}\sum_{i,j=1}^{n}(g_{ij}+\sqrt{-1}g_{i,n+j})\,\mathrm{d}z^i\wedge\mathrm{d}\overline{z^j}. \end{aligned} \tag{2.1}$$

Let $(g^{\alpha\beta})$ and $(h^{\bar{i}j})$ be the inverse matrixes of $(g_{\alpha\beta})$ and $(h_{i\bar{j}})$, respectively, that is,

$$\sum_{\gamma=1}^{2n} g^{\alpha\gamma} g_{\gamma\beta} = \delta_{\alpha\beta}, \quad \sum_{p=1}^{n} h^{\bar{p}i} h_{j\bar{p}} = \delta_{ij}.$$

Then we have $h^{\bar{i}j} = g^{ij} - \sqrt{-1} g^{i,n+j}$.

If the Hermitian manifold (M,h) is Kälerian, the Hermitian connection is the same as Riemannian connection. Therefore, the Laplacian and gradient of Riemannian metric can be rewritten as follows.

$\forall f, h \in C^2(M, \mathbf{C})$, we have

$$\begin{aligned}
\nabla f &= \sum_{\alpha=1}^{2n} \left(\sum_{\gamma=1}^{2n} g^{\alpha\gamma} \frac{\partial f}{\partial x^\gamma} \right) \frac{\partial}{\partial x^\alpha} \\
&= 2 \sum_{i=1}^{n} \left(\sum_{p=1}^{n} h^{\bar{p}i} \frac{\partial f}{\partial \overline{z^p}} \right) \frac{\partial}{\partial z^i} + 2 \sum_{j=1}^{n} \left(\sum_{p=1}^{n} h^{\bar{j}p} \frac{\partial f}{\partial z^p} \right) \frac{\partial}{\partial \overline{z^j}},
\end{aligned}$$

$$\begin{aligned}
\Delta f &= \sum_{\alpha,\beta=1}^{2n} g^{\alpha\beta} f_{\alpha;\beta} \\
&= 4 \sum_{i,j=1}^{n} h^{\bar{j}i} \frac{\partial^2 f}{\partial z^i \partial \overline{z^j}},
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
\nabla f \cdot \nabla h &= \sum_{\alpha,\beta=1}^{2n} g^{\alpha\beta} \frac{\partial f}{\partial x^\alpha} \frac{\partial h}{\partial x^\beta} \\
&= 2 \sum_{i,j=1}^{n} h^{\bar{i}j} \frac{\partial f}{\partial \overline{z^i}} \frac{\partial h}{\partial z^j} + 2 \sum_{i,j=1}^{n} h^{\bar{i}j} \frac{\partial h}{\partial \overline{z^i}} \frac{\partial f}{\partial z^j},
\end{aligned}$$

where $f_{\alpha;\beta}$ is the covariant derivative of f_α with respective to Levi-Civita connection, ∇f is the gradient of f and the $\cdot$ is the Riemannian inner product. In particular,

$$|\nabla f|_g^2 = \nabla f \cdot \nabla \bar{f} = \sum_{\alpha,\beta=1}^{2n} g^{\alpha\beta} \frac{\partial f}{\partial x^\alpha} \frac{\partial \bar{f}}{\partial x^\beta} = 2 \sum_{i,j=1}^{n} h^{\bar{i}j} \frac{\partial f}{\partial \overline{z^i}} \frac{\partial \bar{f}}{\partial z^j} + 2 \sum_{i,j=1}^{n} h^{\bar{i}j} \frac{\partial \bar{f}}{\partial \overline{z^i}} \frac{\partial f}{\partial z^j}.$$

3. A useful lemma

We need the following lemma to prove our results.

LEMMA 3.1. *Let Γ_i be the i-th eigenvalue of the above clamped plate eigenvalue problem* (1.1) *and u_i be the orthonormal eigenvalue corresponding to Γ_i, that is, u_i satisfies*

$$(3.1)\qquad \begin{cases} \Delta^2 u_i = \Gamma_i u_i, & in\ \Omega, \\ u_i = \dfrac{\partial u_i}{\partial v} = 0, & on\ \partial\Omega, \\ \int_\Omega u_i u_j = \delta_{ij}, & for\ any\ i, j. \end{cases}$$

Then for any complex value function $g \in C^3(\Omega, \mathbf{C}) \cap C^2(\partial\Omega, \mathbf{C})$, *we have*

$$(3.2)\qquad \sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)^2 \int_\Omega |\nabla g|^2 u_i^2$$
$$\leq \sum_{i=1}^{k} \frac{\Gamma_{k+1} - \Gamma_i}{\beta} \int_\Omega \left(|\nabla g \cdot \nabla u_i|^2 + \mathrm{Re}(u_i \Delta g \nabla \bar{g} \cdot \nabla u_i) + \frac{1}{4} u_i^2 |\Delta g|^2 \right)$$
$$+ \sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)^2 \beta$$
$$\times \int_\Omega (u_i^2 |\Delta g|^2 + 4(|\nabla g \cdot \nabla u_i|^2 + \mathrm{Re}(u_i \Delta g (\nabla \bar{g} \cdot \nabla u_i))) - 2u_i |\nabla g|^2 \Delta u_i)$$

