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ALGEBRAIC DEPENDENCES OF MEROMORPHIC MAPPINGS SHARING FEW HYPERPLANES COUNTING TRUNCATED MULTIPLICITIES

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Abstract

In this article, we study algebraic dependences of three meromorphic mappings which share few hyperplanes counting truncated multiplicities.

1. Introduction

In 1926, R. Nevanlinna showed that two distinct non-constant meromorphic functions f and g on the complex plane $\mathbf{C}$ cannot have the same inverse images for five distinct values, and that g is a special type of linear fractional transformation of f if they have the same inverse images counted with multiplicities for four distinct values.

In 1975, H. Fujimoto [4] generalized Nevanlinna's result to the case of meromorphic mappings of $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$. He proved that for two linearly non-degenerate meromorphic mappings f and g of $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$, if they have the same inverse images, counted with multiplicities for $(3N+2)$ hyperplanes in $\mathbf{P}^N(\mathbf{C})$ located in general position, then $f \equiv g$, and that there exists a projective linear transformation L of $\mathbf{P}^N(\mathbf{C})$ to itself such that $g = L \cdot f$ if they have the same inverse images counted with multiplicities for $(3N+1)$ hyperplanes in $\mathbf{P}^N(\mathbf{C})$ located in general position. Since that time, the finiteness problem for meromorphic mappings sharing few hyperplanes has been studied intensively by many authors.

We state here the recent best results on this problem of Chen-Yan [2] and Quang [10].

Let f be a linearly non-degenerate meromorphic mapping of $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$. Take q hyperplanes $H_1, \dots, H_q$ in $\mathbf{P}^N(\mathbf{C})$ in general position with

a) $\dim(\mathrm{Zero}(f, H_i) \cap \mathrm{Zero}(f, H_j)) \le n-2$ for all $1 \le i < j \le q$.

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For each positive integer (or $+\infty$) d, denote by $\mathcal{F}(\{H_j\}_{j=1}^q, f, d)$ the set of all linearly non-degenerate meromorphic mappings g of $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$ such that

b) $\min\{v_{(g,H_j)}, M\} = \min\{v_{(f,H_j)}, M\}$, $(1 \le j \le q)$, and

c) $g = f$ *on* $\bigcup_{j=1}^q \mathrm{Zero}(f, H_j)$.

By Lemma 3.1 in [10], we see that if $q \ge 2N+2$ then each meromorphic mapping satisfying the conditions b) and c) will be linearly non-degenerate. Therefore the condition on the linearly non-degeneracy of the mappings g in the definition of the family $\mathcal{F}(\{H_j\}_{j=1}^q, f, d)$ is not necessary in the case where $q \ge 2N+2$.

In 2009, Z. Chen and Q. Yan [2] showed that:

Theorem A (see [2, Main Theorem]). *If* $q \ge 2N+3$ *then* $g_1 = g_2$ *for any* $g_1, g_2 \in \mathcal{F}(\{H_i\}_{i=1}^q, f, 1)$.

Recently, the first author [10] proved that:

Theorem B (see [10, Theorem 1.1]). *If* $q \ge 2N+2$ *and* $N \ge 2$ *then* $\mathcal{F}(\{H_i\}_{i=1}^q, f, 1)$ *contains at most two mappings.*

However, we note that there is a gap in the proof of [10, Theorem 1.1]. For detail, the inequality (3.26) in [10, Lemma 3.20] does not holds. Hence the inequality of [10, Lemma 3.20(ii)] may not hold. In order to fix this gap, we need to slightly change the estimate of this inequality by adding $N^{(1)}_{(f,H_j)}(r)$ to its right-hand side. The rest of the proof is still valid for the case where $N \ge 3$. In the last past of this paper, we would like to give a correction for the proof of this theorem when $N \ge 3$. Theorem B (including the case where $N = 2$) has also been proved and improved in a recent work of the first author [11] by another way.

We would also like to note that so far, all results on the finiteness problem have still been restricted to the case where meromorphic mappings share at least $2N+2$ hyperplanes and they are identity on the inverse images of all these hyperplanes. Then the following questions arise naturally.

a) Is there any relation between meromorphic mappings sharing q hyperplanes regardless of multiplicity with $q < 2N+2$?

b) Is there any relation between meromorphic mappings sharing few hyperplanes regardless of multiplicity with smaller identity set?

In this paper we will show that these mappings are algebraically dependent in some particular cases. Our main results are stated as follows.

For each real number x, denote by $[x]$ the integer part of x, i.e., $[x]$ is the maximal integer which does not exceed x.

Theorem 1.1. *Let* f_1, f_2, f_3 *be three linearly non-degenerate meromorphic mappings of* $\mathbf{C}^n$ *into* $\mathbf{P}^N(\mathbf{C})$. *Let* $\{H_i\}_{i=1}^q$ *be a family of q hyperplanes of* $\mathbf{P}^N(\mathbf{C})$ *in general position with*

$$\dim(\mathrm{Zero}(f_1, H_i) \cap \mathrm{Zero}(f_1, H_j)) \le n-2 \quad (1 \le i < j \le q).$$

Assume that the following conditions are satisfied:

(a) $\min\{\nu_{(f_1,H_i)}, N\} = \min\{\nu_{(f_2,H_i)}, N\} = \min\{\nu_{(f_3,H_i)}, N\}$ $(1 \le i \le q)$,

(b) $f_1 = f_2 = f_3$ *on* $\bigcup_{i=1}^{q} \mathrm{Zero}(f_1, H_i)$.

If $q > \dfrac{2N + 5 + \sqrt{28N^2 + 20N + 1}}{4}$ *then one of the following assertions holds*:

(i) *There exist* $\left[\dfrac{q}{3}\right] + 1$ *hyperplanes* $H_{i_1}, \dots, H_{i_{[q/3]+1}}$ *such that*

$$\frac{(f_u, H_{i_1})}{(f_v, H_{i_1})} = \frac{(f_u, H_{i_2})}{(f_v, H_{i_2})} = \cdots = \frac{(f_u, H_{i_{[q/3]+1}})}{(f_v, H_{i_{[q/3]+1}})}, \quad (1 \le u < v \le 3),$$

(ii) $f_1 \wedge f_2 \wedge f_3 \equiv 0$.

THEOREM 1.2. *Let f_1, f_2, f_3 be three linearly non-degenerate meromorphic mappings of $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$and let $H_1, \dots, H_q$ be q hyperplanes of $\mathbf{P}^N(\mathbf{C})$ in general position with*

$$\dim(\mathrm{Zero}(f_1, H_i) \cap \mathrm{Zero}(f_1, H_j)) \le n - 2 \quad (1 \le i < j \le q).$$

Assume that the following conditions are satisfied:

(a) f_1 *is linearly non-degenerate over* $\mathcal{R}_{f_1}$,

(b) $\min\{\nu_{(f_1,H_i)}, N\} = \min\{\nu_{(f_2,H_i)}, N\} = \min\{\nu_{(f_3,H_i)}, N\}$ $(1 \le i \le q - N - 1)$,

(c) $f_1 = f_2 = f_3$ *on* $\bigcup_{i=q-N}^{q} \bigcup_{u=1}^{3} \mathrm{Zero}(f_u, H_i)$.

If $q > 3\left[\dfrac{N+1}{2}\right] + N + 1$, *then one of the following assertions holds*:

(i) *There exist* $\left[\dfrac{N+1}{2}\right] + 1$ *hyperplanes* $H_{i_1}, \dots, H_{i_{[(N+1)/2]+1}}$ *such that*

$$\frac{(f_u, H_{i_1})}{(f_v, H_{i_1})} = \frac{(f_u, H_{i_2})}{(f_v, H_{i_2})} = \cdots = \frac{(f_u, H_{i_{[(N+1)/2]+1}})}{(f_v, H_{i_{[(N+1)/2]+1}})},$$

(ii) $f_1 \wedge f_2 \wedge f_3 \equiv 0$.

