Y. NAITO
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CONVERGENCE RATE IN THE WEIGHTED NORM FOR A SEMILINEAR HEAT EQUATION WITH SUPERCRITICAL NONLINEARITY

Yūki Naito

Abstract

We study the behavior of solutions to the Cauchy problem for a semilinear heat equation with supercritical nonlinearity. It is known that two solutions approach each other if these initial data are close enough near the spatial infinity. In this paper, we give its sharp convergence rate in the weighted norms for a class of initial data. Proofs are given by a comparison method based on matched asymptotics expansion.

1. Introduction

We consider the Cauchy problem

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u, & x \in \mathbf{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^N, \end{cases} \tag{1.1}$$

where $u = u(x,t)$, Δ is the Laplacian, $p > 1$ and u_0 is a given continuous function on $\mathbf{R}^N$ that decays to zero as $|x| \to \infty$.

We first recall some known facts concerning positive solutions of the elliptic equation

$$\Delta\phi + \phi^p = 0 \quad \text{in } \mathbf{R}^N \tag{1.2}$$

with $N \geq 3$. It is well known that there exists a classical positive radial solution of (1.2) if and only if $p \geq (N+2)/(N-2)$. (See, e.g., [1, 4].) We denote by ϕ_α a solution of the problem

$$\begin{cases} \phi_{rr} + \dfrac{N-1}{r}\phi_r + \phi^p = 0 \quad \text{for } r > 0, \\ \phi(0) = \alpha \quad \text{and} \quad \phi_r(0) = 0. \end{cases}$$

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If $p \geq (N+2)/(N-2)$ then, for each $\alpha > 0$, $\phi_\alpha(r)$ is positive and strictly decreasing for $r \geq 0$, and satisfies $\phi_\alpha(r) \to 0$ as $r \to \infty$.

Define the exponent p_c by

$$p_c = \begin{cases} \dfrac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)}, & N \geq 11, \\ \infty, & 3 \leq N \leq 10. \end{cases}$$

The exponent p_c has appeared in several different studies of (1.1) and related problems, see, e.g., [8, 11, 5, 6]. It was shown by Wang [11] that, for $(N+2)/(N-2) \leq p < p_c$, each pair of positive radial solutions of (1.2) intersect each other, and that, for $p \geq p_c$, these solutions are strictly ordered such that $\phi_\alpha(r)$ is strictly increasing in α for each r and satisfies

$$\lim_{\alpha \to \infty} \phi_\alpha(r) = \phi_\infty(r) \quad \text{for } r > 0,$$

where ϕ_∞ a singular solution given by

$$\phi_\infty(r) = L r^{-m} \quad \text{for } r > 0$$

with

$$m = \frac{2}{p-1} \quad \text{and} \quad L = (m(N-2-m))^{1/(p-1)}.$$

It was also shown in [5] that, for $p > p_c$, the solution ϕ_α has the expansion

$$\phi_\alpha(|x|) = L|x|^{-m} + a(\alpha)|x|^{-m-\lambda_1} + o(|x|^{-m-\lambda_1}) \quad \text{as } |x| \to \infty, \tag{1.3}$$

where λ_1 is a positive constant given by

$$\lambda_1 = \lambda_1(N,p) = \frac{N-2-2m - \sqrt{(N-2-m)^2 - 8(N-2-m)}}{2},$$

and $a(\alpha)$ is a positive number. Note that λ_1 is a smaller root of the quadratic polynomial

$$\lambda^2 - (N-2-2m)\lambda + 2(N-2-m) = 0.$$

We denote by

$$\lambda_2 = \lambda_2(N,p) = \frac{N-2-2m + \sqrt{(N-2-m)^2 - 8(N-2-m)}}{2}$$

a larger root of the quadratic polynomial.

For $\ell \geq 0$, we define the weighted norm

$$\|\psi\|_\ell = \sup_{x \in \mathbf{R}^N} (1+|x|)^\ell |\psi(x)|,$$

where ψ be a continuous function on $\mathbf{R}^N$. It is clear that $\|\cdot\|_\ell = \|\cdot\|_{L^\infty(\mathbf{R}^N)}$ if $\ell = 0$.

The following results have been proved by [5, Theorem 1.15] and [6, Theorem 2].

Theorem A. *Let* $p > p_c$ *and* $\alpha > 0$.

