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INDUCTION FUNCTORS FOR GROUP CORINGS

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Abstract

In the paper, we prove that the induction functor stemming from every morphism of group coring versus coring has a left adjoint, called ad-induction functor. The separability of the induction functor is characterized, extending some results for corings.

1. Introduction

As the generalization of coring, introduced by Sweedler [8] and revised by Brzeziński [1], Caenepeel et al. introduced the group coring and developed Galois theory for group corings in [2], which have become increasingly an interesting subject to study. Some study of the new structure has been carried out in recent papers (see [4], [6] and [10]).

Given an A-coring C, where A is an algebra over a fixed field k, we have the category $\mathcal{M}^C$ of all the right comodule over C. It follows from [1] that there exists a pair of adjoint functor between the category $\mathcal{M}^C$ and the category $\mathcal{M}_A$ of all the right A-modules. The extension of this result to the context of Hopf group-coalgebras was made in the work of the authors in [10], that is, there exists a pair of adjoint functor between the category $\mathcal{M}^{G,C}$ of all the right G-C-comodule and the category $\mathcal{M}_A$ of all the right A-modules. As we know, an algebra A has a canonical A-coring structure over itself. A natural question occurs to us: whether there exists a pair of adjoint functor between the category $\mathcal{M}^{G,C}$ of all the right G-C-comodule and the category $\mathcal{M}^D$ of all the right comodules over a given B-coring, if so, how to characterize its separability. This is done in this paper.

This paper is organized as follows.

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In Section 2, we recall some basic concepts such as group coring and cotensor product. In Section 3, we use the notion of homomorphism of corings to construct a pair of adjoint functors (the induction functor and its adjoint, called here ad-induction functor). Finally, the separability of the induction functor is characterized.

2. Preliminaries

Throughout this paper, we always let G be a group with the unit e and k a field.

2.1. Group corings. First recall from [2] that a G-group A-coring (or shortly a G-A-coring) C is a family $\{C_\alpha\}_{\alpha \in G}$ of A-bimodules together with a family of A-bimodule maps

$$\Delta_{\alpha,\beta} : C_{\alpha\beta} \to C_\alpha \otimes_A C_\beta, \quad \varepsilon : C_e \to A$$

such that

$$(\Delta_{\alpha,\beta} \otimes_A C_\gamma) \circ \Delta_{\alpha\beta,\gamma} = (C_\alpha \otimes_A \Delta_{\beta,\gamma}) \circ \Delta_{\alpha,\beta\gamma}$$

and

$$(C_\alpha \otimes_A \varepsilon) \circ \Delta_{\alpha,e} = C_\alpha = (\varepsilon \otimes_A C_\alpha) \circ \Delta_{e,\alpha}$$

for all $\alpha, \beta, \gamma \in G$.

Remark 2.1. If C is a G-A-coring, then C_e is an ordinary A-coring in sense of [8].

We use the following Sweedler-type notation for the comultiplication maps $\Delta_{\alpha,\beta}$:

$$\Delta_{\alpha,\beta}(c) = c_{(1,\alpha)} \otimes_A c_{(2,\beta)},$$

for all $c \in C_{\alpha\beta}$.

A right G-C-comodule $M = \{M_\alpha\}_{\alpha \in G}$ is a family of right A-modules, together with a family of right A-linear maps $\rho^M = \{\rho^M_{\alpha,\beta}\}_{\alpha,\beta \in G}$,

$$\rho^M_{\alpha,\beta} : M_{\alpha\beta} \to M_\alpha \otimes_A C_\beta$$

such that

$$(M_\alpha \otimes_A \Delta_{\beta,\gamma}) \circ \rho^M_{\alpha,\beta\gamma} = (\rho^M_{\alpha,\beta} \otimes_A C_\gamma) \circ \rho^M_{\alpha\beta,\gamma}$$

and

$$(M_\alpha \otimes_A \varepsilon) \circ \rho^M_{\alpha,e} = M_\alpha$$

for all $\alpha, \beta, \gamma \in G$.

We use the following Sweedler-type notation:

$$\rho^M_{\alpha,\beta}(m) = m_{[0,\alpha]} \otimes_A m_{[1,\beta]}$$

for $m \in M_{\alpha\beta}$.

A morphism between two right G-C-comodules M and N is a family of right A-linear maps $f = \{f_\alpha : M_\alpha \to N_\alpha\}_{\alpha\in G}$ such that

$$(f_\alpha \otimes_A C_\beta) \circ \rho_{\alpha,\beta} = \rho_{\alpha,\beta} \circ f_{\alpha\beta}.$$

The category of right G-C-comodules will be denoted by $\mathscr{M}^{G,C}$.