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2. Basic notions and auxiliary results from Nevanlinna theory

2.1. We set $\|z\| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}$ for $z = (z_1, \dots, z_n) \in \mathbf{C}^n$ and define

$$B(r) := \{z \in \mathbf{C}^n : \|z\| < r\}, \quad S(r) := \{z \in \mathbf{C}^n : \|z\| = r\} \quad (0 < r < \infty).$$

Define

$$d = \partial + \bar{\partial}, \quad d^c = \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial),$$

$$v_{n-1}(z) := (dd^c\|z\|^2)^{n-1} \quad \text{and}$$

$$\sigma_n(z) := d^c \log\|z\|^2 \wedge (dd^c \log\|z\|^2)^{n-1} \quad \text{on } \mathbf{C}^n \backslash \{0\}.$$

2.2. Let F be a nonzero holomorphic function on a domain Ω in $\mathbf{C}^n$. For a set $\alpha = (\alpha_1, \dots, \alpha_n)$ of nonnegative integers, we set $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\mathscr{D}^\alpha F = \dfrac{\partial^{|\alpha|} F}{\partial^{\alpha_1} z_1 \cdots \partial^{\alpha_n} z_n}$. We define the map $\nu_F : \Omega \to \mathbf{Z}$ by

$$\nu_F(z) := \max\{m : \mathscr{D}^\alpha F(z) = 0 \text{ for all } \alpha \text{ with } |\alpha| < m\} \quad (z \in \Omega).$$

We mean by a divisor on a domain Ω in $\mathbf{C}^n$ a map $\nu : \Omega \to \mathbf{Z}$ such that, for each $a \in \Omega$, there are nonzero holomorphic functions F and G on a connected neighborhood $U \subset \Omega$ of a such that $\nu(z) = \nu_F(z) - \nu_G(z)$ for each $z \in U$ outside an analytic set of dimension $\leq n-2$. Two divisors are regarded as the same if they are identical outside an analytic set of dimension $\leq n-2$. For a divisor ν on Ω we define $\mathrm{Supp}(\nu) := \overline{\{z : \nu(z) \neq 0\}}$, which is a purely $(n-1)$-dimensional analytic subset of Ω or empty.

Take a nonzero meromorphic function φ on a domain Ω in $\mathbf{C}^n$. For each $a \in \Omega$, we choose nonzero holomorphic functions F and G on a neighborhood $U \subset \Omega$ such that $\varphi = \dfrac{F}{G}$ on U and $\dim(F^{-1}(0) \cap G^{-1}(0)) \leq n-2$, and we define the divisors ν_φ, ν_φ^∞ by $\nu_\varphi := \nu_F$, $\nu_\varphi^\infty := \nu_G$, which are independent of choices of F and G and so globally well-defined on Ω.

2.3. For a divisor ν on $\mathbf{C}^n$ and for positive integers k, M (maybe $M = \infty$), we define the counting function of ν by

$$\nu^{(M)}(z) = \min\{M, \nu(z)\},$$

$$\nu_{>k}^{(M)}(z) = \begin{cases} \min\{M, \nu(z)\} & \text{if } \nu(z) > k \\ 0 & \text{if } \nu(z) \leq k. \end{cases}$$

$$n(t) = \begin{cases} \int_{\mathrm{Supp}(\nu) \cap B(t)} \nu(z) v_{n-1} & \text{if } n \geq 2, \\ \sum\limits_{|z| \leq t} \nu(z) & \text{if } n = 1. \end{cases}$$

Similarly, we define $n^{(M)}(t)$ and $n_{>k}^{(M)}(t)$.
Define

$$N(r, \nu) = \int_1^r \frac{n(t)}{t^{2n-1}} \, dt \quad (1 < r < \infty).$$

Similarly, we define $N(r, v^{(M)})$ and $N(r, v^{(M)}_{>k})$ and denote them by $N^{(M)}(r, v)$ and $N^{(M)}_{>k}(r, v)$ respectively.

Let φ be a nonzero meromorphic function on $\mathbf{C}^n$. Define

$$N_\varphi(r) = N(r, v_\varphi), \quad N^{(M)}_\varphi(r) = N^{(M)}(r, v_\varphi), \quad N^{(M)}_{\varphi,>k}(r) = N^{(M)}(r, (v_\varphi)_{>k}).$$

For brevity we will omit the character $^{(M)}$ if $M = \infty$.

2.4. Let $f : \mathbf{C}^n \to \mathbf{P}^N(\mathbf{C})$ be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates $(w_0 : \cdots : w_N)$ on $\mathbf{P}^N(\mathbf{C})$, we take a reduced representation $f = (f_0 : \cdots : f_N)$, which means that each f_i is a holomorphic function on $\mathbf{C}^n$ and $f(z) = (f_0(z) : \cdots : f_N(z))$ outside the analytic set $\{f_0 = \cdots = f_N = 0\}$ of co-dimension ≥ 2. Set $\|f\| = (|f_0|^2 + \cdots + |f_N|^2)^{1/2}$.

We define the characteristic function of f as follows

$$T_f(r) = \int_{S(r)} \log\|f\|\sigma_n - \int_{S(1)} \log\|f\|\sigma_n.$$

Let H be a hyperplanes $\mathbf{P}^N(\mathbf{C})$ defined by $H = \{(w_0 : \cdots w_N) : \sum_{i=0}^N a_i w_i = 0\}$. We define

$$m_{f,H}(r) = \int_{S(r)} \log\frac{\|f\| \cdot \|H\|}{|(f,a)|}\sigma_n - \int_{S(1)} \log\frac{\|f\| \cdot \|a\|}{|(f,a)|}\sigma_n,$$

where $(f, H) = \sum_{i=0}^N a_i f_i$ and $\|H\| = (\sum_{i=0}^N |a_i|^2)^{1/2}$.

Let φ be a nonzero meromorphic function on $\mathbf{C}^n$, which are occasionally regarded as a meromorphic map into $\mathbf{P}^1(\mathbf{C})$. The proximity function of φ is defined by

$$m(r, \varphi) := \int_{S(r)} \log \max(|\varphi|, 1)\sigma_n.$$

The Nevanlinna characteristic function of φ defined by

$$T(r, \varphi) = m(r, \varphi) + N_{1/\varphi}(r).$$

Then

$$T_\varphi(r) = T(r, \varphi) + O(1).$$

The meromorphic function φ is said to be "small" (with respect to f) if $\| \ T_\varphi(r) = o(T_f(r))$. Here by the notation "$\| \ P$" we mean the assertion P holds for all $r \in [0, \infty)$ excluding a Borel subset E of the interval $[0, \infty)$ with $\int_E dr < \infty$.

We denote by $\mathscr{R}_f$ the field of all small (with respect to f) meromorphic functions on $\mathbf{C}^n$. The mapping f is said to be linearly non-degenerate over $\mathscr{R}_f$ if the family $\{f_0, \ldots, f_n\}$ is independent over $\mathscr{R}_f$ for some its representations $(f_0 : \cdots : f_n)$.

Let $\{H_i\}_{i=1}^q$ $(q \geq N+1)$ be a set of q hyperplanes in $\mathbf{P}^N(\mathbf{C})$. We say that $\{H_i\}_{i=1}^q$ are in general position if for any $1 \leq i_1 < \cdots < i_{N+1} \leq q$ we have $\bigcap_{j=1}^{N+1} H_{i_j} = \emptyset$.

Theorem 2.5 (Second Main Theorem for meromorphic mappings with hyperplanes). *Let $f : \mathbf{C}^n \to \mathbf{P}^N(\mathbf{C})$ be a linearly non-degenerate meromorphic mapping. Let $\{H_i\}_{i=1}^q$ $(q \geq N+2)$ be a set of q hyperplanes in $\mathbf{P}^N(\mathbf{C})$ in general position. Then*

$$\| \ (q-N-1)T_f(r) \leq \sum_{i=1}^q N_{(f,H_i)}^{(N)}(r) + o(T_f(r)).$$

3. Proof of Main Theorems

In order to prove the main theorems, we need the following lemma.