and

$$(3.3)\qquad \sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)^2 \int_\Omega |\nabla g|^2 u_i^2$$
$$\leq \sum_{i=1}^{k} \frac{\Gamma_{k+1} - \Gamma_i}{\beta} \int_\Omega \left(|\nabla g \cdot \nabla u_i|^2 - \frac{1}{2} \int_\Omega u_i^2\ \mathrm{Re}(\nabla \bar{g} \cdot \nabla(\Delta g)) - \frac{1}{4} \int_\Omega |\Delta g|^2 u_i^2 \right)$$
$$+ \beta \sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)^2 \int_\Omega (-u_i^2 |\Delta g|^2 + 4|\nabla g \cdot \nabla u_i|^2$$
$$- 2u_i^2\ \mathrm{Re}(\nabla \bar{g} \cdot \nabla(\Delta g)) - 2u_i |\nabla g|^2 \Delta u_i).$$

where β *is any positive constant.*

Proof. Define

$$\begin{cases} a_{ij} = \displaystyle\int_\Omega g u_i u_j = u a_{ji}, \\ b_{ij} = \displaystyle\int_\Omega \left(\nabla g \cdot \nabla u_i + \frac{1}{2} u_i \Delta g \right) u_j = -b_{ji}, \\ c_{ij} = \displaystyle\int_\Omega u_j (\Delta(u_i \Delta g) + 2\Delta(\nabla g \cdot \nabla w_i) + 2\nabla g \cdot \nabla(\Delta u_i) + \Delta g \Delta u_i). \end{cases}$$

Then from the Stokes' theorem, we have

$$(3.4)\quad (\Gamma_j - \Gamma_i)a_{ij} = \int_\Omega (gu_i\Delta^2 u_j - gu_j\Delta^2 u_i)$$
$$= \int_\Omega (\Delta(gu_i)\Delta u_j - \Delta(gu_j)\Delta u_i)$$
$$= \int_\Omega (u_i\Delta g + 2\nabla g\cdot\nabla u_i)\Delta u_j - (u_j\Delta g + 2\nabla g\cdot\nabla u_j)\Delta u_i$$
$$= \int_\Omega u_j(\Delta(u_i\Delta g) + 2\Delta(\nabla g\cdot\nabla u_i) + \Delta g\Delta u_i + 2\nabla(\Delta u_i)\cdot\nabla g)$$
$$= c_{ij}.$$

From (3.4), the Stokes' theorem and the Paseval Identity, we have

$$(3.5)\quad \int_\Omega gu_i\overline{(\Delta(u_i\Delta g) + 2\Delta(\nabla g\cdot\nabla u_i) + \Delta g\Delta u_i + 2\nabla(\Delta u_i)\cdot\nabla g)}$$
$$= \int_\Omega (\Delta(gu_i)u_i\Delta\bar{g} + 2\Delta(gu_i)\nabla\bar{g}\cdot\nabla u_i - 2\Delta u_i\,\mathrm{div}(gu_i\nabla\bar{g}) + gu_i\Delta\bar{g}\Delta u_i)$$
$$= \int_\Omega (u_i^2|\Delta g|^2 + 4(|\nabla g\cdot\nabla u_i|^2 + \mathrm{Re}(u_i\Delta g(\nabla\bar{g}\cdot\nabla u_i))) - 2u_i|\nabla g|^2\Delta u_i)$$
$$= \int_\Omega (-u_i^2|\Delta g|^2 + 4|\nabla g\cdot\nabla u_i|^2 - 2\,\mathrm{Re}(u_i^2\nabla\bar{g}\cdot\nabla(\Delta g)) - 2u_i|\nabla g|^2\Delta u_i)$$
$$= \sum_{j=1}^{\infty} a_{ij}\overline{c_{ij}}$$
$$= \sum_{j=1}^{\infty}(\Gamma_j - \Gamma_i)|a_{ij}|^2.$$

From (3.5), we have

$$(3.6)\quad (\Gamma_{k+1} - \Gamma_i)\sum_{j=1}^{\infty}|a_{ij}|^2$$
$$\le \sum_{j=1}^{\infty}(\Gamma_j - \Gamma_i)|a_{ij}|^2 + \sum_{j=1}^{k}(\Gamma_{k+1} - \Gamma_j)|a_{ij}|^2$$
$$= \int_\Omega (u_i^2|\Delta g|^2 + 4(|\nabla g\cdot\nabla u_i|^2 + \mathrm{Re}(u_i\Delta g(\nabla\bar{g}\cdot\nabla u_i))) - 2u_i|\nabla g|^2\Delta u_i)$$
$$+ \sum_{j=1}^{k}(\Gamma_{k+1} - \Gamma_j)|a_{ij}|^2$$

$$= \int_\Omega (-u_i^2|\Delta g|^2 + 4|\nabla g \cdot \nabla u_i|^2 - 2\operatorname{Re}(u_i^2 \nabla \bar{g} \cdot \nabla(\Delta g)) - 2u_i|\nabla g|^2 \Delta u_i) + \sum_{j=1}^{k} (\Gamma_{k+1} - \Gamma_j)|a_{ij}|^2.$$

From the definitions of b_{ij}'s and a_{ij}'s and the Paseval Identity, we have

$$\int_\Omega |\nabla g|^2 u_i^2 = -2\int_\Omega \bar{g} u_i \left(\nabla g \cdot \nabla u_i + \frac{1}{2} u_i \Delta g\right) = -2\sum_{j=1}^{\infty} \overline{a_{ij}} b_{ij}. \tag{3.7}$$