2.2. The cotensor product. Let D be a B-coring. Let $M \in \mathscr{M}^D$ and $N \in {}^D\mathscr{M}$. First recall that the cotensor product $M\square_D N$ of M and N is given by

$$M\square_D N = \left\{ \sum_i m_i \otimes_B n_i \in M \otimes N \,\middle|\, \sum_i m_{i[0]} \otimes_B m_{i[1]} \otimes_B n_i = \sum_i m_i \otimes_B n_{i[-1]} \otimes_B n_{i[0]} \right\},$$

that is, $M\square_D N$ fits an exact sequence

$$0 \to M\square_D N \to M \otimes_B N \rightrightarrows M \otimes_B D \otimes_B N,$$

where the two maps $M \otimes_B N \to M \otimes_B D \otimes_B N$ are $\rho^M \otimes_B N$ and $M \otimes_B \rho^N$.

3. Separable homomorphisms of G-A-corings versus corings

Consider a G-A-coring C and a B-coring D, where A and B are both k-algebra.

DEFINITION 3.1. A *coring homomorphism* is a pair (φ, μ), where $\mu : A \to B$ is a homomorphism of algebras and $\varphi : C_e \to D$ is a homomorphism of A-bimodules, and such that the following equations

$$\vartheta_{D,D} \circ (\varphi \otimes_A \varphi) \circ \Delta_{e,e} = \Delta_D \circ \varphi,$$

$$\mu \circ \varepsilon_e = \varepsilon_D \circ \varphi,$$

where $\vartheta_{D,D} : D \otimes_A D \to D \otimes_B D$ is the canonical map induced by μ.

Throughout the rest of this section, we always assume that there exists a coring homomorphism (φ, μ) between two corings C_e and D.

3.1. The induction functor.

PROPOSITION 3.2. *The assignment* $M \mapsto M_e \otimes_A B$ *establishes a functor* $(-)_e \otimes_A B : \mathscr{M}^{G,C} \to \mathscr{M}^D$.

Proof. Let $\rho^M = \{\rho^M_{\alpha,\beta} : M_{\alpha\beta} \to M_\alpha \otimes_A C_\beta\}_{\alpha,\beta \in G}$ be a right G-C-comodule. Define

$$\rho^{M_e \otimes_A B} : M_e \otimes_A B \to M_e \otimes_A B \otimes_B D, \quad m \otimes_A b \mapsto m_{[0,e]} \otimes_A 1_B \otimes_B \varphi(m_{[1,e]}) \cdot b.$$

It is straightforward to check that $M_e \otimes_A B$ is an object of $\mathscr{M}^D$. In order to show the assignment $M \mapsto M_e \otimes_A B$ is functorial, we will prove that $f_e \otimes_A B$ is a homomorphism of right D-comodules for every morphism $f = \{f_\alpha : M_\alpha \to N_\alpha\}_{\alpha \in G}$ in $\mathscr{M}^{G,C}$. In fact, for all $m \in M_e$ and $b \in B$, we have

$$\begin{aligned}
\rho^{N_e \otimes_A B} \circ (f_e \otimes_A B)(m \otimes_A b) &= f(m)_{[0,e]} \otimes_A 1_B \otimes_B \varphi(f(m)_{[1,e]}) \cdot b \\
&= f(m_{[0,e]}) \otimes_A 1_B \otimes_B \varphi(m_{[1,e]}) \cdot b \\
&= (f_e \otimes_A B \otimes_B D) \circ \rho^{M_e \otimes_A B}(m \otimes_A b).
\end{aligned}$$

This ends the proof. □

3.2. The ad-induction functor. For each $\alpha \in G$, define

$$\rho^{B \otimes_A C_\alpha} : B \otimes_A C_\alpha \to D \otimes_B B \otimes_A C_\alpha, \quad b \otimes_A c \mapsto b \cdot \varphi(c_{(1,e)}) \otimes_B 1_B \otimes_A c_{(2,\alpha)}.$$

Lemma 3.3. *$B \otimes_A C = \{B \otimes_A C_\alpha\}_{\alpha \in G}$ is a D-C-bicomodule.*

Proof. It is sufficient to prove that the following diagram is commutative,

$$\begin{array}{ccc}
B \otimes_A C_{\alpha\beta} & \xrightarrow{B \otimes_A \Delta_{\alpha,\beta}} & B \otimes_A C_\alpha \otimes_A C_\beta \\
\Big\downarrow{\scriptstyle \rho^{B \otimes_A C_{\alpha\beta}}} & & \Big\downarrow{\scriptstyle \rho^{B \otimes_A C_\alpha} \otimes_A C_\beta} \\
D \otimes_B B \otimes_A C_{\alpha\beta} & \xrightarrow{D \otimes_B B \otimes_A \Delta_{\alpha,\beta}} & D \otimes_B B \otimes_A C_\alpha \otimes_A C_\beta
\end{array}$$