(i) *The stationary solution* ϕ_α *is stable with respect to* $\|\cdot\|_{m+\lambda_1}$, *that is, for any* $\varepsilon > 0$ *there is* $\delta > 0$ *such that, if* $\|u_0 - \phi_\alpha\|_{m+\lambda_1} < \delta$, *then the solution* u *of* (1.1) *satisfies* $\|u(\cdot,t) - \phi_\alpha\|_{m+\lambda_1} < \varepsilon$ *for all* $t > 0$.

(ii) *Let* $\ell \in (m+\lambda_1, m+\lambda_2]$. *Then* ϕ_α *is stable with respect to* $\|\cdot\|_\ell$ *and there exists* $\delta > 0$ *such that, if* $\|u_0 - \phi_\alpha\|_\ell < \delta$, *then the solution* u *of* (1.1) *satisfies* $\|u(\cdot,t) - \phi_\alpha\|_{\ell'} \to 0$ *as* $t \to \infty$ *for any* $\ell' \in (0,\ell)$.

Poláčik and Yanagida [10] improved the above results and proved that the solutions approach a set of stationary solutions for a wide class of initial data. Later, Fila, Winkler and Yanagida [3] and Hoshino and Yanagida [7] studied more general problem. Denote by u and $\tilde{u}$ solutions of (1.1) with initial data u_0 and $\tilde{u}_0$, respectively. In [3, 7] they showed that how fast two solutions u and $\tilde{u}$ approach each other as $t \to \infty$ if u_0 and $\tilde{u}_0$ are close enough near the spatial infinity. Clearly, in the case $\tilde{u}_0(x) = \phi_\alpha(|x|)$, the rate of approach of these solutions corresponds to the convergence rate to the steady state. Precisely, the following results were shown by [3, 7].

Theorem B. *Let* $p > p_c$ *and* $m+\lambda_1 < \ell < m+\lambda_2+2$. *Assume that* u_0 *and* $\tilde{u}_0$ *satisfy*

$$-\phi_\alpha(|x|) \le u_0(x), \quad \tilde{u}_0(x) \le \phi_\alpha(|x|) \quad \text{for } x \in \mathbf{R}^N \tag{1.4}$$

with some $\alpha > 0$. *If*

$$\limsup_{|x|\to\infty} |x|^\ell |u_0(x) - \tilde{u}_0(x)| < \infty, \tag{1.5}$$

then the solutions u *and* $\tilde{u}$ *of* (1.1) *satisfy*

$$\limsup_{t\to\infty} t^{(\ell-m-\lambda_1)/2} \|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} < \infty.$$

In this paper, we consider the optimal rate of approach of these solutions in the weighted norm $\|\cdot\|_\ell$, and verify that the rate depends on the order ℓ. We also show the stability of solutions with respect to the norm $\|\cdot\|_\ell$ for $m+\lambda_1 < \ell < m+\lambda_2+2$.

Our first result is the following.

Theorem 1.1. *Let* $p > p_c$ *and* $\ell \in (m+\lambda_1, m+\lambda_2+2)$. *Assume that* $u_0, \tilde{u}_0 \in C(\mathbf{R}^N)$ *satisfy* (1.4) *with some* $\alpha > 0$. *If* (1.5) *holds, then, for* $\ell' \in [0,\ell)$, *the solutions* u *and* $\tilde{u}$ *of* (1.1) *satisfy*

$$\limsup_{t\to\infty} t^\nu \|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{\ell'} < \infty, \tag{1.6}$$

where

$$\nu = \begin{cases} (\ell - m - \lambda_1)/2 & \text{if } 0 \le \ell' \le m + \lambda_1, \\ (\ell - \ell')/2 & \text{if } m + \lambda_1 < \ell' < \ell. \end{cases} \tag{1.7}$$

Furthermore, for any constants $c > 0$ *and* $\mu \ge 1/2$, *the solutions* u *and* $\tilde{u}$ *satisfy*

$$\limsup_{t\to\infty} t^{(\ell-\ell')\mu}\left(\sup_{|x|\ge ct^{\mu}} (1+|x|)^{\ell'}|u(x,t)-\tilde{u}(x,t)|\right) < \infty. \tag{1.8}$$

The next result shows that the upper estimates given in Theorem 1.1 are optimal.