From (3.5), (3.6), (3.7) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
(\Gamma_{k+1} - \Gamma_i)^2 \int_\Omega |\nabla g|^2 u_i^2 \tag{3.8}
&= -2(\Gamma_{k+1} - \Gamma_i)^2 \sum_{j=1}^{\infty} \overline{a_{ij}} b_{ij} \\
&= -2(\Gamma_{k+1} - \Gamma_i)^2 \sum_{j=1}^{k} \overline{a_{ij}} b_{ij} - 2(\Gamma_{k+1} - \Gamma_i)^2 \sum_{j=k+1}^{\infty} \overline{a_{ij}} b_{ij} \\
&\le -2(\Gamma_{k+1} - \Gamma_i)^2 \sum_{j=1}^{k} \overline{a_{ij}} b_{ij} + (\Gamma_{k+1} - \Gamma_i)^3 \beta_i \sum_{j=k+1}^{\infty} |a_{ij}|^2 \\
&\quad + \frac{\Gamma_{k+1} - \Gamma_i}{\beta_i} \sum_{j=k+1}^{\infty} |b_{ij}|^2 \\
&= -2(\Gamma_{k+1} - \Gamma_i)^2 \sum_{j=1}^{k} \overline{a_{ij}} b_{ij} - (\Gamma_{k+1} - \Gamma_i)^3 \beta_i \sum_{j=1}^{k} |a_{ij}|^2 \\
&\quad - \frac{\Gamma_{k+1} - \Gamma_i}{\beta_i} \sum_{j=1}^{k} |b_{ij}|^2 \\
&\quad + \frac{\Gamma_{k+1} - \Gamma_i}{\beta_i} \sum_{j=1}^{\infty} |b_{ij}|^2 + (\Gamma_{k+1} - \Gamma_i)^2 \beta_i (\Gamma_{k+1} - \Gamma_i) \sum_{j=1}^{\infty} |a_{ij}|^2 \\
&\le -2(\Gamma_{k+1} - \Gamma_i)^2 \sum_{j=1}^{k} \overline{a_{ij}} b_{ij} - (\Gamma_{k+1} - \Gamma_i)^3 \beta_i \sum_{j=1}^{k} |a_{ij}|^2 \\
&\quad - \frac{\Gamma_{k+1} - \Gamma_i}{\beta_i} \sum_{j=1}^{k} |b_{ij}|^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\Gamma_{k+1}-\Gamma_i}{\beta_i}\sum_{j=1}^{\infty}|b_{ij}|^2+(\Gamma_{k+1}-\Gamma_i)^2\beta_i\sum_{j=1}^{\infty}(\Gamma_j-\Gamma_i)|a_{ij}|^2\\
&+(\Gamma_{k+1}-\Gamma_i)^2\beta_i\sum_{j=1}^{k}(\Gamma_{k+1}-\Gamma_j)|a_{ij}|^2\\
&=-2(\Gamma_{k+1}-\Gamma_i)^2\sum_{j=1}^{k}\overline{a_{ij}}b_{ij}+(\Gamma_{k+1}-\Gamma_i)^2\beta_i\sum_{j=1}^{k}(\Gamma_i-\Gamma_j)|a_{ij}|^2\\
&-\frac{\Gamma_{k+1}-\Gamma_i}{\beta_i}\sum_{j=1}^{k}|b_{ij}|^2\\
&+\frac{\Gamma_{k+1}-\Gamma_i}{\beta_i}\sum_{j=1}^{\infty}|b_{ij}|^2+(\Gamma_{k+1}-\Gamma_i)^2\beta_i\sum_{j=1}^{\infty}(\Gamma_j-\Gamma_i)|a_{ij}|^2\\
&=-2(\Gamma_{k+1}-\Gamma_i)^2\sum_{j=1}^{k}\overline{a_{ij}}b_{ij}+(\Gamma_{k+1}-\Gamma_i)^2\beta_i\sum_{j=1}^{k}(\Gamma_i-\Gamma_j)|a_{ij}|^2\\
&-\frac{\Gamma_{k+1}-\Gamma_i}{\beta_i}\sum_{j=1}^{k}|b_{ij}|^2+\frac{\Gamma_{k+1}-\Gamma_i}{\beta_i}\int_\Omega\left|\nabla g\cdot\nabla u_i+\frac{1}{2}u_i\Delta g\right|^2\\
&+(\Gamma_{k+1}-\Gamma_i)^2\beta_i\int_\Omega(u_i^2|\Delta g|^2+4(|\nabla g\cdot\nabla u_i|^2\\
&+\mathrm{Re}(u_i\Delta g(\nabla\bar{g}\cdot\nabla u_i)))-2u_i|\nabla g|^2\Delta u_i)\\
&=-2(\Gamma_{k+1}-\Gamma_i)^2\sum_{j=1}^{k}\overline{a_{ij}}b_{ij}+(\Gamma_{k+1}-\Gamma_i)^2\beta_i\sum_{j=1}^{k}(\Gamma_i-\Gamma_j)|a_{ij}|^2\\
&-\frac{\Gamma_{k+1}-\Gamma_i}{\beta_i}\sum_{j=1}^{k}|b_{ij}|^2\\
&+\frac{\Gamma_{k+1}-\Gamma_i}{\beta_i}\int_\Omega\left(|\nabla g\cdot\nabla u_i|^2-\frac{1}{2}\int_\Omega u_i^2\,\mathrm{Re}(\nabla\bar{g}\cdot\nabla(\Delta g))-\frac{1}{4}\int_\Omega|\Delta g|^2u_i^2\right)\\
&+(\Gamma_{k+1}-\Gamma_i)^2\beta_i\int_\Omega(-u_i^2|\Delta g|^2+4|\nabla g\cdot\nabla u_i|^2\\
&-2u_i^2\,\mathrm{Re}(\nabla\bar{g}\cdot\nabla(\Delta g))-2u_i|\nabla g|^2\Delta u_i)
\end{aligned}
$$

where β_i is any positive constant.