Indeed, for all $b \in B$ and $c \in C_{\alpha\beta}$,

$$\begin{aligned}
&(\rho^{B \otimes_A C_\alpha} \otimes_A C_\beta) \circ (B \otimes_A \Delta_{\alpha,\beta})(b \otimes_A c) \\
&\quad = b \cdot \varphi(c_{(1,\alpha)(1,e)}) \otimes_B 1_B \otimes_A c_{(1,\alpha)(2,\alpha)} \otimes_A c_{(2,\beta)} \\
&\quad = b \cdot \varphi(c_{(1,e)}) \otimes_B 1_B \otimes_A c_{(2,\alpha\beta)(1,\alpha)} \otimes_A c_{(2,\alpha\beta)(2,\beta)} \\
&\quad = (D \otimes_B B \otimes_A \Delta_{\alpha,\beta}) \circ \rho^{B \otimes_A C_{\alpha\beta}}(b \otimes_A c).
\end{aligned}$$

This shows that $B \otimes_A C = \{B \otimes_A C_\alpha\}_{\alpha \in G}$ is a D-C-bicomodule. □

Proposition 3.4. *We have a pair of adjoint functors (F, U) between the categories $\mathscr{M}^{G,C}$ and $\mathscr{M}_B$ (the category of right B-module).*

Proof. Take $M = \{M_\alpha\}_{\alpha\in G}$, and define

$$F : \mathscr{M}^{G,C} \to \mathscr{M}_B, \quad M \mapsto M_e \otimes_A B.$$

For a morphism $f = \{f_\alpha : M_\alpha \to M'_\alpha\}_{\alpha\in G}$ in $\mathscr{M}^{G,C}$, we simply define

$$F(f) = f_e \otimes_A B.$$

Let us now define U. For $N \in \mathscr{M}_B$, and define

$$U : \mathscr{M}_B \to \mathscr{M}^{G,C}, \quad N \mapsto N \otimes_B (B \otimes_A C),$$

where $N \otimes_B (B \otimes_A C) = \{N \otimes_B (B \otimes_A C_\alpha)\}_{\alpha\in G}$ with the G-comodule structure maps

$$\rho^{G(N)} = \{\rho^{G(N)}_{\alpha,\beta} = N \otimes_B (B \otimes_A \Delta_{\alpha,\beta})\}_{\alpha,\beta\in G}.$$

Consider the map

$$\phi : \mathrm{Hom}_B(M_e \otimes_A B, N) \to \mathrm{Hom}^C(M, N \otimes_B (B \otimes_A C)), \tag{3.1}$$

sending f to $\phi(f) = \{\phi(f)_\alpha\}_{\alpha\in G}$, where

$$\phi(f)_\alpha : M_\alpha \to N \otimes_B (B \otimes_A C_\alpha), \phi(f)_\alpha(m) = f(m_{[0,e]} \otimes_A 1_B) \otimes_B (1_B \otimes_A m_{[1,\alpha]})$$

and

$$\varphi : \mathrm{Hom}^C(M, N \otimes_B (B \otimes_A C)) \to \mathrm{Hom}_B(M_e \otimes_A B, N), g \to \varphi(g),$$

where

$$\varphi(g)(m \otimes_A b) = (N \otimes_B (B \otimes_A \varepsilon)(g_e(m))) \cdot b.$$

Let us check that ϕ and φ are mutually inverse:

$$\begin{aligned}
\phi(\varphi(g))_\alpha(m) &= \varphi(g)(m_{[0,e]} \otimes_A 1_B) \otimes_B (1_B \otimes_A m_{[1,\alpha]})\\
&= (N \otimes_B (B \otimes_A \varepsilon)(g_e(m_{[0,e]}))) \otimes_B (1_B \otimes_A m_{[1,\alpha]})\\
&= ((N \otimes_B (B \otimes_A \varepsilon)) \otimes_B (B \otimes_A C_\alpha))(g_e(m_{[0,e]}) \otimes_B 1_B \otimes_A m_{[1,\alpha]})\\
&= ((N \otimes_B (B \otimes_A \varepsilon)) \otimes_B (B \otimes_A C_\alpha))(g_\alpha(m)_{[0,e]} \otimes_B 1_B \otimes_A g_\alpha(m)_{[1,\alpha]})\\
&\quad (g_\alpha(m) = n_i \otimes_B (b_i \otimes_A c_i) \in N \otimes_B (B \otimes_A C_\alpha))\\
&= ((N \otimes_B (B \otimes_A \varepsilon)) \otimes_B (B \otimes_A C_\alpha))(n_i \otimes_B (b_i \otimes_A c_{i(1,e)}) \otimes_B 1_B \otimes_A c_{i(2,\alpha)})\\
&= n_i \cdot b_i\mu(\varepsilon(c_{i(1,e)})) \otimes_B 1_B \otimes_A c_{i(2,\alpha)}\\
&= n_i \otimes_B b_i \otimes_A \varepsilon(c_{i(1,e)}) \cdot c_{i(2,\alpha)}\\
&= n_i \otimes_B b_i \otimes_A c_i = g_\alpha(m)
\end{aligned}$$