THEOREM 1.2. *Let* $p > p_c$ *and* $\ell \in (m+\lambda_1, m+\lambda_2+2)$. *Assume that* $u_0, \tilde{u}_0 \in C(\mathbf{R}^N)$ *satisfy*

$$\phi_\alpha(|x|) \le \tilde{u}_0(x) < u_0(x) \le \phi_\infty(|x|) \quad \text{for } x \in \mathbf{R}^N\setminus\{0\} \tag{1.9}$$

with some $\alpha > 0$. *If*

$$\liminf_{|x|\to\infty} |x|^{\ell}(u_0(x)-\tilde{u}_0(x)) > 0, \tag{1.10}$$

then, for $\ell' \in [0,\ell)$, *the solutions* u *and* $\tilde{u}$ *of* (1.1) *satisfy*

$$\liminf_{t\to\infty} t^{\nu}\|u(\cdot,t)-\tilde{u}(\cdot,t)\|_{\ell'} > 0, \tag{1.11}$$

where ν *is the constant defined by* (1.7). *Furthermore, the following* (i) *and* (ii) *hold.*

(i) *For any constant* $c > 0$, *the solutions* u *and* $\tilde{u}$ *satisfy*

$$\liminf_{t\to\infty} t^{(\ell-m-\lambda_1)/2}\left(\inf_{|x|\le ct^{1/2}} (1+|x|)^{m+\lambda_1}|u(x,t)-\tilde{u}(x,t)|\right) > 0. \tag{1.12}$$

(ii) *Let* $\ell' \in [0,\ell)$. *For any constants* $c > 0$ *and* $\mu \ge 1/2$, *the solutions* u *and* $\tilde{u}$ *satisfy*

$$\liminf_{t\to\infty} t^{(\ell-\ell')\mu}\left(\sup_{|x|\ge ct^{\mu}} (1+|x|)^{\ell'}|u(x,t)-\tilde{u}(x,t)|\right) > 0. \tag{1.13}$$

Remark 1.1. In Theorem 1.2, we also assume that $u_0(x), \tilde{u}_0(x) \le \phi_{\tilde{\alpha}}(|x|)$ for $x \in \mathbf{R}^N$ with some $\tilde{\alpha} > \alpha$, and that (1.5) holds. Then, by applying Theorem 1.1 with $\alpha = \tilde{\alpha}$, we obtain (1.6) with (1.7) and (1.8). Thus, from (1.6) with $\ell' = m + \lambda_1$, we obtain, for any constant $c > 0$,

$$\limsup_{t\to\infty} t^{(\ell-m-\lambda_1)/2}\left(\sup_{|x|\le ct^{1/2}} (1+|x|)^{m+\lambda_1}(u(x,t)-\tilde{u}(x,t))\right) < \infty. \tag{1.14}$$

Thus, in the range $|x| \le ct^{1/2}$, the solutions u and $\tilde{u}$ approach uniformly in the senses (1.12) and (1.14). In particular, for any constant $c > 0$, we have

$$0 < \liminf_{t\to\infty} t^{(\ell-m-\lambda_1)/2}\left(\inf_{|x|\le c}(u(x,t)-\tilde{u}(x,t))\right)$$
$$\le \limsup_{t\to\infty} t^{(\ell-m-\lambda_1)/2}\left(\sup_{|x|\le c}(u(x,t)-\tilde{u}(x,t))\right) < \infty.$$

On the other hand, in the range $|x| \ge ct^{\mu}$ with $\mu > 1/2$, the solutions approaches each other in the different rate by (1.8) and (1.13). In particular, for any constant $c > 0$, we obtain

$$0 < \liminf_{t\to\infty} t^{(\ell-\ell')/2}\left(\sup_{|x|\ge ct^{1/2}}(1+|x|)^{\ell'}(u(x,t)-\tilde{u}(x,t))\right)$$
$$\le \limsup_{t\to\infty} t^{(\ell-\ell')/2}\left(\sup_{|x|\ge ct^{1/2}}(1+|x|)^{\ell'}(u(x,t)-\tilde{u}(x,t))\right) < \infty.$$

Note here that the constant ν defined by (1.7) fulfills the property that $\nu = \min\{(\ell-m-\lambda_1)/2, (\ell-\ell')/2\}$.

We consider the stability of solutions with respect to the norm $\|\cdot\|_\ell$.