Since

$$a_{ij}=a_{ji},\quad b_{ij}=-b_{ji},\tag{3.9}$$

from the Cauchy-Schwarz inequality, we have

$$
(3.10)\qquad -2\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\overline{a_{ij}}b_{ij} = -2\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)(\Gamma_j-\Gamma_i)\overline{a_{ij}}b_{ij}
$$
$$
\le \sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)(\Gamma_j-\Gamma_i)^2\beta_i|a_{ij}|^2 + \sum_{i,j=1}^{k}\frac{\Gamma_{k+1}-\Gamma_i}{\beta_i}|b_{ij}|^2.
$$

From (3.10), (3.8) and the Stokes' theorem, we have

$$
(3.11)\qquad \sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\int_\Omega |\nabla g|^2 u_i^2
$$
$$
\le \sum_{i=1}^{k}\frac{\Gamma_{k+1}-\Gamma_i}{\beta_i}\int_\Omega\left(|\nabla g\cdot\nabla u_i|^2+\mathrm{Re}(u_i\Delta g\nabla\bar{g}\cdot\nabla u_i)+\frac{1}{4}u_i^2|\Delta g|^2\right)
$$
$$
+\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\beta_i
$$
$$
\times\int_\Omega (u_i^2|\Delta g|^2+4(|\nabla g\cdot\nabla u_i|^2+\mathrm{Re}(u_i\Delta g(\nabla\bar{g}\cdot\nabla u_i)))-2u_i|\nabla g|^2\Delta u_i)
$$
$$
+\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)(\Gamma_{k+1}-\Gamma_j)(\Gamma_i-\Gamma_j)\beta_i|a_{ij}|^2
$$
$$
=\sum_{i=1}^{k}\frac{\Gamma_{k+1}-\Gamma_i}{\beta_i}
$$
$$
\times\int_\Omega\left(|\nabla g\cdot\nabla u_i|^2-\frac{1}{2}\int_\Omega u_i^2\,\mathrm{Re}(\nabla\bar{g}\cdot\nabla(\Delta g))-\frac{1}{4}\int_\Omega|\Delta g|^2u_i^2\right)
$$
$$
+(\Gamma_{k+1}-\Gamma_i)^2\beta_i\int_\Omega(-u_i^2|\Delta g|^2+4|\nabla g\cdot\nabla u_i|^2
$$
$$
-2u_i^2\,\mathrm{Re}(\nabla\bar{g}\cdot\nabla(\Delta g))-2u_i|\nabla g|^2\Delta u_i).
$$
$$
+\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)(\Gamma_{k+1}-\Gamma_j)(\Gamma_i-\Gamma_j)\beta_i|a_{ij}|^2.
$$

Taking $\beta_1=\cdots=\beta_k=\beta$ in (3.11), we have (3.2) and (3.3). □

Remark 3.1. (3.2) for $f\in C^\infty(\Omega,\mathbf{R})$ can be found in [22], and the proof there is different from ours.

4. Eigenvalue problems on $\mathbf{C}P^n(c)$

Proof of Theorem 1.1. Let $Z = (Z^0, Z^1, \dots, Z^n)$ be a homogeneous coordinate system of $\mathbf{C}P^n(c)$ $(Z^p \in \mathbf{C})$. Then the Fubini-Study metric is

$$h = \frac{4}{c}\frac{\sum_{\alpha=0}^n |Z^\alpha|^2 \sum_{\beta=0}^n \mathrm{d}Z^\beta \mathrm{d}\overline{Z^\beta} - \sum_{\alpha=0}^n \overline{Z^\alpha}\,\mathrm{d}Z^\alpha \sum_{\beta=0}^n Z^\beta\,\mathrm{d}\overline{Z^\beta}}{(\sum_{\alpha=0}^n |Z^\alpha|^2)^2}. \tag{4.1}$$

Defining $f_{p\bar{q}}$, for $p, q = 0, 1, \dots n$, by

$$f_{p\bar{q}} = \frac{Z^p\overline{Z^q}}{\sum_{r=0}^n Z^r\overline{Z^r}}, \tag{4.2}$$

we have

$$f_{p\bar{q}} = \overline{f_{q\bar{p}}}, \qquad \sum_{p,q=0}^n f_{p\bar{q}}\overline{f_{p\bar{q}}} = 1. \tag{4.3}$$