For all $m \in M_\alpha$, and

$$\begin{aligned}\varphi \circ \phi(f)(m \otimes_A b) &= (N \otimes_B (B \otimes_A \varepsilon)(\phi(f)_e(m))) \cdot b \\ &= (N \otimes_B (B \otimes_A \varepsilon)(f(m_{[0,e]} \otimes_A 1_B) \otimes_B (1_B \otimes_A m_{[1,e]})) \cdot b \\ &= f(m_{[0,e]} \otimes_A 1_B) \cdot \mu(\varepsilon(m_{[1,e]}))b \\ &= f(m_{[0,e]} \cdot \varepsilon(m_{[1,e]}) \otimes_A b) = f(m \otimes_A b).\end{aligned}$$

This ends the proof. □

For a G-A-coring C, recall from [10, Lemma 3.1] that there exists a pair of adjoint functors (F_1, U_1) between the categories $\mathcal{M}^{G,C}$ and $\mathcal{M}_A$ (the category of right A-modules). Notice that the adjoint functors (F, U) in Proposition 3.4 are the composition of the functors (F_1, U_1) and the restriction/induction functor induced by $A \to B$:

$$\mathcal{M}^{G,C} \leftrightarrows \mathcal{M}_A \leftrightarrows \mathcal{M}_B.$$

Next, take $N \in \mathcal{M}^D$ and $N\square_D(B \otimes_A C_\alpha)$ denotes the cotensor product of N and $B \otimes_A C_\alpha$. Let $N\square_D(B \otimes_A C) = \{N\square_D(B \otimes_A C_\alpha)\}_{\alpha \in G}$. From the proof of Prop. 3.4, we have

Proposition 3.5. *If C is flat as a left A-module* (*means that each C_α is flat*), *and $N \in \mathcal{M}^D$, $M = \{M_\alpha\}_{\alpha \in G} \in \mathcal{M}^{G,C}$ then $N\square_D(B \otimes_A C) = \{N\square_D(B \otimes_A C_\alpha)\}_{\alpha \in G}$ is an object of $\mathcal{M}^{G,C}$ via the structure map $\{N \otimes_B B \otimes_A \Delta_{\alpha,\beta}\}_{\alpha,\beta \in G}$.* (3.1) *restricts to an isomorphism*

$$\mathrm{Hom}^D(M_e \otimes_A B, N) \cong \mathrm{Hom}^C(M, N\square_D(B \otimes_A C)).$$

Therefore, $-\square_D(B \otimes_A C)$ is right adjoint to $(-)_e \otimes_A B$.

Proof. We have to show that, for all $\sum_i n_i \otimes_B (b_i \otimes c_i) \in N\square_D(B \otimes_A C_{\alpha\beta})$ with $\alpha, \beta \in G$:

$$x = \sum_i (n_i \otimes_B (b_i \otimes_A c_{i(1,\alpha)})) \otimes c_{i(2,\beta)} \in (N\square_D(B \otimes_A C_\alpha)) \otimes_A C_\beta.$$

For each $\alpha \in G$, we have an exact sequence

$$0 \to N\square_D(B \otimes_A C_\alpha) \to N \otimes_B (B \otimes_A C_\alpha) \rightrightarrows N \otimes_B D \otimes_B (B \otimes_A C_\alpha).$$

Since C_β is flat, we have another exact sequence

$$\begin{aligned}0 \to (N\square_D(B \otimes_A C_\alpha)) \otimes_A C_\beta &\to N \otimes_B (B \otimes_A C_\alpha) \otimes_A C_\beta \\ &\rightrightarrows N \otimes_B D \otimes_B (B \otimes_A C_\alpha) \otimes_A C_\beta.\end{aligned}$$

Therefore, in order to show that $x \in (N\square_D(B \otimes_A C_\alpha)) \otimes_A C_\beta$, it suffices to show that

$$(\rho^N \otimes_B B \otimes_A C_\alpha \otimes_A C_\beta)(x) = (N \otimes_B \rho^{B \otimes_A C_\alpha} \otimes_A C_\beta)(x).$$

Indeed, we have

$$
\begin{aligned}
(\rho^N \otimes_B B \otimes_A C_\alpha \otimes_A C_\beta)(x) &= \sum_i n_{i(0)} \otimes_B n_{i(1)} \otimes_B b_i \otimes_A c_{i(1,\alpha)} \otimes_A c_{i(2,\beta)} \\
&= \sum_i n_{i(0)} \otimes_B n_{i(1)} \otimes_B b_i \otimes_A c_{i(1,\alpha)} \otimes_A c_{i(2,\beta)} \\
&= \sum_i n_i \otimes_B b_i \cdot \varphi(c_{i(1,\alpha)(1,e)}) \otimes_A c_{i(1,\alpha)(2,\alpha)} \otimes_A c_{i(2,\beta)} \\
&= (N \otimes_B \rho^{B \otimes_A C_\alpha} \otimes_A C_\beta)(x).
\end{aligned}
$$

So $N \square_D (B \otimes_A C) = \{N \square_D (B \otimes_A C_\alpha)\}_{\alpha \in G}$ is an object of $\mathscr{M}^{G,C}$.