Lemma 3. *Let f and g be linearly non-degenerate meromorphic mappings of $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$. Let $\{H_i\}_{i=1}^q$ be a set of q hyperplanes in $\mathbf{P}^N(\mathbf{C})$ in general position. Assume that*

$$\mathrm{Zero}(f, H_i) = \mathrm{Zero}(g, H_i) \quad (1 \leq i \leq q).$$

If $q \geq N+2$, then

$$\| \ T_f(r) = O(T_g(r)) \quad \textit{and} \quad \| \ T_g(r) = O(T_f(r)).$$

Proof. By the Second Main Theorem, we have

$$\begin{aligned} \| \ T_f(r) &\leq \sum_{i=1}^{N+2} N_{(f,H_i)}^{(N)}(r) + o(T_f(r)) \\ &\leq \sum_{i=1}^{N+2} N N_{(f,H_i)}^{(1)}(r) + o(T_f(r)) \\ &= \sum_{i=1}^{N+2} N N_{(g,H_i)}^{(1)}(r) + o(T_f(r)) \leq N(N+2)T_g(r) + o(T_f(r)). \end{aligned}$$

Therefore, $\| \ T_f(r) = O(T_g(r))$. Similarly, we have $\| \ T_g(r) = O(T_f(r))$. ■

Proof of Theorem 1.1. Suppose that $f_1 \wedge f_2 \wedge f_3 \not\equiv 0$. For each $1 \leq i \leq q$, we set

$$N_i(r) = \sum_{u=1}^3 (N_{(f_u,H_i)}^{(N)}(r) - N_{(f_u,H_i)}^{(1)}(r)).$$

We denote by $\mathscr{I}$ the set of all permutations of the q-tuple $(1,\dots,q)$, that means

$$\mathscr{I} = \{I = (i_1,\dots,i_q) : \{i_1,\dots,i_q\} = \{1,\dots,q\}\}.$$

For each $I = (i_1,\dots,i_q) \in \mathscr{I}$ we define the subset E_I of $[1,+\infty)$ as follows

$$E_I = \{r \geq 1 : N_{i_1}(r) \geq \cdots \geq N_{i_q}(r)\}.$$

It is clear that $\bigcup_{I\in\mathscr{I}} E_I = [1,+\infty)$. Therefore, there exists an element I_0 of $\mathscr{I}$ satisfying $\int_{E_{I_0}} dr = +\infty$. We may assume $I_0 = (1,2,\dots,q)$ by rearranging if necessary. Then, we have $N_1(r) \geq N_2(r) \geq \cdots \geq N_q(r)$ for all $r \in E_{I_0}$.

We consider $\mathscr{M}^3$ as a vector space over the field $\mathscr{M}$, where by $\mathscr{M}$ we denote the field of all meromorphic functions on $\mathbf{C}^n$. For each $i = 1,\dots,q$, we set

$$V_i = ((f_1,H_i),(f_2,H_i),(f_3,H_i)) \in \mathscr{M}^3.$$

We put

$$t = \min\{i : V_1 \wedge V_i \not\equiv 0\}.$$

Then we have $V_i \wedge V_j \equiv 0$ for all $1 \leq i < j < t$.

We distinguish the following two cases.

Case 1. $t > \left[\frac{q}{3}\right] + 1$. This implies that

$$\frac{(f_k,H_1)}{(f_l,H_1)} = \frac{(f_k,H_2)}{(f_l,H_2)} = \cdots = \frac{(f_k,H_{[q/3]+1})}{(f_l,H_{[q/3]+1})} \quad (1 \leq k,l \leq 3).$$

The assertion (i) holds in this case.

Case 2. $t \leq \left[\frac{q}{3}\right] + 1$. We have $V_1 \wedge V_t \not\equiv 0$. Since $f_1 \wedge f_2 \wedge f_3 \not\equiv 0$, there exists an index s $(t < s \leq N+1)$ such that $V_1 \wedge V_t \wedge V_s \not\equiv 0$. This means that

$$P := \det\begin{pmatrix} (f_1,H_1) & (f_1,H_t) & (f_1,H_s) \\ (f_2,H_1) & (f_2,H_t) & (f_2,H_s) \\ (f_3,H_1) & (f_3,H_t) & (f_3,H_s) \end{pmatrix} \not\equiv 0.$$

For $z \notin \bigcup_{u=1}^{3} I(f_u) \cup \bigcup_{i' \neq j'} (\mathrm{Zero}(f_1,H_{i'}) \cap \mathrm{Zero}(f_1,H_{j'}))$, we consider the following four subcases:

Subcase 1. Let z be a zero of (f_1,H_1). We set $m = \min\{\nu_{(f_1,H_1)}(z), \nu_{(f_2,H_1)}(z), \nu_{(f_2,H_1)}(z)\}$. Then there exist a neighborhood U of z and holomorphic function h defined on U such that $\mathrm{Zero}(h) = U \cap \mathrm{Zero}(f_1,H_1)$ and dh has no zero. Moreover we may assume that $U \cap (\bigcup_{u=1}^{3} I(f_u) \cup \bigcup_{i' \neq j'} (\mathrm{Zero}(f_1, H_{i'}) \cap \mathrm{Zero}(f_1, H_{j'}))) = \emptyset$. Then there exist holomorphic functions φ_1, φ_2, φ_3 defined on U such that

$$(f_u, H_1) = h^m \varphi_u \quad \text{on } U \quad (1 \le u \le 3).$$

On the other hand, since $f_1 = f_2 = f_3$ on $\mathrm{Zero}(f_1, H_1)$, we have

$$\frac{(f_u, H_t)}{(f_1, H_t)} = \frac{(f_u, H_s)}{(f_1, H_s)} \quad \text{on } \mathrm{Zero}(f_1, H_1), \quad u = 2, 3.$$

Therefore, there exist holomorphic functions ψ_2 and ψ_3 satisfying

$$\frac{(f_u, H_t)}{(f_1, H_t)} - \frac{(f_u, H_s)}{(f_1, H_s)} = h\psi_u \quad \text{on } U, u = 2, 3.$$

We rewrite P on U as follows

$$\begin{aligned} P &= h^m \det \begin{pmatrix} \varphi_1 & (f_1, H_t) & (f_1, H_s) \\ \varphi_2 & (f_2, H_t) & (f_2, H_s) \\ \varphi_3 & (f_3, H_t) & (f_3, H_s) \end{pmatrix} \\ &= h^m (f_1, H_t)(f_1, H_s) \det \begin{pmatrix} \varphi_1 & 1 & 1 \\ \varphi_2 & \dfrac{(f_2, H_t)}{(f_1, H_t)} & \dfrac{(f_2, H_s)}{(f_1, H_s)} \\ \varphi_3 & \dfrac{(f_3, H_t)}{(f_1, H_t)} & \dfrac{(f_3, H_s)}{(f_1, H_s)} \end{pmatrix} \\ &= -h^{m+1} (f_1, H_t)(f_1, H_s) \det \begin{pmatrix} \varphi_1 & 1 & 0 \\ \varphi_2 & \dfrac{(f_2, H_t)}{(f_1, H_t)} & \psi_2 \\ \varphi_3 & \dfrac{(f_3, H_t)}{(f_1, H_t)} & \psi_3 \end{pmatrix}. \end{aligned}$$

This yields that

$$\nu_P(z) \ge m + 1 = \min\{\nu_{(f_1, H_1)}(z), \nu_{(f_2, H_1)}(z), \nu_{(f_3, H_1)}(z)\} + 1.$$

Subcase 2. Let z be a zero of (f_1, H_t). Repeating the same argument as in Subcase 1, we have

$$\nu_P(z) \ge \min\{\nu_{(f_1, H_t)}(z), \nu_{(f_2, H_t)}(z), \nu_{(f_3, H_t)}(z)\} + 1.$$

Subcase 3. Let z be a zero of (f_1, H_s). Repeating the same argument as in Subcase 1, we have

$$\nu_P(z) \ge \min\{\nu_{(f_1, H_s)}(z), \nu_{(f_2, H_s)}(z), \nu_{(f_3, H_s)}(z)\} + 1.$$