Theorem 1.3. *Let $p > p_c$ and $\ell \in (m+\lambda_1, m+\lambda_2+2)$.*

(i) *Assume that $u_0, \tilde{u}_0 \in C(\mathbf{R}^N)$ satisfy (1.4) with some $\alpha > 0$. For any $\varepsilon > 0$ there exists $\delta > 0$ such that, if $\|u_0(\cdot) - \tilde{u}_0(\cdot)\|_\ell < \delta$, then the solutions u and $\tilde{u}$ of (1.1) satisfy $\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_\ell < \varepsilon$ for all $t \ge 0$.*

(ii) *Assume that u_0 and $\tilde{u}_0$ satisfy (1.9) with some $\alpha > 0$. If (1.10) holds, then the solutions u and $\tilde{u}$ satisfy*

$$\liminf_{t\to\infty} \|u(\cdot,t) - \tilde{u}(\cdot,t)\|_\ell > 0. \tag{1.15}$$

Let us consider the convergence rate of solutions to the steady state. Assume that $u_0 \in C(\mathbf{R}^N)$ satisfies

$$-\phi_\infty(x) < u_0(x) < \phi_\infty(x) \quad \text{for } x \in \mathbf{R}^N\setminus\{0\}. \tag{1.16}$$

and

$$\limsup_{|x|\to\infty} |x|^\ell |u_0(x) - \phi_\alpha(|x|)| < \infty \tag{1.17}$$

with some $\alpha > 0$. Then there exists $\beta > \alpha$ such that $-\phi_\beta(x) \le u_0(x) \le \phi_\beta(x)$ for $x \in \mathbf{R}^N$. Applying Theorem 1.1 with $\alpha = \beta$ and $\tilde{u}_0 = \phi_\alpha$, and Theorem 1.2 with $\tilde{u}_0 = \phi_\alpha$, we obtain the following

Corollary 1.1. *Let* $p > p_c$ *and* $\ell \in (m + \lambda_1, m + \lambda_2 + 2)$. *Assume that* $u_0 \in C(\mathbf{R}^N)$ *satisfies* (1.16) *and* (1.17) *with some* $\alpha > 0$. *Then the solution* u *of* (1.1) *satisfies*

$$\|u(\cdot, t) - \phi_\alpha(|\cdot|)\|_{\ell'} \to 0 \quad as \ t \to \infty \tag{1.18}$$

for any $\ell' \in [0, \ell)$. *Precisely, the solution* u *satisfies*

$$\limsup_{t\to\infty} \ t^\nu \|u(\cdot, t) - \phi_\alpha(|\cdot|)\|_{\ell'} < \infty,$$

where ν *is the constant defined by* (1.7). *Furthermore, if* u_0 *satisfies* $u_0(x) > \phi_\alpha(|x|)$ *for* $x \in \mathbf{R}^N$ *and*

$$\liminf_{|x|\to\infty} |x|^\ell (u_0(x) - \phi_\alpha(|x|)) > 0, \tag{1.19}$$

then the solution u *satisfies*

$$\liminf_{t\to\infty} \ t^\nu \|u(\cdot, t) - \phi_\alpha(|\cdot|)\|_{\ell'} > 0$$

for $\ell' \in [0, \ell)$ *with the constant* ν *defined by* (1.7).

We consider the stability of the steady state with respect to the norm $\|\cdot\|_\ell$ by applying Theorem 1.3. Let $p > p_c$ and $\alpha > 0$. Put $\beta > \alpha$. Then there exists $\tilde{\delta} = \tilde{\delta}(\beta) > 0$ such that, if $\|u_0(\cdot) - \phi_\alpha(|\cdot|)\|_\ell < \tilde{\delta}$, then $-\phi_\beta(|x|) \le u_0(x) \le \phi_\beta(|x|)$ for $x \in \mathbf{R}^N$. Applying Theorem 1.3 (i) with $\alpha = \beta$, $\tilde{u}_0 = \phi_\alpha$, and $\min\{\delta, \tilde{\delta}\}$ instead of δ, and Theorem 1.3 (ii) with $\tilde{u}_0 = \phi_\alpha$, we obtain the following

Corollary 1.2. *Let* $p > p_c$, $\alpha > 0$, *and let* $\ell \in (m + \lambda_1, m + \lambda_2 + 2)$. *For any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that, if* $\|u_0(\cdot) - \phi_\alpha(|\cdot|)\|_\ell < \delta$, *then the solution* u *of* (1.1) *satisfy* $\|u(\cdot, t) - \phi_\alpha(|\cdot|)\|_\ell < \varepsilon$ *for all* $t \ge 0$. *Furthermore, if* u_0 *satisfies* $u_0(x) > \phi_\alpha(|x|)$ *for* $x \in \mathbf{R}^N$ *and* (1.19), *then the solution* u *satisfy*

$$\liminf_{t\to\infty} \|u(\cdot, t) - \phi_\alpha(|\cdot|)\|_\ell > 0. \tag{1.20}$$