Take $f \in \mathrm{Hom}^D(M_e \otimes_A B, N)$, for all $\alpha \in G$ and $m \in M_\alpha$, since

$$
\begin{aligned}
&(\rho^N \otimes_B (B \otimes_A C_\alpha)) \circ \phi(f)_\alpha(m) \\
&\quad = (\rho^N \otimes_B (B \otimes_A C_\alpha))(f(m_{[0,e]} \otimes_A 1_B) \otimes_B (1_B \otimes_A m_{[1,\alpha]})) \\
&\quad = (f \otimes_B D \otimes_B (B \otimes_A C_\alpha))(\rho^{M_e \otimes_A B}(m_{[0,e]} \otimes_A 1_B) \otimes_B (1_B \otimes_A m_{[1,\alpha]})) \\
&\quad = (f \otimes_B D \otimes_B (B \otimes_A C_\alpha))(m_{[0,e][0,e]} \otimes_A 1_B \otimes_B \varphi(m_{[0,e][1,e]}) \otimes_B (1_B \otimes_A m_{[1,\alpha]})) \\
&\quad = (f \otimes_B D \otimes_B (B \otimes_A C_\alpha))(m_{[0,e]} \otimes_A 1_B \otimes_B \varphi(m_{[1,\alpha](1,e)}) \otimes_B (1_B \otimes_A m_{[1,\alpha](2,\alpha)})) \\
&\quad = f(m_{[0,e]} \otimes_A 1_B) \otimes_B \varphi(m_{[1,\alpha](1,e)}) \otimes_B (1_B \otimes_A m_{[1,\alpha](2,\alpha)}) \\
&\quad = (N \otimes_B \rho^{B \otimes_A C_\alpha}) \circ \phi(f)_\alpha(m).
\end{aligned}
$$

Hence it follows $\phi(f)_\alpha(m) \in N \square_D (B \otimes_A C_\alpha)$. Conversely, let $f \in \mathrm{Hom}_B(M_e \otimes_A B, N)$. Assume that $\phi(f)_\alpha(m) \in N \square_D (B \otimes_A C_\alpha)$, by (3.1), it is sufficient to check that

$$
\phi(\rho^N \circ f) = \phi((f \otimes_B D) \circ \rho^{M_e \otimes_A B}). \tag{3.2}
$$

In fact, for all $\alpha \in G$ and $m \in M_\alpha$, since

$$
\begin{aligned}
&\phi((f \otimes_B D) \circ \rho^{M_e \otimes_A B})_\alpha(m) \\
&\quad = (f \otimes_B D) \circ \rho^{M_e \otimes_A B}(m_{[0,e]} \otimes_A 1_B) \otimes_B (1_B \otimes_A m_{[1,\alpha]}) \\
&\quad = f(m_{[0,e][0,e]} \otimes_A 1_B) \otimes_B \varphi(m_{[0,e][1,e]}) \otimes_B (1_B \otimes_A m_{[1,\alpha]}) \\
&\quad = f(m_{[0,e]} \otimes_A 1_B) \otimes_B \varphi(m_{[1,\alpha](1,e)}) \otimes_B (1_B \otimes_A m_{[1,\alpha](2,\alpha)}) \\
&\quad = (N \otimes_B \rho^{B \otimes_A C_\alpha}) \circ \phi(f)_\alpha(m)
\end{aligned}
$$

and

$$
\begin{aligned}
\phi(\rho^N \circ f)_\alpha(m) &= \rho^N(f(m_{[0,e]} \otimes_A 1_B)) \otimes_B (1_B \otimes_A m_{[1,\alpha]}) \\
&= (\rho^N \otimes_B (B \otimes_A C_\alpha)) \circ \phi(f)_\alpha(m).
\end{aligned}
$$

So we can get relation (3.2) from the assumption. □

Let us finally describe the unit η of this adjunction in Prop. 3.5. Taking $M = \{M_\alpha\}_{\alpha \in G} \in \mathscr{M}^{G,C}$, the unit $\eta^M = \{\eta^M_\alpha\}_{\alpha\in G}$ for $(-)_e \otimes_A B \dashv -\Box_D(B \otimes_A C)$ at M is given by

$$\eta^M_\alpha(m) = (m_{[0,e]} \otimes_A 1_B)\Box_D(1_B \otimes_A m_{[1,\alpha]}).$$

Now, we shall achieve the main goal in this section. Before presenting the main theorem, we first give the following remark which is necessary.