SUBCASE 4. Let z be a zero point of (f_1, H_v) with $v \notin \{1, t, s\}$. We have

$$(3.2)\qquad P = \det\begin{pmatrix} (f_1,H_1) & (f_1,H_t) & (f_1,H_s) \\ (f_2,H_1) & (f_2,H_t) & (f_2,H_s) \\ (f_3,H_1) & (f_3,H_t) & (f_3,H_s) \end{pmatrix}$$

$$= \prod_{i=1,t,s} (f_1, H_i) \cdot \det\begin{pmatrix} 1 & 1 & 1 \\ \dfrac{(f_2,H_1)}{(f_1,H_1)} & \dfrac{(f_2,H_t)}{(f_1,H_t)} & \dfrac{(f_2,H_s)}{(f_1,H_s)} \\ \dfrac{(f_3,H_1)}{(f_1,H_1)} & \dfrac{(f_3,H_t)}{(f_1,H_t)} & \dfrac{(f_3,H_s)}{(f_1,H_s)} \end{pmatrix}$$

$$= \prod_{i=1,t,s} (f_1, H_i) \cdot \det\begin{pmatrix} \dfrac{(f_2,H_t)}{(f_1,H_t)} - \dfrac{(f_2,H_1)}{(f_1,H_1)} & \dfrac{(f_2,H_s)}{(f_1,H_s)} - \dfrac{(f_2,H_1)}{(f_1,H_1)} \\ \dfrac{(f_3,H_t)}{(f_1,H_t)} - \dfrac{(f_3,H_1)}{(f_1,H_1)} & \dfrac{(f_3,H_s)}{(f_1,H_s)} - \dfrac{(f_3,H_1)}{(f_1,H_1)} \end{pmatrix}.$$

Since $f_1(z) = f_2(z) = f_3(z)$, we have

$$\frac{(f_2,H_t)}{(f_1,H_t)}(z) - \frac{(f_2,H_1)}{(f_1,H_1)}(z) = \frac{(f_2,H_s)}{(f_1,H_s)}(z) - \frac{(f_2,H_1)}{(f_1,H_1)}(z) = 0,$$

$$\text{and}\quad \frac{(f_3,H_t)}{(f_1,H_t)}(z) - \frac{(f_3,H_1)}{(f_1,H_1)}(z) = \frac{(f_3,H_s)}{(f_1,H_s)}(z) - \frac{(f_3,H_1)}{(f_1,H_1)}(z) = 0.$$

Therefore, the equality (3.2) implies that z is a zero of P with multiplicity at least 2.

Thus, from the above four subcases we have

$$v_P(z) \ge \sum_{v=1,\,t,\,s} (\min\{v_{(f_1,H_v)}(z), v_{(f_2,H_v)}(z), v_{(f_3,H_v)}(z)\}$$
$$+ v^{(1)}_{(f_1,H_v)}(z)) + 2 \sum_{\substack{v=1 \\ v\neq 1,\,t,\,s}}^{q} v^{(1)}_{(f_1,H_v)}(z),$$

for all z outside the analytic set $I(f_1) \cup I(f_2) \cup I(f_3) \bigcup_{i' \neq j'} f_1^{-1}(H_{i'} \cap H_{j'})$ of co-dimension two.

Since $\min\{v_{(f_1,H_v)}(z), v_{(f_2,H_v)}(z), v_{(f_3,H_v)}(z)\} \ge v^{(N)}_{(f_u,H_v)}(z)$ and $v^{(1)}_{(f_1,H_v)}(z) = v^{(1)}_{(f_u,H_v)}(z)$ for all $1 \le u \le 3$, the above inequality implies that

$$v_P(z) \ge \frac{1}{3}\sum_{u=1}^{3}\left(\sum_{v=1,\,t,\,s}(v^{(N)}_{(f_u,H_v)}(z) - v^{(1)}_{(f_u,H_v)}(z)) + 2\sum_{v=1}^{q} v^{(1)}_{(f_u,H_v)}(z)\right),$$

for all z outside an analytic subset of co-dimension two.

Integrating both sides of the above inequality, we get

$$N_P(r) \geq \frac{1}{3}\sum_{u=1}^{3}\left(\sum_{v=1,t,s}(N^{(N)}_{(f_u,H_v)}(r) - N^{(1)}_{(f_u,H_v)}(r)) + 2\sum_{v=1}^{q}N^{(1)}_{(f_u,H_v)}(r)\right)$$
$$= \frac{1}{3}\sum_{u=1}^{3}\left(\sum_{v=1,t,s}N_v(r) + 2\sum_{v=1}^{q}N^{(1)}_{(f_u,H_v)}(r)\right).$$

Then for all $r \in E_{I_0}$, we have

$$\begin{aligned}
N_P(r) &\geq \frac{1}{3}\sum_{u=1}^{3}\left(\sum_{v=1,t,s}N_v(r) + 2\sum_{v=1}^{q}N^{(1)}_{(f_u,H_v)}(r)\right) \\
&\geq \frac{1}{3}\sum_{u=1}^{3}\left(\sum_{v=1,[q/3]+1,2[q/3]+1}N_v(r) + 2\sum_{v=1}^{q}N^{(1)}_{(f_u,H_v)}(r)\right) \\
&\geq \frac{1}{3}\sum_{u=1}^{3}\left(\frac{1}{\left[\frac{q}{3}\right]}\sum_{v=1}^{3[q/3]}N_v(r) + 2\sum_{v=1}^{q}N^{(1)}_{(f_u,H_v)}(r)\right) \\
&\geq \frac{1}{3}\sum_{u=1}^{3}\left(\frac{3}{q}\sum_{v=1}^{q}N_v(r) + 2\sum_{v=1}^{q}N^{(1)}_{(f_u,H_v)}(r)\right) \\
&\geq \frac{1}{3q}\sum_{u=1}^{3}\sum_{v=1}^{q}(3N^{(N)}_{(f_u,H_v)}(r) + (2q-3)N^{(1)}_{(f_u,H_v)}(r)) \\
&\geq \frac{2q+3N-3}{3Nq}\sum_{u=1}^{3}\sum_{v=1}^{q}N^{(N)}_{(f_u,H_v)}(r).
\end{aligned}$$

On the other hand, by Jensen's formula and the definition of the characteristic function we have

$$\begin{aligned}
N_P(r) &= \int_{S(r)}\log|P|\sigma_n + O(1) \\
&\leq \sum_{u=1}^{3}\int_{S(r)}\log(|(f_u,H_1)|^2 + |(f_u,H_t)|^2 + |(f_u,H_s)|^2)^{1/2}\sigma_n + O(1) \\
&\leq \sum_{u=1}^{3}\int_{S(r)}\log\|f_u\|\sigma_n + O(1) = \sum_{u=1}^{3}T_{f_u}(r) + o(T_{f_1}(r)).
\end{aligned}$$

By these inequalities and by the Second Main Theorem, we have

$$\sum_{u=1}^{3} T_{f_u}(r) \geq \frac{2q+3N-3}{3Nq} \sum_{u=1}^{3} \sum_{v=1}^{q} N_{(f_u,H_v)}^{(N)}(r) + o(T_{f_1}(r))$$
$$\geq \frac{(2q+3N-3)(q-N-1)}{3Nq} \sum_{u=1}^{3} T_{f_u}(r) + o(T_{f_1}(r))$$

for every $z \in E_{I_0}$ outside a Borel finite measure set.

Letting $r \to +\infty$ $(r \in E_{I_0})$ we get

$$\frac{(2q+3N-3)(q-N-1)}{3Nq} \leq 1.$$

This implies that

$$q \leq \frac{2N+5+\sqrt{28N^2+20N+1}}{4}.$$

This is a contradiction. Thus, $f_1 \wedge f_2 \wedge f_3 = 0$. We complete the proof of the theorem. ∎

In order to prove Theorem 1.2, we need the following.

Lemma 3.3. *Let f and g be two linearly non-degenerate meromorphic mappings of $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$ and let $\{H_i\}_{i=1}^q$ be a family of q $(q \geq 2N+3)$ hyperplanes of $\mathbf{P}^N(\mathbf{C})$ in general position with*

$$\dim((\mathrm{Zero}(f,H_i) \cup \mathrm{Zero}(g,H_i)) \cap (\mathrm{Zero}(f,H_j) \cup \mathrm{Zero}(g,H_j))) \leq n-2 \quad (1 \leq i < j \leq q).$$

Assume that the following conditions are satisfied:

(a) *f is linearly non-degenerate over $\mathcal{R}_f$,*

(b) $\| \ N(r, |v_{(f,H_i)} - v_{(g,H_i)}|) = o(T_f(r))$ $(1 \leq i \leq q-N-1)$,

(c) $\| \ N_{(f,H_i)}(r) = N_{(g,H_i)}(r) + o(T_f(r)) = o(T_f(r))$ $(q-N \leq i \leq q)$.

Then $f = g$.