For any fixed point $P \in \Omega$, since any two points in $\mathbf{C}P^n(c)$ are holomorphic isometric, without loss of generality, we can assume that at P,

$$Z^0 \neq 0, \quad Z^1 = \cdots = Z^n = 0. \tag{4.4}$$

We know that

$$\left(z^1 = \frac{Z^1}{Z^0}, \dots, z^n = \frac{Z^n}{Z^0}\right)$$

is a local holomorphic coordinate system of $\mathbf{C}P^n(c)$ in a neighborhood U of $P \in \Omega$. Then in U we have

$$z^1(P) = \cdots = z^n(P) = 0, \tag{4.5}$$

for $p, q = 0, 1, \dots, n$,

$$f_{p\bar{q}} = \frac{z^p\overline{z^q}}{1 + \sum_{r=1}^n |z^r|^2}, \tag{4.6}$$

where $z^0 = 1$ and the Fubini-Study metric can be rewritten as $h = h_{i\bar{j}}\,\mathrm{d}z^i \otimes \mathrm{d}\overline{z^j}$, where

$$h_{i\bar{j}} = \frac{4}{c}\left(\frac{\delta_{ij}}{1 + \sum_{r=1}^n |z^r|^2} - \frac{z^j\overline{z^i}}{(1 + \sum_{r=1}^n |z^r|^2)^2}\right). \tag{4.7}$$

The inverse matrix of $(h_{i\bar{j}})$ is

$$h^{\bar{i}j} = \frac{c}{4}\left(1 + \sum_{r=1}^n |z^r|^2\right)(\delta_{ij} + \overline{z^i}z^j), \tag{4.8}$$

that is,

$$\sum_{p=1}^{n} h^{\bar{i}p} h_{p\bar{j}} = \delta_{ij}.$$

From (2.2) and (4.8), at the point P, we have

$$\nabla f_{p\bar{q}} = \begin{cases} \frac{c}{2} \frac{\partial}{\partial z^q}, & p = 0,\ q = 1, \ldots, n, \\ \frac{c}{2} \frac{\partial}{\partial \overline{z^p}}, & q = 0,\ p = 1, \ldots, n, \\ 0, & \text{otherwises}, \end{cases} \tag{4.9}$$

$$\Delta f_{p\bar{q}} = \begin{cases} -cn, & p = q = 0, \\ c\delta_{pq}, & 1 \le p, q \le n, \\ 0, & \text{otherwises} \end{cases} \tag{4.10}$$

and

$$|\nabla u_i|^2 = c \sum_{l=1}^{n} \left| \frac{\partial u_i}{\partial z^l} \right|^2. \tag{4.11}$$

From (4.9), (4.10) and (4.11), at the point P, we have

$$\begin{cases} \sum_{p,q=0}^{n} |\nabla f_{p\bar{q}}|^2 = cn, \\ \sum_{p,q=0}^{n} \Delta f_{p\bar{q}} \nabla \overline{f_{p\bar{q}}} = \vec{0}, \\ \sum_{p,q=0}^{n} |\Delta f_{p\bar{q}}|^2 = c^2 n(n+1), \\ \sum_{p,q=0}^{n} |\nabla f_{p\bar{q}} \cdot \nabla u_i|^2 = \frac{c}{2} |\nabla u_i|^2. \end{cases} \tag{4.12}$$

Since the point P is arbitrary, (4.12) holds on Ω.

From the Cauchy-Schwarz inequality and the Stokes' theorem, we have

$$\int_\Omega |\nabla u_i|^2 = \int_\Omega (-u_i)\Delta u_i \le \left(\int_\Omega u_i^2 \int_\Omega (\Delta u_i)^2 \right)^{1/2} = \left(\int_\Omega u_i(\Delta^2 u_i) \right)^{1/2} = \Gamma_i^{1/2}. \tag{4.13}$$

Taking $g = f_{p\bar{q}}$, $p, q = 0, 1, \ldots n$ in (3.2) and making summation on p and q from 0 to n, from (4.13), we have

$$(4.14)\qquad cn\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \le \sum_{i=1}^{k}\frac{\Gamma_{k+1}-\Gamma_i}{\beta}\left(\frac{c}{2}\Gamma_i^{1/2}+\frac{1}{4}c^2n(n+1)\right) + \beta\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2(2c(n+1)\Gamma_i^{1/2}+c^2n(n+1)).$$

Taking

$$\beta = \left(\frac{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2(2c(n+1)\Gamma_i^{1/2}+c^2n(n+1))}{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left(\frac{c}{2}\Gamma_i^{1/2}+\frac{1}{4}c^2n(n+1)\right)}\right)^{1/2},$$

we have (1.10). □

We need the following lemma in [22].

LEMMA 4.1. *Let $\{a_i\}_{i=1}^m$, $\{b_i\}_{i=1}^m$ and $\{c_i\}_{i=1}^m$ be three sequences of non-negative read numbers with $\{a_i\}_{i=1}^m$ decreasing and $\{b_i\}_{i=1}^m$ and $\{c_i\}_{i=1}^m$ increasing. Then we have*

$$(4.15)\qquad \left(\sum_{i=1}^{m}a_i^2b_i\right)\left(\sum_{i=1}^{m}a_ic_i\right) \le \left(\sum_{i=1}^{m}a_i^2\right)\left(\sum_{i=1}^{m}a_ib_ic_i\right).$$