Remark 3.6. (1) Let $M = \{M_\alpha\}_{\alpha\in G} \in \mathscr{M}^{G,C}$. For each $\alpha \in G$, $M^\alpha = \{M_{\alpha\beta}\}_{\beta\in G}$ is a G-C-comodule via the structure map $\{\rho^{M_\alpha}_{\beta,\gamma} = \rho^M_{\alpha\beta,\gamma}\}_{\beta,\gamma\in G}$.

(2) Let $M = \{M_\alpha\}_{\alpha\in G} \in \mathscr{M}^{G,C}$ be flat as a right A-module. Then the map

$$(M_{\alpha\beta} \otimes_A B) \otimes_B (B \otimes_A C_\gamma) \xrightarrow{(\rho^M_{\alpha,\beta}\otimes_A B)\otimes_B(B\otimes_A C_\gamma)} M_\alpha \otimes_A (C_\beta \otimes_A B) \otimes_B (B \otimes_A C_\gamma)$$

restricts to

$$\begin{aligned}(M_{\alpha\beta} \otimes_A B)\Box_D(B \otimes_A C_\gamma) &\to M_\alpha \otimes_A (C_\beta \otimes_A B)\Box_D(B \otimes_A C_\gamma)\\ &= (M_\alpha \otimes_A C_\beta \otimes_A B)\Box_D(B \otimes_A C_\gamma)\end{aligned}$$

for all $\alpha, \beta, \gamma \in G$, where $M_{\alpha\beta} \otimes_A B$, $C_\beta \otimes_A B$ are considered as right D-comodules by applying the functor $(-)_e \otimes_A B$ to $M^{\alpha\beta}$, C^β, and $M_\alpha \otimes_A C_\beta \otimes_A B$ via $M_\alpha \otimes_A \rho^{C_\beta\otimes_A B}$.

Theorem 3.7. *Assume that C is flat as a left A-module and every object in $\mathscr{M}^{G,C}$ is flat as right A-module. The functor $(-)_e \otimes_A B : \mathscr{M}^{G,C} \to \mathscr{M}^D$ is separable if and only if there is a family of homomorphisms of A-bimodule*

$$\theta = \{\theta^{(\alpha)} : (C_{\alpha^{-1}} \otimes_A B)\Box_D(B \otimes_A C_\alpha) \to A\}_{\alpha\in G},$$

such that

(1) $\theta^{(\alpha)} \circ \eta^{C^{\alpha^{-1}}}_\alpha = \varepsilon$,

(2) *θ satisfies the following commutative diagram*:

$$\begin{array}{ccc}(C_{\alpha^{-1}} \otimes_A B)\Box_D(B \otimes_A C_{\alpha\beta}) & \xrightarrow{(\Delta_{\beta,(\alpha\beta)^{-1}}\otimes_A B)\otimes_B(B\otimes_A C_{\alpha\beta})} & C_\beta \otimes_A ((C_{(\alpha\beta)^{-1}} \otimes_A B)\Box_D(B \otimes_A C_{\alpha\beta}))\\ \Big\downarrow{\scriptstyle (C_{\alpha^{-1}}\otimes_A B)\otimes_B(B\otimes_A \Delta_{\alpha,\beta})} & & \Big\downarrow{\scriptstyle C_\beta\otimes_A\theta^{(\alpha\beta)}}\\ ((C_{\alpha^{-1}} \otimes_A B)\Box_D(B \otimes_A C_\alpha)) \otimes_A C_\beta & \xrightarrow{\theta^{(\alpha)}\otimes_A C_\beta} & C_\beta\end{array}$$

Proof. Assume that $(-)_e \otimes_A B$ is separable. By Rafael's Theorem (see [7]), there exists a natural transformation $\omega : ((-)_e \otimes_A B)\Box_D(B \otimes_A C) \to 1_{\mathscr{M}^{G,C}}$ such that $\omega \circ \eta = 1$ (the identiy natural transformation.) Specially, considering G-C-comodule $C^{\alpha^{-1}} = \{C_{\alpha^{-1}\beta}\}_{\beta\in G}$ via $\Delta_{\alpha^{-1}\beta,\gamma}$ and applying ω to it, we have

$$\omega^{C^{\alpha^{-1}}} = \{\omega^{C^{\alpha^{-1}}}_\beta : (C_{\alpha^{-1}} \otimes_A B)\Box_D(B \otimes_A C_\beta) \to C_{\alpha^{-1}\beta}\}_{\beta\in G}.$$

Then we construct a family of k-linear maps $\theta = \{\theta^{(\alpha)}\}_{\alpha\in G}$,

$$\theta^{(\alpha)} = \varepsilon \circ \omega_\alpha^{C^{\alpha^{-1}}} : (C_{\alpha^{-1}} \otimes_A B)\Box_D(B \otimes_A C_\alpha) \to A.$$