Proof. We assume that f and g have reduced representations $f = (f_0 : \cdots : f_N)$ and $g = (g_0 : \cdots : g_N)$ respectively. Each hyperplane H_i $(1 \leq i \leq q)$ is given by

$$H_i = \left\{ w = (w_0 : \cdots : w_N) : \sum_{v=0}^{N} a_{iv} w_v = 0 \right\}.$$

For each i $(1 \leq i \leq q-N-1)$, using the Second Main Theorem we have

$$\| \ T_f(r) \leq \sum_{v=q-N}^{q} N_{(f,H_v)}(r) + N_{(f,H_i)}(r) + o(T_f(r)) = N_{(f,H_i)}(r) + o(T_f(r)).$$

On the other hand, by the assumption (b) we have

$$\| \ T_f(r) = N_{(f,H_i)}(r) + o(T_f(r)) \leq N_{(g,H_i)}(r) + N(r, |\nu_{(f,H_i)} - \nu_{(g,H_i)}|) + o(T_f(r))$$
$$\leq T_g(r) + o(T_f(r)).$$

Therefore, $\| \ T_f(r) = O(T_g(r))$. Similarly, we also have

$$\| \ T_g(r) = N_{(g,H_i)}(r) + o(T_f(r)) \quad \text{and} \quad \| \ T_g(r) = O(T_f(r)).$$

For each i $(1 \leq i \leq N+2)$, we set $h_i = \dfrac{(f,H_1)}{(g,H_1)} \cdot \dfrac{(g,H_i)}{(f,H_i)}$. We will show that $h_i \in \mathscr{R}_f$. Indeed, we see that

$$\| \ m(r,h_i) \leq m\left(r, \frac{(f,H_1)}{(f,H_i)}) + m(r, \frac{(g,H_i)}{(g,H_1)}\right) + O(1)$$
$$\leq T_{(f,H_1)/(f,H_i)}(r) - N_{(f,H_i)}(r) + T_{(g,H_i)/(g,H_1)}(r) - N_{(g,H_1)}(r) + O(1)$$
$$\leq T_f(r) - N_{(f,H_i)}(r) + T_g(r) - N_{(g,H_1)}(r) + o(T_f(r)) = o(T_f(r)).$$

On the other hand, we also have

$$\| \ N_{1/h_i}(r) \leq N(r, |\nu_{(f,H_1)} - \nu_{(g,H_1)}|) + N(r, |\nu_{(f,H_i)} - \nu_{(g,H_i)}|) = o(T_f(r)).$$

Thus $\| \ T_{h_i}(r) = m(r,h_i) + N_{1/h_i}(r) = o(T_f(r))$. This means $h_i \in \mathscr{R}_f$ for all $1 \leq i \leq N+2$. Then we have

$$\frac{(f,H_1)}{(g,H_1)} = h_2 \frac{(f,H_2)}{(g,H_2)} = \cdots = h_{N+2} \frac{(f,H_{N+2})}{(g,H_{N+2})}.$$

Since $\{H_i\}_{i=1}^{N+2}$ are in general position, there exist nozero constants $c_1, \ldots, c_{N+1}$ such that

$$a_{(N+2)j} = \sum_{i=1}^{N+1} c_i a_{ij} \quad (0 \leq j \leq N).$$

Thus $(f, H_{N+2}) = \sum_{i=1}^{N+1} c_i(f,H_i)$ and $(g,H_{N+2}) = \sum_{i=1}^{N+1} c_i(g,H_i)$. This implies that

$$\frac{(f,H_1)}{(g,H_1)} = h_{N+2} \frac{(f,H_{N+2})}{(g,H_{N+2})} = h_{N+2} \frac{\sum_{i=1}^{N+1} c_i(f,H_i)}{\sum_{i=1}^{N+1} c_i(g,H_i)} = h_{N+2} \frac{\sum_{i=1}^{N+1} c_i(f,H_i)}{\sum_{i=1}^{N+1} c_i \dfrac{h_i(f,H_i)(g,H_1)}{(f,H_1)}}.$$

Thus

$$\sum_{i=1}^{N+1} c_i h_i (f,H_i) = h_{N+2} \sum_{i=1}^{N+1} c_i (f,H_i) \Leftrightarrow \sum_{i=1}^{N+1} c_i (h_i - h_{N+2})(f,H_i) = 0.$$

Since f is linearly non-degenerate over $\mathscr{R}_f$, the above equality yields that

$$h_1 = h_2 = h_3 = \cdots = h_{N+2}.$$

This means that

$$\frac{(f,H_1)}{(g,H_1)} = \frac{(f,H_2)}{(g,H_2)} = \cdots = \frac{(f,H_{N+2})}{(g,H_{N+2})}.$$

This implies that $f = g$. The lemma is proved. ∎

Proof of Theorem 1.2. Suppose that $f_1 \wedge f_2 \wedge f_3 \not\equiv 0$. Denote by $\mathscr{I}$ the set of all permutations of the $(q-N-1)$-tuple $(1,\dots,q-N-1)$, that means

$$\mathscr{I} = \{I = (i_1,\dots,i_{q-N-1}) : \{i_1,\dots,i_{q-N-1}\} = \{1,2,\dots,q-N-1\}\}.$$

For each $I = (i_1,\dots,i_{q-N-1}) \in \mathscr{I}$ we define a subset E_I of $[1,+\infty)$ as follows

$$E_I = \{r \ge 1 : N^{(N)}_{(f_1,H_{i_1})}(r) \ge \cdots \ge N^{(N)}_{(f_1,H_{i_{q-N-1}})}(r)\}.$$

By rearranging if necessary, we may assume that the element $I_0 = (1,2,\dots,q-N-1)$ of $\mathscr{I}$, satisfying

$$\int_{E_{I_0}} dr = +\infty.$$

We have $N^{(N)}_{(f_1,H_1)}(r) \ge N^{(N)}_{(f_1,H_2)}(r) \ge \cdots \ge N^{(N)}_{(f_1,H_{q-N-1})}(r)$ for all $r \in E_{I_0}$.

For each $i = 1,\dots,q-N-1$, we set

$$V_i = ((f_1,H_i),(f_2,H_i),(f_3,H_i)) \in \mathscr{M}^3$$

and

$$t = \min\{i : V_1 \wedge V_i \not\equiv 0\}.$$

Then we have $V_i \wedge V_j \equiv 0$ for all $1 \le i < j < t$.

We consider the following two cases.

Case 1. $t > \left[\frac{N+1}{2}\right] + 1$. Then we have

$$\frac{(f_k,H_1)}{(f_l,H_1)} = \frac{(f_k,H_2)}{(f_l,H_2)} = \cdots = \frac{(f_k,H_{[(N+1)/2]+1})}{(f_l,H_{[(N+1)/2]+1})} \quad (1 \le k,l \le 3).$$

The the assertion (i) holds in this case.

Case 2. $t \le \left[\frac{N+1}{2}\right] + 1$. In this case, we have $V_1 \wedge V_t \not\equiv 0$. Since $f_1 \wedge f_2 \wedge f_3 \not\equiv 0$, there exists an index s $(t < s \le N+1)$ such that $V_1 \wedge V_t \wedge V_s \not\equiv 0$. Therefore

$$P := \det\begin{pmatrix} (f_1,H_1) & (f_1,H_t) & (f_1,H_s) \\ (f_2,H_1) & (f_2,H_t) & (f_2,H_s) \\ (f_3,H_1) & (f_3,H_t) & (f_3,H_s) \end{pmatrix} \not\equiv 0.$$

For $z \notin \bigcup_{u=1}^{3} I(f_u) \cup \bigcup_{i' \neq j'} (\mathrm{Zero}(f_1, H_{i'}) \cap \mathrm{Zero}(f_1, H_{j'}))$, we consider the following two subcases:

SUBCASE 1. Let z be a zero of (f_1, H_v) with $v \in \{1, t, s\}$. It is easy to see that

$$\nu_P(z) \geq \min\{\nu_{(f_1,H_v)}(z), \nu_{(f_2,H_v)}(z), \nu_{(f_3,H_v)}(z)\} \geq \nu^{(N)}_{(f_u,H_v)}(z) \quad (1 \leq u \leq 3).$$