Remark 1.2. (i) By (1.20) we can not expect that (1.18) holds with $\ell' = \ell$.
(ii) It was shown in [9] that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, if

$$\limsup_{|x|\to\infty} |x|^{m+\lambda_1} |u_0(x) - \phi_\alpha(|x|)| < \delta,$$

then the solution u of (1.1) satisfies $\limsup_{t\to\infty} \|u(\cdot, t) - \phi_\alpha(|\cdot|)\|_{m+\lambda_1} \le \varepsilon$. In the case $\ell > m + \lambda_1$, it is an open and interesting question, for any $\varepsilon > 0$, whether there exists $\delta > 0$ such that, if

$$\limsup_{|x|\to\infty} |x|^\ell |u_0(x) - \phi_\alpha(|x|)| < \delta,$$

then $\limsup_{t\to\infty} \|u(\cdot, t) - \phi_\alpha(|\cdot|)\|_\ell < \varepsilon$ holds.

For the attractivity property of steady states and its convergence rate in the norm $\|\cdot\|_\ell$ with $\ell = m + \lambda_1$, we refer to [9].

Proofs of the above theorems are obtained by a comparison technique for the linearized equation. Our approach is mainly based on the ideas of [3, 7], we however need some additional ingredients to obtain the convergence properties in the weighted norms.

This paper is organized as follows. In Section 2 we recall some results of [11, 3] concerning super and sub-solution methods and certain linearized problems. In Sections 3 and 4, we give the proof of Theorems 1.1 and 1.2, respectively, by deriving suitable upper and lower bound for solutions of these linearized problems. Finally, in Section 5, we prove Theorem 1.3.

2. Preliminary results

We first recall the definition of continuous weak super and sub-solutions to the following general problem

$$\begin{cases} u_t = \Delta u + f(|x|, u), & x \in \mathbf{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^N, \end{cases} \tag{2.1}$$

where $f(r,u)$ is continuous on $([0,\infty) \times \mathbf{R})$, locally Hölder continuous in $r \in [0,\infty)$ locally uniformly with respect to $u \in \mathbf{R}$, and locally Lipschitz continuous in u locally uniformly with respect to r. We say that u is a continuous weak super-solution of (2.1) for $0 \le t \le T$ if u is continuous on $\mathbf{R}^N \times [0,T]$, $u(x,0) \ge u_0(x)$ and satisfies, for any $\xi \in C^{2,1}(\mathbf{R}^N \times [0,T])$ with $\xi \ge 0$ and $\operatorname{supp} \xi(\cdot,t)$ being compact in $\mathbf{R}^N$ for all $t \in [0,T]$,

$$\int_{\mathbf{R}^N} u(x,t)\xi(x,t)\,dx\Big|_{t=0}^{t=T'} \ge \int_0^{T'}\int_{\mathbf{R}^N} u(x,t)(\xi_t + \Delta\xi)(x,t) + f(|x|,u)\xi(x,t)\,dxdt$$

for all $T' \in [0,T]$. Continuous weak subsolutions are defined in a similar way by reversing the inequalities. Consider the corresponding elliptic equation

$$\Delta u + f(|x|,u) = 0 \quad \text{in } \mathbf{R}^N. \tag{2.2}$$

We call a function u a continuous weak supersolution of (2.2) in $\mathbf{R}^N$ if u is continuous in $\mathbf{R}^N$ and satisfies, for any nonnegative function $\eta \in C_0^\infty(\mathbf{R}^N)$,

$$\int_{\mathbf{R}^N} u(x)\Delta\eta(x) + f(|x|,u)\eta(x)\,dx \le 0. \tag{2.3}$$

Continuous weak sub-solutions are defined in a similar way by reversing the inequality in (2.3).

The following results are shown by Wang [11].