From ε and $\omega_\alpha^{C^{\alpha^{-1}}}$ being both right A-linear, it follows that the map $\theta^{(\alpha)}$ is a right A-module morphism. Next for all $a \in A$, we consider a family of k-linear maps $f^{(a,\alpha^{-1})} = \{f_\beta^{(a,\alpha^{-1})}\}_{\beta\in G}$,

$$f_\beta^{(a,\alpha^{-1})} : C_{\alpha^{-1}\beta} \to C_{\alpha^{-1}\beta}, \quad f_\beta^{(a,\alpha^{-1})}(c) = a \cdot c.$$

It is checked easily that $f^{a,\alpha}$ is a morphism of $\mathscr{M}^{G,C}$. By the naturality of ω, we have the following commutative diagram

$$\begin{array}{ccc}
(C_{\alpha^{-1}} \otimes_A B)\Box_D(B \otimes_A C_\beta) & \xrightarrow{\omega_\beta^{C^{\alpha^{-1}}}} & C_{\alpha^{-1}\beta} \\
{\scriptstyle (f_e^{(a,\alpha^{-1})}\otimes_A B)\Box_D(B\otimes_A C_\beta)}\downarrow & & \downarrow{\scriptstyle f_\beta^{(a,\alpha^{-1})}} \\
(C_{\alpha^{-1}} \otimes_A B)\Box_D(B \otimes_A C_\beta) & \xrightarrow{\omega_\beta^{C^{\alpha^{-1}}}} & C_{\alpha^{-1}\beta}
\end{array}$$

for all $\alpha \in G$. It follows from the above commutative diagram that $\omega_\beta^{C^{\alpha^{-1}}}$ is left A-linear, thus $\theta^{(\alpha)}$ is left A-linear. Since

$$\omega_\alpha^{C^{\alpha^{-1}}}((c_{(1,\alpha^{-1})} \otimes_A 1_B)\Box_D(1_B \otimes_A c_{(2,\alpha)})) = c$$

for any $c \in C_e$, we have

$$\begin{aligned}
\theta^{(\alpha)} \circ \eta_\alpha^{C^{\alpha^{-1}}}(c) &= \theta^\alpha((c_{(1,\alpha^{-1})} \otimes_A 1_B)\Box_D(1_B \otimes_A c_{(2,\alpha)})) \\
&= \varepsilon \circ \omega_\alpha^{C^{\alpha^{-1}}}((c_{(1,\alpha^{-1})} \otimes_A 1_B)\Box_D(1_B \otimes_A c_{(2,\alpha)})) = \varepsilon(c).
\end{aligned}$$

Now, for all $c \in C_\beta$, we consider the morphism

$$l_\gamma^{(c,\alpha\beta)} : C_{(\alpha\beta)^{-1}\gamma} \to C_\beta \otimes_A C_{(\alpha\beta)^{-1}\gamma}, \quad c' \mapsto c \otimes_A c'.$$

By the naturality of ω, we have the following commutative diagram

$$\begin{array}{ccc}
(C_{(\alpha\beta)^{-1}} \otimes_A B)\Box_D(B \otimes_A C_\gamma) & \xrightarrow{\omega_\gamma^{C^{(\alpha\beta)^{-1}}}} & C_{(\alpha\beta)^{-1}\gamma} \\
\downarrow{\scriptstyle (l_e^{(c,\alpha\beta)}\otimes_A B)\Box_D(B\otimes_A C_\beta)} & & \downarrow{\scriptstyle l_\gamma^{(c,\alpha\beta)}} \\
(C_\beta \otimes_A C_{(\alpha\beta)^{-1}} \otimes_A B)\Box_D(B \otimes_A C_\gamma) & \xrightarrow{\omega_\gamma^{\mathfrak{R}^{(\beta,\alpha)}}} & C_\beta \otimes_A C_{(\alpha\beta)^{-1}\gamma}
\end{array}$$

where

$$\mathfrak{R}^{(\beta,\alpha)} = \{\mathfrak{R}_\gamma^{(\beta,\alpha)} = C_\beta \otimes_A C_{(\alpha\beta)^{-1}\gamma}\}_{\gamma\in G}.$$