SUBCASE 2. Let z is a zero of (f, H_v) with $v \geq q - N$. We have

$$(3.4) \qquad P = \det\begin{pmatrix} (f_1,H_1) & (f_1,H_t) & (f_1,H_s) \\ (f_2,H_1) & (f_2,H_t) & (f_2,H_s) \\ (f_3,H_1) & (f_3,H_t) & (f_3,H_s) \end{pmatrix}$$

$$= \prod_{i=1,t,s} (f_1, H_i) \cdot \det\begin{pmatrix} 1 & 1 & 1 \\ \dfrac{(f_2,H_1)}{(f_1,H_1)} & \dfrac{(f_2,H_t)}{(f_1,H_t)} & \dfrac{(f_2,H_s)}{(f_1,H_s)} \\ \dfrac{(f_3,H_1)}{(f_1,H_1)} & \dfrac{(f_3,H_t)}{(f_1,H_t)} & \dfrac{(f_3,H_s)}{(f_1,H_s)} \end{pmatrix}$$

$$= \prod_{i=1,t,s} (f_1, H_i) \cdot \det\begin{pmatrix} \dfrac{(f_2,H_t)}{(f_1,H_t)} - \dfrac{(f_2,H_1)}{(f_1,H_1)} & \dfrac{(f_2,H_s)}{(f_1,H_s)} - \dfrac{(f_2,H_1)}{(f_1,H_1)} \\ \dfrac{(f_3,H_t)}{(f_1,H_t)} - \dfrac{(f_3,H_1)}{(f_1,H_1)} & \dfrac{(f_3,H_s)}{(f_1,H_s)} - \dfrac{(f_3,H_1)}{(f_1,H_1)} \end{pmatrix}.$$

Since $f_1(z) = f_2(z) = f_3(z)$, we have

$$\frac{(f_2,H_t)}{(f_1,H_t)}(z) - \frac{(f_2,H_1)}{(f_1,H_1)}(z) = \frac{(f_2,H_s)}{(f_1,H_s)}(z) - \frac{(f_2,H_1)}{(f_1,H_1)}(z) = 0,$$

$$\text{and} \quad \frac{(f_3,H_t)}{(f_1,H_t)}(z) - \frac{(f_3,H_1)}{(f_1,H_1)}(z) = \frac{(f_3,H_s)}{(f_1,H_s)}(z) - \frac{(f_3,H_1)}{(f_1,H_1)}(z) = 0.$$

Therefore, the equality (3.4) implies that z is a zero of P with multiplicity at least 2. Hence

$$\nu_P(z) \geq 2\nu^{(1)}_{(f_u,H_v)}(z) \quad (1 \leq u \leq 3).$$

Thus, from the above two subcases we have

$$\nu_P(z) \geq \sum_{v=1,t,s} \nu^{(N)}_{(f_u,H_v)}(z) + 2\sum_{v=q-N}^{q} \nu^{(1)}_{(f_u,H_v)}(z) \quad (1 \leq u \leq 3)$$

for all z outside the analytic set $I(f_1) \cup I(f_2) \cup I(f_3) \bigcup_{i' \neq j'} (\mathrm{Zero}(f_1, H_{i'}) \cap \mathrm{Zero}(f_1, H_{j'}))$ of co-dimension two. This inequality implies that

$$\nu_P(z) \geq \frac{1}{3}\sum_{u=1}^{3}\left(\sum_{v=1,t,s} \nu^{(N)}_{(f_u,H_v)}(z) + 2\sum_{v=q-N}^{q} \nu^{(1)}_{(f_u,H_v)}(z)\right),$$

for all z outside an analytic subset of co-dimension two.

Integrating both sides of the above inequality, we get

$$N_P(r) \geq \frac{1}{3}\sum_{u=1}^{3}\left(\sum_{v=1,t,s} N^{(N)}_{(f_u,H_v)}(r) + 2\sum_{v=q-N}^{q} N^{(1)}_{(f_u,H_v)}(r)\right).$$

Then for all $r \in E_{I_0}$, we have

$$\begin{aligned}
N_P(r) &\geq \frac{1}{3}\sum_{u=1}^{3}\left(\sum_{v=1,t,s} N^{(N)}_{(f_u,H_v)}(r) + 2\sum_{v=q-N}^{q} N^{(1)}_{(f_u,H_v)}(r)\right)\\
&\geq \frac{1}{3}\sum_{u=1}^{3}\left(\sum_{v=1,[(N+1)/2]+1,2[(N+1)/2]+1} N^{(N)}_{(f_u,H_v)}(r) + 2\sum_{v=q-N}^{q} N^{(1)}_{(f_u,H_v)}(r)\right)\\
&\geq \frac{1}{3}\sum_{u=1}^{3}\left(\frac{1}{\left[\frac{N+1}{2}\right]}\sum_{v=1}^{3[(N+1)/2]} N^{(N)}_{(f_u,H_v)}(r) + 2\sum_{v=q-N}^{q} N^{(1)}_{(f,H_v)}(r)\right)\\
&\geq \frac{1}{3}\sum_{u=1}^{3}\left(\frac{3}{q-N-1}\sum_{v=1}^{q-N-1} N^{(N)}_{(f_u,H_v)}(r) + 2\sum_{v=q-N}^{q} N^{(1)}_{(f_u,H_v)}(r)\right)\\
&\geq \frac{1}{3}\sum_{u=1}^{3}\left(\frac{3}{q-N-1}\sum_{v=1}^{q-N-1} N^{(N)}_{(f_u,H_v)}(r) + \frac{2}{N}\sum_{v=q-N}^{q} N^{(N)}_{(f_u,H_v)}(r)\right)\\
&= \sum_{u=1}^{3}\frac{1}{q-N-1}\sum_{v=1}^{q} N^{(N)}_{(f_u,H_v)}(r) + \frac{1}{3}\sum_{u=1}^{3}\left(\frac{2}{N} - \frac{3}{q-N-1}\right)\sum_{v=q-N}^{q} N^{(N)}_{(f_u,H_v)}(r).
\end{aligned}$$

Here we note that since $q > 3\left[\frac{N+1}{2}\right] + N + 2 \geq 3\frac{N}{2} + N + 2$ then $\frac{2}{N} - \frac{3}{q-N-1} > 0$.

On the other hand, by Jensen's formula and the definition of the characteristic function we have

$$\begin{aligned}
N_P(r) &= \int_{S(r)} \log|P|\sigma_n + O(1)\\
&\leq \sum_{u=1}^{3}\int_{S(r)} \log(|(f_u,H_1)|^2 + |(f_u,H_t)|^2 + |(f_u,H_s)|^2)^{1/2}\sigma_n + O(1)\\
&\leq \sum_{u=1}^{3}\int_{S(r)} \log\|f_u\|\sigma_n + O(1) = \sum_{u=1}^{3} T_{f_u}(r) + o(T_{f_1}(r)).
\end{aligned}$$

By this inequality and by the Second Main Theorem, we have

$$\begin{aligned}\sum_{u=1}^{3} T_{f_u}(r) &\geq N_P(r) + o(T_{f_1}(r)) \\ &\geq \sum_{u=1}^{3} \frac{1}{q-N-1} \sum_{v=1}^{q} N^{(N)}_{(f_u,H_v)}(r) \\ &\quad + \frac{1}{3}\sum_{u=1}^{3}\left(\frac{2}{N} - \frac{3}{q-N-1}\right) \sum_{v=q-N}^{q} N^{(N)}_{(f_u,H_v)}(r) + o(T_{f_1}(r)) \\ &\geq \sum_{u=1}^{3} T_{f_u}(r) + \frac{1}{3}\sum_{u=1}^{3}\left(\frac{2}{N} - \frac{3}{q-N-1}\right) \sum_{v=q-N}^{q} N^{(N)}_{(f_u,H_v)}(r) + o(T_{f_1}(r))\end{aligned}$$

for every $r \in E_{I_0}$ outside a Borel set with finite measure. Thus

$$N^{(N)}_{(f_u,H_v)}(r) = o(T_{f_u}(r)) \quad (1 \leq u \leq 3, q-N \leq v \leq q) \tag{3.5}$$

for every $r \in E_{I_0}$ outside a Borel set with finite measure.