Lemma 2.1. *Let $\bar{u}$ and $\underline{u}$ be bounded continuous weak super and sub-solutions of* (2.1), *respectively. Then* (2.1) *has a unique classical solution u with $\underline{u} \le u \le \bar{u}$ in $\mathbf{R}^N \times (0,\infty)$.*

We say that $\bar{u}$ is a classical supersolution of (2.1) if $\bar{u}$ satisfies

$$\bar{u}_t \geq \Delta\bar{u} + f(|x|, \bar{u}) \quad \text{for } x \in \mathbf{R}^N,\ t > 0$$

and $\bar{u}(x, 0) \geq u_0(x)$ for $x \in \mathbf{R}^N$. Classical subsolutions are defined in a similar way by reversing the inequalities.

By the similar argument as in [11, Proposition 3.8], we obtain the following result.

LEMMA 2.2. *Let $r(t) \in (0, \infty)$ be a continuous function for $t \geq 0$, and define $D_1 = \{(x, t) \in \mathbf{R}^N \times [0, \infty) : |x| < r(t), t \geq 0\}$ and $D_2 = \{(x, t) \in \mathbf{R}^N \times [0, \infty) : |x| > r(t), t \geq 0\}$.*

(i) *Suppose that $u_1(r, t)$ and $u_2(r, t)$, with $r = |x|$, are classical supersolutions of (2.1) in D_1 and D_2, respectively. Assume that $u_1 = u_2$ and $\partial u_1/\partial r \geq \partial u_2/\partial r$ at $(r, t) = (r(t), t)$ for all $t \geq 0$. For each $t \geq 0$, put*

$$\bar{u}(r, t) = \begin{cases} u_1(r, t), & \text{for } 0 \leq r \leq r(t), \\ u_2(r, t), & \text{for } r > r(t). \end{cases}$$

Then $\bar{u}(|x|, t)$ is a continuous weak supersolution to (2.1) in $\mathbf{R}^N \times [0, \infty)$.

(ii) *Suppose that $v_1(r, t)$ and $v_2(r, t)$, with $r = |x|$, are classical sub-solutions of (2.1) in D_1 and D_2, respectively. Assume that $v_1 = v_2$ and $\partial v_1/\partial r \leq \partial v_2/\partial r$ at $(r, t) = (r(t), t)$ for all $t \geq 0$. For each $t \geq 0$, put*

$$\underline{u}(r, t) = \begin{cases} v_1(r, t), & \text{for } 0 \leq r \leq r(t), \\ v_2(r, t), & \text{for } r > r(t). \end{cases}$$

Then $\underline{u}(|x|, t)$ is a continuous weak subsolution to (2.1) in $\mathbf{R}^N \times (0, \infty)$.

Proof. Since the proof of (ii) is similar to (i), we will show (i) only. Take any $\xi \in C^{2,1}(\mathbf{R}^N \times [0, T])$ with $\xi \geq 0$ and supp $\xi(\cdot, t)$ being compact in $\mathbf{R}^N$ for all $t \in [0, T]$. For each $i = 1, 2$, $u_i = u_i(x, t)$ satisfies

$$(u_i)_t \geq \Delta u_i + f(|x|, u_i), \quad (x, t) \in D_i. \tag{2.4$_i$}$$

Fix $t \in (0, T)$ arbitrarily. Multiplying $(2.4)_1$ by $\xi(x, t)$, and integrating it on $\{x : |x| < r(r)\}$, we obtain

$$\int_{|x|<r(t)} (u_1)_t \xi\, dx \geq \int_{|x|<r(t)} (\Delta u_1)\xi\, dx + \int_{|x|<r(t)} f(|x|, u_1)\xi\, dx.$$

By using of the integration by parts, we have

$$\int_{|x|<r(t)} (\Delta u_1)\xi\, dx = \int_{|x|=r(t)} \left(\xi \frac{\partial u_1}{\partial n} - u_1 \frac{\partial \xi}{\partial n}\right) dS + \int_{|x|<r(t)} u_1(\Delta\xi)\, dx,$$

where n is the outward unit normal vector to $|x| = r(t)$ and dS denotes the surface measure on $|x| = r(t)$. Thus we obtain

$$\int_{|x|<r(t)} (u_1)_t \xi \, dx \geq \int_{|x|=r(t)} \left(\xi \frac{\partial u_1}{\partial n} - u_1 \frac{\partial \xi}{\partial n} \right) dS + \int_{|x|<r(t)} (u_1(\Delta \xi) + f(|x|, u_1)\xi) \, dx. \tag{2.5}$$