Proof of Corollary 1.2. Taking

$$a_i = \Gamma_{k+1}-\Gamma_i,$$
$$b_i = 2(n+1)\Gamma_i^{1/2}+cn(n+1),$$
$$c_i = 2\Gamma_i^{1/2}+cn(n+1),$$

from (1.10) and Lemma 4.1, we have (1.11). From (1.11), we can deduce (1.12). From the Young's inequality ($\varepsilon > 0$), we have

$$\begin{aligned}(4.16)\qquad &(2\Gamma_i^{1/2}+cn(n+1))(2(n+1)\Gamma_i^{1/2}+cn(n+1))\\ &= 4(n+1)\Gamma_i+2cn(n+1)(n+2)\Gamma_i^{1/2}+c^2n^2(n+1)^2\\ &\le 4(n+1)\Gamma_i+4\varepsilon\Gamma_i+\frac{c^2n^2(n+1)^2(n+2)^2}{4\varepsilon}+c^2n^2(n+1)^2\\ &= 4(n+1+\varepsilon)\left(\Gamma_i+\frac{c^2n^2(n+1)^2(n^2+4n+4+4\varepsilon)}{16\varepsilon(n+1+\varepsilon)}\right)\end{aligned}$$

From (1.11) and (4.16), we have

$$(4.17)\quad \sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \leq \frac{4(n+1+\varepsilon)}{n^2}\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left(\Gamma_i+\frac{c^2n^2(n+1)^2(n^2+4n+4+4\varepsilon)}{16\varepsilon(n+1+\varepsilon)}\right).$$

From (4.17) and Corollary 2.1 in Cheng and Yang [9] (see also [10]), we can deduce (1.13). □

5. Eigenvalue problems on the submanifold of $\mathbf{C}P^{n+m}(c)$

Proof of Theorem 1.3. Let $\imath : \Omega \hookrightarrow \mathbf{C}P^{n+m}(c)$ be the inclusion map and an imbedding. Defining $d_{p\bar{q}}$, for $p, q = 0, 1, \dots n+m$, by

$$(5.1)\quad d_{p\bar{q}} = \frac{Z^p\overline{Z^q}}{\sum_{r=0}^{n+m} Z^r\overline{Z^r}},$$

we have

$$(5.2)\quad d_{p\bar{q}} = \overline{d_{q\bar{p}}}, \quad \sum_{p,q=0}^{n+m} d_{p\bar{q}}\overline{d_{p\bar{q}}} = 1.$$

For any fixed point $P \in \Omega$, since any two points in $\mathbf{C}P^{n+m}(c)$ are holomorphic isometric, without loss of generality, we can assume that at P,

$$(5.3)\quad Z^0 \neq 0, \quad Z^1 = \cdots = Z^{n+m} = 0,$$
$$\left(z^1 = \frac{Z^1}{Z^0}, \dots, z^{n+m} = \frac{Z^{n+m}}{Z^0}\right)$$

is a local holomorphic coordinate system of $\mathbf{C}P^{n+m}(c)$ in a neighborhood U of $P \in \Omega$, and

$$(5.4)\quad \imath : \Omega \cap U \hookrightarrow U,$$
$$(z^1, \dots, z^n) \mapsto (z^1, \dots, z^n, h_1(z^1, \dots, z^n), \dots, h_m(z^1, \dots, z^n)),$$

where h_i's are holomorphic functions defined in $\Omega \cap U$ satisfying

$$\left.\frac{\partial h_i}{\partial z^k}\right|_P = 0, \quad i = 1, \dots, m, \ k = 1, \dots, n.$$

Then in U we have

$$(5.5)\quad z^1(P) = \cdots = z^{n+m}(P) = 0,$$

for $p, q = 0, 1, \ldots, n+m$,

$$d_{p\bar{q}} = \frac{z^p \overline{z^q}}{1 + \sum_{r=1}^{n+m} |z^r|^2}, \tag{5.6}$$

where $z^0 = 1$ and the Kählerian metric on Ω at the point P induced from the Fubini-Study metric h on $\mathbf{C}P^{n+m}(c)$ by ι can be rewritten as

$$g|_P = (\iota^* h)|_P = g_{i\bar{j}}|_P \, \mathrm{d}z^i|_P \otimes \mathrm{d}\overline{z^j}|_P = \frac{4}{c}\delta_{ij} \, \mathrm{d}z^i|_P \otimes \mathrm{d}\overline{z^j}|_P.$$

Therefore, the inverse matrix of $(g_{i\bar{j}}|_P)$ is

$$g^{\bar{i}j}|_P = \frac{c}{4}\delta_{ij}. \tag{5.7}$$

that is,

$$\sum_{r=1}^{n} g^{\bar{r}i} g_{j\bar{r}} = \delta_{ij}.$$

From (5.6) and (5.7), at the point P, we have

$$\nabla d_{p\bar{q}} = \begin{cases} \dfrac{c}{2}\dfrac{\partial}{\partial z^q}, & p = 0, \ q = 1, \ldots, n, \\ \dfrac{c}{2}\dfrac{\partial}{\partial \overline{z^p}}, & q = 0, \ p = 1, \ldots, n, \\ 0, & \text{otherwises} \end{cases} \tag{5.8}$$

and

$$\Delta d_{p\bar{q}} = \begin{cases} -cn, & p = q = 0, \\ c\delta_{pq}, & 1 \le p, q \le n, \\ 0, & \text{otherwises.} \end{cases} \tag{5.9}$$