Since $[1, +\infty) = \bigcup_{I \in \mathscr{I}} E_I$ and the inequality (3.5) holds for all $r \in E_I$ with $\int_{E_I} dr = \infty$ outside a Borel set with finite measure, this equality also holds for all r outside a Borel set with finite measure. This means that, for all $r \in [1, +\infty)$ we have

$$\| \ N^{(N)}_{(f_u,H_v)}(r) = o(T_f(r)) \quad (1 \leq u \leq 3, q-N \leq v \leq q). \tag{3.6}$$

For each index i $(1 \leq i \leq q-N-1)$, by the Second Main Theorem we have

$$\begin{aligned}\| \ T_{f_u}(r) &\leq \sum_{v=q-N}^{q} N^{(N)}_{(f_u,H_v)}(r) + N^{(N)}_{(f_u,H_i)}(r) + o(T_{f_u}(r)) \\ &= N^{(N)}_{(f_u,H_i)}(r) + o(T_{f_u}(r)) \quad (1 \leq u \leq 3).\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}\| \ N_{(f_u,H_i),>N}(r) &\leq (N+1)(N_{(f_u,H_i)}(r) - N^{(N)}_{(f_u,H_i)}(r)) \\ &\leq (N+1)(T_{f_u}(r) - N^{(N)}_{(f_u,H_i)}(r)) + o(T_{f_u}(r)) \\ &= o(T_{f_u}(r)) \quad (1 \leq u \leq 3).\end{aligned}$$

Then for each $1 \leq i \leq q-N-1$, we have

$$\begin{aligned}\| \ N(r, |v_{(f_u,H_i)} - v_{(f_k,H_i)}|) &\leq N_{(f_u,H_i),>N}(r) + N_{(f_k,H_i),>N}(r) \\ &= o(T_{f_1}(r)) \quad (1 \leq u, k \leq 3).\end{aligned} \tag{3.7}$$

From (3.6) and (3.7) and applying Lemma 3.3, we have $f_1 \equiv f_2 \equiv f_3$. This is a contradiction to the supposition that $f_1 \wedge f_2 \wedge f_3 \not\equiv 0$.

Thus, $f_1 \wedge f_2 \wedge f_3 = 0$. Hence the assertion (ii) holds in this case.

We complete the proof of the theorem. ■

4. Correction of the proof of Theorem B in [10]

Let f be a linearly non-degenerate meromorphic mapping of $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$ $(N \geq 3)$. Let $H_1, \dots, H_{2N+2}$ be hyperplanes of $\mathbf{P}^N(\mathbf{C})$ in general position with

$$\dim(\mathrm{Zero}(f, H_i) \cap \mathrm{Zero}(f, H_j)) \leq n - 2 \quad (1 \leq i < j \leq q).$$

Now for three mappings $f_1, f_2, f_3 \in \mathscr{F}(f, \{H_j\}_{j=1}^{2N+2}, 1)$, we set

$$F_u^{ij} = \frac{(f_u, H_i)}{(f_u, H_j)} \quad (0 \leq k \leq 2, 1 \leq i, j \leq 2N+2).$$

For meromorphic functions F, G, H on $\mathbf{C}^n$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n$ with $|\alpha| = \sum_{i=1}^n = 1$, we put

$$\Phi^\alpha(F, G, H) := F \cdot G \cdot H \cdot \begin{vmatrix} 1 & 1 & 1 \\ \dfrac{1}{F} & \dfrac{1}{G} & \dfrac{1}{H} \\ \mathscr{D}^\alpha\left(\dfrac{1}{F}\right) & \mathscr{D}^\alpha\left(\dfrac{1}{G}\right) & \mathscr{D}^\alpha\left(\dfrac{1}{H}\right) \end{vmatrix}$$

Lemma 4.1 (see [10, Lemma 3.8]). *Let f and $\{H_i\}_{i=1}^{2N+2}$ be as above. If there are two distinct maps f_1 and f_2 in $\mathscr{F}(f, \{H_i\}_{i=1}^{2N+2}, 1)$ then the following assertion holds.*

$$\| \; T_{f_1}(r) + T_{f_2}(r) = 2(N^{(N)}_{(f_1, H_t)}(r) + N^{(N)}_{(f_2, H_t)}(r) - (N+1)N^{(1)}_{(f, H_t)}(r)) + \sum_{v=1}^{q} N^{(1)}_{(f, H_v)}(r) + o(T_f(r)) \quad (1 \leq t \leq 2N+2).$$

Lemma 4.2 (see [10, Lemma 3.16]). *Let f_1, f_2, f_3 be three distinct maps in $\mathscr{F}(f, \{H_i\}_{i=1}^{2N+2}, 1)$. Assume that there exist $i, j \in \{1, 2, \dots, 2N+2\}$ $(i \neq j)$ such that $\Phi^\alpha := \Phi^\alpha(F_0^{ij}, F_1^{ij}, F_2^{ij}) \equiv 0$ for all $|\alpha| = 1$. Then the following assertion holds*

$$\| \; 2 \sum_{v=i,j} N^{(1)}_{(f, H_v)}(r) \geq \sum_{v=1}^{2N+2} N^{(1)}_{(f, H_v)}(r) + o(T_f(r)).$$

The following lemma is the correction of [10, Lemma 3.20].

Lemma 4.3. *Let f_1, f_2, f_3 be three maps in $\mathscr{F}(f, \{H_i\}_{i=1}^{2N+2}, 1)$. Assume that there exist $i, j \in \{1, 2, \ldots, 2N+2\}$ $(i \neq j)$ and $|\alpha| = 1$ such that $\Phi^\alpha := \Phi^\alpha(F_0^{ij}, F_1^{ij}, F_2^{ij}) \not\equiv 0$. Then, for each $1 \leq u \leq 3$, the following assertions hold*

(i) $\| \sum_{u=1}^{3} N^{(N)}_{(f_u, H_i)}(r) + 2\sum_{\substack{t=1\\ t\neq i,j}}^{2N+2} N^{(1)}_{(f, H_t)}(r) - (2N+1)N^{(1)}_{(f, H_i)}(r) \leq N_{\Phi^\alpha}(r)$,

(ii) $\| N_{\Phi^\alpha}(r) \leq \sum_{u=1}^{3} T_{f_u}(r) - \sum_{u=1}^{3} N^{(N)}_{(f_u, H_j)}(r) + (N+1)N^{(1)}_{(f, H_j)}(r) + o(T_f(r))$,

(iii) *Moreover, if we assume further that $\Phi^\alpha(F_0^{ji}, F_1^{ji}, F_2^{ji}) \not\equiv 0$ for all $|\alpha| = 1$ then*

$$\| \; 3(N^{(1)}_{(f, H_i)}(r) + N^{(1)}_{(f, H_j)}(r)) \geq \sum_{t=1}^{2n+2} N^{(1)}_{(f, H_t)}(r) + o(T_f(r)).$$

Proof. (i) The first assertion is due to [10, Lemma 3.20(i)].

(ii) We now prove the second assertion of the lemma. Denote by S the set of all singularities of $f^{-1}(H_t)$ $(1 \leq t \leq 2N+2)$. Then S is an analytic subset of codimension at least two in $\mathbf{C}^n$. We set

$$I = S \cup \bigcup_{1 \leq s < t \leq 2N+2} (f^{-1}(H_s) \cap f^{-1}(H_t)).$$

Similarly as in the proof of [10, Lemma 3.20(ii)], we have

$$\| \; m(r, \Phi^\alpha) \leq \sum_{v=0}^{2} m(r, F_v^{ij}) + o(T_f(r)) \tag{4.4}$$

and that Φ^α is holomorphic at all zeros of (f, H_i), which are outside I. Hence a zero of (f, H_i) outside I is not pole of Φ^α. Thus, a pole of Φ^α outside I is a zero of (f, H_j). Assume that z_0 is a zero of (f, H_j), and $z_0 \notin I$. We may assume that

$$v^0_{F_1^{ji}}(z_0) = d_1 \geq v^0_{F_2^{ji}}(z_0) = d_2 \geq v^0_{F_3^{ji}}(z_0) = d_3.$$