Multiplying $(2.4)_2$ by ξ, and integrating by parts on $\{x : |x| > r(t)\}$, we obtain

$$\int_{|x|>r(t)} (u_2)_t \xi \, dx \geq \int_{|x|=r(t)} \left(-\xi \frac{\partial u_2}{\partial n} + u_2 \frac{\partial \xi}{\partial n} \right) dS + \int_{|x|>r(t)} (u_2(\Delta \xi) + f(|x|, u_2)\xi) \, dx. \tag{2.6}$$

Adding (2.5) and (2.6), we obtain

$$\int_{\mathbf{R}^N} (\bar{u})_t \xi \, dx \geq \int_{|x|=r(t)} \xi \left(\frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} \right) dS + \int_{\mathbf{R}^N} (\bar{u}(\Delta \xi) + f(|x|, \bar{u})\xi) \, dx.$$

Since $\partial u_1/\partial r \geq \partial u_2/\partial r$ on $r = r(t)$, we have $\partial u_1/\partial n - \partial u_2/\partial n \geq 0$ on $|x| = r(t)$. Thus we obtain

$$\int_{\mathbf{R}^N} (\bar{u})_t \xi \, dx \geq \int_{\mathbf{R}^N} (\bar{u}(\Delta \xi) + f(|x|, \bar{u})\xi) \, dx.$$

Integrating by parts on $t \in [0, T']$, we obtain

$$\int_{\mathbf{R}^N} \bar{u}(|x|, t)\xi(x, t) \, dx \Big|_{t=0}^{t=T'} \geq \int_0^{T'} \int_{\mathbf{R}^N} (\bar{u}(|x|, t)(\xi_t + \Delta \xi)(x, t) + f(|x|, \bar{u})\xi(x, t)) \, dxdt.$$

Since u_1 and u_2 are classical supersolution of (2.1), we have $\bar{u}(|x|, t) \geq u_0(x)$ for $x \in \mathbf{R}^N$. Thus $\bar{u}$ is a continuous weak supersolution of (2.1). □

We next summarize previous results of [3] on the problem for a linearized equation of (1.1) at a steady state ϕ_α. For $\alpha > 0$ we define the linear operator P_α by

$$P_\alpha U = U_{rr} + \frac{N-1}{r} U_r + p\phi_\alpha(r)^{p-1} U$$

and consider the solutions $U = U(r, t)$ of the problem

$$\begin{cases} U_t = P_\alpha U & \text{for } r > 0, \ t > 0, \\ U_r(0, t) = 0 & \text{for } t > 0, \\ U(r, 0) = U_0(r) & \text{for } r \geq 0, \end{cases} \tag{2.7}$$

where $U_0(r)$ is a continuous function decaying to zero as $r \to \infty$. By the maximum principle, we see that $U(\cdot, t) > 0$ for all $t > 0$ if $U_0 \geq 0$ and $U_0 \not\equiv 0$.

Let $\psi(r)$ satisfy

$$\begin{cases} P_\alpha \psi = 0 & \text{for } r > 0, \\ \psi(0) = 1 & \text{and} \quad \psi_r(0) = 0, \end{cases} \tag{2.8}$$

and let Ψ denote the solution to

$$\begin{cases} P_\alpha \Psi = \psi & \text{for } r > 0, \\ \Psi(0) = 0 & \text{and} \quad \Psi_r(0) = 0. \end{cases} \tag{2.9}$$

We state some useful properties of ψ, which were proved in [3, Lemma 2.3].

LEMMA 2.3. *Let $p > p_c$ and $\alpha > 0$. Then the solution ψ of* (2.8) *is given by $\psi(r) = \partial\phi_\alpha(r)/\partial\alpha$ for $r \geq 0$, $\psi(r)$ is positive for all $r \geq 0$, and the solution $\Psi(r)$ of* (2.9) *is positive for all $r > 0$. Moreover, $\psi(r)$ is decreasing for $r > 0$ and satisfies*

$$\lim_{r\to\infty} r^{m+\lambda_1}\psi(r) = c_\alpha \tag{2.10}$$

for some positive constant c_α.

We recall comparison results in [3, Lemmas 2.1 and 2.2]. Let u and $\tilde{u}$ denote solutions of (1.1) with initial data u_0, $\tilde{u}_0$, respectively.