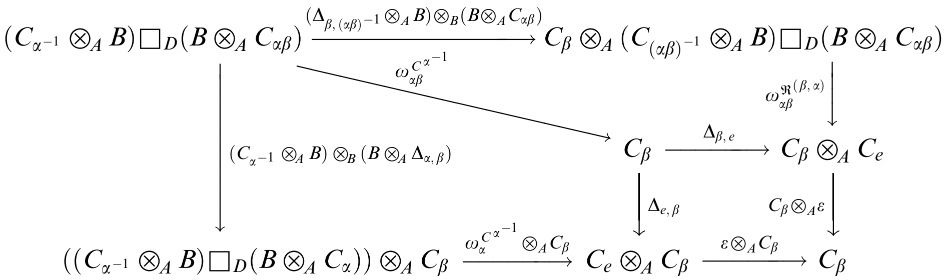
It follows from the commutative diagram above that

$$\omega_\gamma^{\mathfrak{R}^{(\beta,\alpha)}}(c \otimes_A x) = c \otimes_A \omega_\gamma^{C^{(\alpha\beta)^{-1}}}(x)$$

for any $c \in C_\beta$ and $x \in (C_{(\alpha\beta)^{-1}} \otimes_A B)\Box_D(B \otimes_A C_\gamma)$. Since c is arbitrary, we have

$$\omega_\gamma^{\mathfrak{R}^{(\beta,\alpha)}} = C_\beta \otimes_A \omega_\gamma^{C^{(\alpha\beta)^{-1}}}.$$

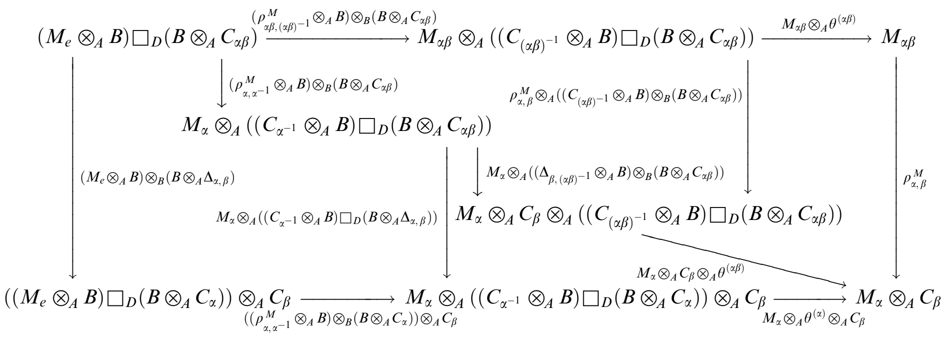
The condition (2) in Theorem 3.7 follows from the following commutative diagram



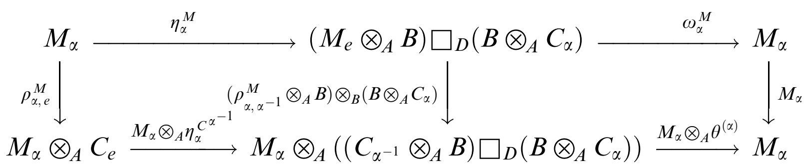
To prove the converse, we need to construct a natural transformation ω from the A-bimodule θ. Given a right G-C-comodule $M = \{M_\alpha\}_{\alpha\in G}$, we define a family of k-linear maps $\omega^M = \{\omega^M_\alpha\}_{\alpha\in G}$, where ω^M_α can be defined by the composition

$$(M_e \otimes_A B)\Box_D(B\otimes_A C_\alpha) \xrightarrow{(\rho^M_{\alpha,\alpha^{-1}}\otimes_A B)\otimes_B(B\otimes_A C_\alpha)} M_\alpha \otimes_A ((C_{\alpha^{-1}}\otimes_A B)\Box_D(B\otimes_A C_\alpha)) \xrightarrow{M_\alpha\otimes_A\theta^{(\alpha)}} M_\alpha$$

It follows from $\theta^{(\alpha)}$ being A-linear that each ω^M_α is right A-linear. Using the following commutative diagrams



shows that ω^M is a morphism in $\mathscr{M}^{G,C}$, and



shows $\omega^M_\alpha \circ \eta^M_\alpha = id_{M_\alpha}$. It is easily to check that ω is natural at M.

The proof is completed. □

By considering on A the canonical A-coring structure, as a corollary of Theorem 3.7, we have the main result of [10].

COROLLARY 3.8 ([10]). *For a G-A-coring* $\mathscr{C}$, *the forgetful functor* $F : \mathscr{M}^{G,C} \to \mathscr{M}_A$ *is separable if and only if there exists a family of A-bimodules* $\theta = \{\theta^{(\alpha)} : C_{\alpha^{-1}} \otimes_A C_\alpha \to A\}_{\alpha \in G}$ *such that*

$$\theta^{(\alpha)}(c'_{(1,\alpha^{-1})} \otimes_A c'_{(2,\alpha)}) = \varepsilon(c'),$$
$$c_{(1,\beta)} \cdot \theta^{(\alpha\beta)}(c_{(2,\beta^{-1}\alpha^{-1})} \otimes_A d) = \theta^{(\alpha)}(c \otimes_A d_{(1,\alpha)}) \cdot d_{(2,\beta)}$$

for all $c' \in C_e$, $c \in C_{\alpha^{-1}}$, $d \in C_{\alpha\beta}$.

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