From (5.8) and (5.9), we have

$$\begin{cases} \displaystyle\sum_{p,q=0}^{n+m} |\nabla d_{p\bar{q}}|^2 = cn, \\ \displaystyle\sum_{p,q=0}^{n+m} \Delta d_{p\bar{q}} \nabla \overline{d_{p\bar{q}}} = \vec{0}, \\ \displaystyle\sum_{p,q=0}^{n+m} |\Delta d_{p\bar{q}}|^2 = c^2 n(n+1), \\ \displaystyle\sum_{p,q=0}^{n+m} |\nabla d_{p\bar{q}} \cdot \nabla u_i|^2 = \frac{c}{2}|\nabla u_i|^2. \end{cases} \tag{5.10}$$

Since the point P is arbitrary, (5.10) holds on Ω. Similar to the process of obtaining (3.2), we have

$$
\begin{aligned}
(5.11)\qquad & \sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\int_\Omega |\nabla g|^2 u_i^2 \\
&\le \sum_{i=1}^{k}\frac{\Gamma_{k+1}-\Gamma_i}{\beta}\int_\Omega\left(|\nabla g\cdot\nabla u_i|^2+\mathrm{Re}(u_i\Delta g\nabla\bar g\cdot\nabla u_i)+\frac14 u_i^2|\Delta g|^2\right)\\
&\quad+\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\beta\\
&\quad\times\int_\Omega (u_i^2|\Delta g|^2+4(|\nabla g\cdot\nabla u_i|^2+\mathrm{Re}(u_i\Delta g(\nabla\bar g\cdot\nabla u_i)))-2u_i|\nabla g|^2\Delta u_i).
\end{aligned}
$$

Taking $g=d_{p\bar q}$, $p,q=0,1,\dots n+m$ in (5.11) and making summation on p and q from 0 to $n+m$, from (4.13) and (5.10), we have

$$
\begin{aligned}
(5.12)\qquad cn\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 &\le \sum_{i=1}^{k}\frac{\Gamma_{k+1}-\Gamma_i}{\beta}\left(\frac{c}{2}\Gamma_i^{1/2}+\frac14 c^2 n(n+1)\right)\\
&\quad+\beta\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2(2c(n+1)\Gamma_i^{1/2}+c^2n(n+1)).
\end{aligned}
$$

Taking

$$
\beta=\left(\frac{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2(2c(n+1)\Gamma_i^{1/2}+c^2n(n+1))}{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left(\frac{c}{2}\Gamma_i^{1/2}+\frac14 c^2n(n+1)\right)}\right)^{1/2},
$$

we have (1.14). □

Proof of Corollary 1.4. The proof of this corollary is similar to the one of Corollary 1.2. □

6. Eigenvalue problems on complete noncompact Riemannian manifold

Proof of Theorem 1.5. For $p\notin\overline{\Omega}$ fixed point, the distance function $\rho(x)$ is defined by $\rho(x)=\mathrm{distance}(x,p)$. From $|\nabla\rho|=1$ and Proposition 2.2 of [19], we have

$$
(6.1)\qquad \nabla\rho\cdot\nabla(\Delta\rho)=-|\mathrm{Hess}\,\rho|^2-\mathrm{Ric}(\nabla\rho,\nabla\rho).
$$

From the discussion in [5], we have

$$
(6.2)\qquad 2|\mathrm{Hess}\,\rho|^2+2\,\mathrm{Ric}(\nabla\rho,\nabla\rho)-(\Delta\rho)^2\le-(n-1)^2b^2+(a^2-b^2).
$$

From (4.13) and (6.2), we have

$$\int_\Omega\left(|\nabla\rho\cdot\nabla u_i|^2-\frac{1}{2}\int_\Omega u_i^2(\nabla\rho\cdot\nabla(\Delta\rho))-\frac{1}{4}\int_\Omega|\Delta\rho|^2u_i^2\right) \tag{6.3}$$
$$\le\int_\Omega|\nabla u_i|^2+\frac{1}{4}(-(n-1)^2b^2+(a^2-b^2))$$
$$\le\Gamma_i^{1/2}-\frac{(n-1)^2}{4}b^2+\frac{a^2-b^2}{4}$$

and

$$\int_\Omega(-u_i^2|\Delta\rho|^2+4|\nabla\rho\cdot\nabla u_i|^2-2u_i^2(\nabla\rho\cdot\nabla(\Delta\rho))-2u_i|\nabla\rho|^2\Delta u_i) \tag{6.4}$$
$$\le 6\int_\Omega|\nabla u_i|^2+\int_\Omega u_i^2(2|\mathrm{Hess}\,\rho|^2+2\,\mathrm{Ric}(\nabla\rho,\nabla\rho)-(\Delta\rho)^2)$$
$$\le 6\left(\Gamma_i^{1/2}-\frac{(n-1)^2}{6}b^2+\frac{a^2-b^2}{6}\right).$$

Taking $g=\rho$ in (3.3), from (6.3) and (6.4), we have

$$\sum_{i=1}^k(\Gamma_{k+1}-\Gamma_i)^2\le\sum_{i=1}^k\frac{\Gamma_{k+1}-\Gamma_i}{\beta}\left(\Gamma_i^{1/2}-\frac{(n-1)^2}{4}b^2+\frac{a^2-b^2}{4}\right) \tag{6.5}$$
$$+\beta 6\sum_{i=1}^k(\Gamma_{k+1}-\Gamma_i)^2\left(\Gamma_i^{1/2}-\frac{(n-1)^2}{6}b^2+\frac{a^2-b^2}{6}\right).$$