Choose a holomorphic function h on $\mathbf{C}^n$ with zero multiplicity at z_0 equal to 1 such that $F_u^{ji} = h^{d_u}\varphi_u$ $(1 \leq u \leq 3)$, where φ_u are meromorphic on $\mathbf{C}^n$ and holomorphic on a neighborhood of z_0. Then

$$\begin{aligned}\Phi^\alpha &= F_1^{ij} \cdot F_2^{ij} \cdot F_3^{ij} \cdot \begin{vmatrix} F_2^{ji} - F_1^{ji} & F_3^{ji} - F_1^{ji} \\ \mathscr{D}^\alpha(F_2^{ji} - F_1^{ji}) & \mathscr{D}^\alpha(F_3^{ji} - F_1^{ji}) \end{vmatrix} \\ &= F_1^{ij} \cdot F_2^{ij} \cdot F_3^{ij} \cdot h^{d_2+d_3} \cdot \begin{vmatrix} \varphi_2 - h^{d_1-d_2}\varphi_1 & \varphi_3 - h^{d_1-d_3}\varphi_1 \\ \dfrac{\mathscr{D}^\alpha(h^{d_2-d_3}\varphi_2 - h^{d_1-d_3}\varphi_1)}{h^{d_2-d_3}} & \mathscr{D}^\alpha(\varphi_3 - h^{d_1-d_3}\varphi_1) \end{vmatrix}.\end{aligned}$$

We see that the pole multiplicity of the function $\dfrac{\mathscr{D}^\alpha(h^{d_2-d_3}\varphi_2 - h^{d_1-d_3}\varphi_1)}{h^{d_2-d_3}}$ at z_0 is at most 1. This yields that

$$(4.5)\qquad \nu^{\infty}_{\Phi^{\alpha}_{ij}}(z_0) \le \sum_{u=1}^{3} \nu^{\infty}_{F^{ij}_u}(z_0) - d_2 - d_3 + 1$$
$$\le \sum_{u=1}^{3} \nu^{\infty}_{F^{ij}_u}(z_0) - \min\{N, d_1\} - \min\{N, d_2\} - \min\{N, d_3\} + (N+1)$$
$$= \sum_{u=1}^{3} \nu^{\infty}_{F^{ij}_u}(z_0) - \sum_{u=1}^{3} \min\{N, \nu^{0}_{(f_u, H_j)}(z_0)\}$$
$$+ (N+1)\min\{1, \nu^{0}_{(f, H_j)}(z_0)\}.$$

This yields that

$$(4.6)\qquad N_{1/\Phi^{\alpha}}(r) \le \sum_{u=1}^{3} N_{F^{ji}_u}(r) - \sum_{u=1}^{3} N^{(N)}_{(f_u, H_j)}(r) + (N+1) N^{(1)}_{(f, H_j)}(r).$$

From (4.4) and (4.6) we get

$$\| \; N_{\Phi^{\alpha}}(r) \le T(r, \Phi^{\alpha}) + O(1) = m(r, \Phi^{\alpha}) + N_{1/\Phi^{\alpha}}(r) + O(1)$$
$$\le \sum_{u=1}^{3} (m(r, F^{ij}_u) + N_{F^{ji}_u}(r)) - \sum_{u=1}^{3} N^{(N)}_{(f_u, H_j)}(r)$$
$$+ (N+1) N^{(1)}_{(f, H_j)}(r) + o(T_f(r))$$
$$= \sum_{u=1}^{3} T(r, F^{ij}_u) - \sum_{u=1}^{3} N^{(N)}_{(f_u, H_j)}(r) + (N+1) N^{(1)}_{(f, H_j)}(r) + o(T_f(r))$$
$$\le \sum_{u=1}^{3} T_{f_u}(r) - \sum_{u=1}^{3} N^{(N)}_{(f_u, H_j)}(r) + (N+1) N^{(1)}_{(f, H_j)}(r) + o(T_f(r)).$$

This implies the second assertion of the lemma.

(iii) Now we assume that $\Phi^{\alpha}(F^{ji}_1, F^{ji}_2, F^{ji}_3) \not\equiv 0$. By the second assertion of the lemma, we have

$$\| \; \sum_{u=1}^{3} T_{f_u}(r) \ge \sum_{u=1}^{3} (N^{(N)}_{(f_u, H_i)}(r) + N^{(N)}_{(f_u, H_j)}(r)) + 2 \sum_{t=1}^{2N+2} N^{(1)}_{(f, H_t)}(r)$$
$$- (2N+3) N^{(1)}_{(f, H_i)}(r) - (N+3) N^{(1)}_{(f, H_j)}(r) + o(T_f(r))$$

and

$$\| \sum_{u=1}^{3} T_{f_u}(r) \ge \sum_{u=1}^{3} (N^{(N)}_{(f_u, H_i)}(r) + N^{(N)}_{(f_u, H_j)}(r)) + 2 \sum_{t=1}^{2N+2} N^{(1)}_{(f, H_t)}(r)$$
$$- (2N+3) N^{(1)}_{(f, H_j)}(r) - (N+3) N^{(1)}_{(f, H_i)}(r) + o(T_f(r)).$$

Summing-up both sides of these above inequalities, we get

$$\begin{aligned}
(4.7)\qquad \| \, 2\sum_{u=1}^{3} T_{f_u}(r) &\geq 2\sum_{u=1}^{3}(N^{(N)}_{(f_u,H_i)}(r) + N^{(N)}_{(f_u,H_j)}(r)) \\
&\quad + 4\sum_{t=1}^{2N+2} N^{(1)}_{(f,H_t)}(r) - (3N+6)N^{(1)}_{(f,H_i)}(r) \\
&\quad - (3N+6)N^{(1)}_{(f,H_j)}(r) + o(T_f(r)) \\
&= \sum_{1\leq u<l\leq 3}\left(\sum_{v=i,j}(N^{(N)}_{(f_u,H_v)}(r) + N^{(N)}_{(f^l,H_v)}(r)\right. \\
&\qquad \left. - (N+1)N^{(1)}_{(f,H_v)}) + \sum_{t=1}^{2N+2} N^{(1)}_{(f,H_t)}(r)\right) \\
&\quad - \sum_{v=i,j} 3N^{(1)}_{(f,H_v)}(r) + \sum_{t=1}^{2N+2} N^{(1)}_{(f,H_t)}(r) + o(T_f(r)).
\end{aligned}$$

From Lemma 4.1 and the inequality (4.7), it follows that

$$\| \, 2\sum_{u=1}^{3} T_{f_u}(r) \geq 2\sum_{u=1}^{3} T_{f_u}(r) - 3\sum_{v=i,j} N^{(1)}_{(f,H_v)}(r) + \sum_{t=1}^{2N+2} N^{(1)}_{(f,H_t)}(r) + o(T_f(r)).$$

Thus

$$\| \, 3\sum_{v=i,j} N^{(1)}_{(f,H_v)}(r) \geq \sum_{t=1}^{2N+2} N^{(1)}_{(f,H_t)}(r) + o(T_f(r)).$$

The third assertion is proved. ■

Proof of Theorem B for the case where $N \geq 3$. Suppose that there exist three distinct maps f_1, f_2, f_3 in $\mathscr{F}(f, \{H_i\}_{i=1}^{2N+2}, 1)$. By Lemma 4.2 and Lemma 4.3(iii), we always have

$$\| \, 3(N^{(1)}_{(f,H_i)}(r) + N^{(1)}_{(f,H_j)}(r)) \geq \sum_{t=1}^{2N+2} N^{(1)}_{(f,H_t)}(r) + o(T_f(r)) \quad (1 \leq i < j \leq 2N+2).$$

Summing-up both sides of the above inequalities over all $1 \leq i < j \leq 2N+2$, we get

$$\| \, 6\sum_{t=1}^{2N+2} N^{(1)}_{(f,H_t)}(r) \geq (2N+2)\sum_{t=1}^{2N+2} N^{(1)}_{(f,H_t)}(r) + o(T_f(r)).$$

Thus

$$\| \sum_{t=1}^{2N+2} N^{(1)}_{(f,H_t)}(r) = o(T_f(r)).$$

By the second main theorem, we have

$$\begin{aligned} \| \ (N+1)T_f(r) &\le \sum_{i=1}^{2N+2} N^{(N)}_{(f,H_t)}(r) + o(T_f(r)) \\ &\le N \sum_{i=1}^{2N+2} N^{(1)}_{(f,H_t)}(r) + o(T_f(r)) = o(T_f(r)). \end{aligned}$$

This is a contradiction.

Hence $\#\mathscr{F}(f, \{H_i\}_{i=1}^{2N+2}, 1) \le 2$. We complete the proof of the theorem. ∎

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