Taking

$$\beta=\left(\frac{6\sum_{i=1}^k(\Gamma_{k+1}-\Gamma_i)^2\left(\Gamma_i^{1/2}-\frac{(n-1)^2}{6}b^2+\frac{a^2-b^2}{6}\right)}{\sum_{i=1}^k(\Gamma_{k+1}-\Gamma_i)\left(\Gamma_i^{1/2}-\frac{(n-1)^2}{4}b^2+\frac{a^2-b^2}{4}\right)}\right)^{1/2},$$

we have (1.21). □

Proof of Corollary 1.6. Taking

$$a_i=\Gamma_{k+1}-\Gamma_i,$$
$$b_i=6\left(\Gamma_i^{1/2}-\frac{(n-1)^2}{6}b^2+\frac{a^2-b^2}{6}\right),$$
$$c_i=4\left(\Gamma_i^{1/2}-\frac{(n-1)^2}{4}b^2+\frac{a^2-b^2}{4}\right),$$

from the Lemma 4.1 and (1.21), we have (1.22). □

Acknowledgements. The author thanks Professor Hongcang Yang for his encouragements, suggestions and support and also thanks the referees for their helpful comments and suggestions.

References

[1] S. Agmon, On kernels, eigenvalues, and eigenfunctions of operators related to elliptic problems, Comm. Pure Appl. Math. **18** (1965), 627–663.

[2] M. S. Ashbaugh, Isoperimetric and universal inequalities for eigenvalues, Spectral theory and geometry, London Math. Soc. lecture notes **273**, Cambridge Univ. Press, Cambridge, 1999, 95–139.

[3] Z.-C. Chen and C.-L. Qian, Estimates for discrete spectrum of operator with any order, J. China Univ. Sci. Tech. **20** (1990), 259–266.

[4] D. Chen, T. Zheng and M. Lu, Eigenvalue estimates on domains in complete noncompact Riemannian manifolds, Pacific J. Math. **255** (2012), 41–54.

[5] D. Chen, T. Zheng and H. Yang, Estimates of the gaps between consecutive eigenvalues of Laplacian, preprint, arXiv:1309.7446.

[6] Q.-M. Cheng, T. Ichikawa and S. Mametsuka, Estimates for eigenvalues of a clamped plate problem on Riemannian manifolds, J. Math. Soc. Japan **62** (2010), 673–686.

[7] Q.-M. Cheng and H. Yang, Inequalities for eigenvalues of Laplacian on domains and compact hypersurfaces in complex projective spaces, J. Math. Soc. Japan **58** (2006), 545–561.

[8] Q.-M. Cheng and H. Yang, Inequalities for eigenvalues of a clamped plate problem, Trans. Amer. Math. Soc. **358** (2006), 2625–2635.

[9] Q.-M. Cheng and H. Yang, Bounds on eigenvalues of Dirichlet Laplacian, Math. Ann. **337** (2007), 159–175.

[10] Q.-M. Cheng and H. Yang, Estimates for eigenvalues on Riemannian manifolds, J. Differential Equations **247** (2009), 2270–2281.

[11] Q.-M. Cheng and H. Yang, Universal inqualities for eigenvalues of a calmpled plate problem on a hyperbolic space, Proc. Amer. Math. Soc. **139** (2011), 461–471.

[12] G. N. Hile and R. Z. Yeh, Inequalities for eigenvalues of the biharmonic operator, Pacific J. Math. **112** (1984), 115–133.

[13] S. M. Hook, Domain independent upper bounds for eigenvalues of elliptic operator, Trans. Amer. Math. Soc. **318** (1990), 615–642.

[14] Q. Lu, The new results of the classical manifolds and the classical domains, Shanghai Scientific and Technical Publishers, Shanghai, 1997 (in China).

[15] L. E. Payne, G. Pólya and H. F. Weinberger, Sur le quotient de deux fréquences propres consécutives, Comptes Rendus Acad. Sci. Paris **241** (1955), 917–919.

[16] G. E. Payne, G. Pólya and H. F. Weinberger, On the ratio of consecutive eigenvalue, J. Math. Phys. **35** (1956), 289–298.

[17] Å. Pleijel, On the eigenvalues and eigenfunctions of elastic plates, Comm. Pure Appl. Math. **3** (1950), 1–10.

[18] M. H. Protter, Universal inequalities for eigenvalues, Maximum principles and eigenvalue problems in partial differential equations, Pitman res. notes math. ser. **175**, Longman Sci. Tech., Harlow, Essex, United Kingdom, 1988, 111–120.

[19] R. Schoen and S. T. Yau, Lectures on differntial geometry, International Press, Boston, 1994.

[20] H. Sun, Q.-M. Cheng and H. Yang, Lower eigenvalue of Dirichlet Laplacian, Manuscripta Math. **125** (2008), 139–156.

[21] Q. Wang and C. Xia, Universal bounds for eigenvalues of the biharmonic operator on Riemannian manifolds, J. Funct. Anal. **245** (2007), 334–352.

[22] Q. Wang and C. Xia, Inequlities for eigenvalues of a clamped plate problem, Calc. Var. Partial Differential Equations **40** (2011), 273–289.

[23] C. Xia, Universal inequalities for eigenvalues of the vibration problem for a clamped plate on Riemannian manifolds, Q. J. Math. **62** (2011), 235–258.

Tao Zheng
School of Mathematics and Statistics
Beijing Institute of Technology
Beijing 100081
P. R. China
E-mail: zhengtao08@amss.ac.cn