LEMMA 2.4. *Let $p \geq p_c$ and $\alpha > 0$.*

(i) *Assume that u_0 and $\tilde{u}_0$ satisfy*

$$-\phi_\alpha(x) \leq u_0(x), \tilde{u}_0(x) \leq \phi_\alpha(x) \quad \text{for } x \in \mathbf{R}^N.$$

If $|u_0(x) - \tilde{u}_0(x)| \leq U_0(|x|)$ for $x \in \mathbf{R}^N$, then the solutions U of (2.7) *and u, $\tilde{u}$ of* (1.1) *satisfy*

$$|u(x,t) - \tilde{u}(x,t)| \leq U(|x|,t) \quad \text{for all } x \in \mathbf{R}^N,\ t > 0. \tag{2.11}$$

(ii) *Assume that u_0 and $\tilde{u}_0$ satisfy*

$$\phi_\alpha(|x|) \leq \tilde{u}_0(x) \leq u_0(x) \leq \phi_\infty(|x|) \quad \text{for } x \in \mathbf{R}^N\backslash\{0\}.$$

If $u_0(x) - \tilde{u}_0(x) \geq U_0(|x|) \geq 0$ for $x \in \mathbf{R}^N$, then the solutions U of (2.7) *and u, $\tilde{u}$ of* (1.1) *satisfy*

$$u(x,t) - \tilde{u}(x,t) \geq U(|x|,t) \geq 0 \quad \text{for all } x \in \mathbf{R}^N,\ t > 0. \tag{2.12}$$

3. Proof of Theorem 1.1

In the proof of Theorem 1.1, we construct suitable super- and subsolutions of (2.7). We use the ideas presented in [3, 7], and this section is similar to Section 3 in [7]. But in order to obtain the convergence rate in the weighted norms, we need some additional ideas.

We first construct a supersolution of (2.7). We recall the result by [2, Lemma 3.1].

Lemma 3.1. *If $m+\lambda_1 < \ell < m+\lambda_2+2$, then there exists a positive solution F of*

$$F_{\eta\eta} + \frac{N-1}{\eta}F_\eta + \frac{\eta}{2}F_\eta + \frac{\ell}{2}F + \frac{pL^{p-1}}{\eta^2}F = 0 \quad \textit{for } \eta > 0$$

satisfying

$$\lim_{\eta\to 0} \eta^{m+\lambda_1}F(\eta) = a_0 > 0 \tag{3.1}$$

and

$$0 < \liminf_{\eta\to\infty} \eta^{\ell}F(\eta) \le \limsup_{\eta\to\infty} \eta^{\ell}F(\eta) < \infty. \tag{3.2}$$

We construct an outer supersolution in the same manner as [7, Lemma 3.1].

Lemma 3.2. *Let $m+\lambda_1 < \ell < m+\lambda_2+2$. Put*

$$U_{\rm out}(r,t) = (t+\tau)^{-\ell/2}F(\eta) \quad \textit{with } \eta = \frac{r}{(t+\tau)^{1/2}}, \tag{3.3}$$

where $\tau > 0$ is a constant and F is the positive solution obtained in Lemma 3.1. Then $U_{\rm out}$ satisfies $(U_{\rm out})_t \ge P_\alpha U_{\rm out}$ for all $t > 0$ and $r > 0$.

Recall that ψ and Ψ denote the solutions of (2.8) and (2.9), respectively.

Lemma 3.3. *Let $m+\lambda_1 < \ell < m+\lambda_2+2$, and set*

$$U_{\rm in}(r,t) = (t+\tau)^{-q}\psi(r) - q(t+\tau)^{-q-1}\Psi(r), \tag{3.4}$$

where $q = (\ell - m - \lambda_1)/2$ and $\tau > 0$ is a constant. Define $U_{\rm out}$ by (3.3). Then there are positive constants B, C_0, τ_0 and R_0 with $R_0 < B\tau_0^{1/2}$ such that, for $\tau \ge \tau_0$, the following inequalities hold:

(i) $(U_{\rm in})_t \ge P_\alpha U_{\rm in}$ *for all $t > 0$ and $r > 0$.*
(ii) $U_{\rm in}(r,t) > 0$ *for all $t \ge 0$ and $r \in [0, B(t+\tau)^{1/2}]$.*
(iii) $U_{\rm in}(r,t) > C_0 U_{\rm out}(r,t)$ *at $r = B(t+\tau)^{1/2}$ for all $t \ge 0$.*
(iv) $U_{\rm in}(r,t) < C_0 U_{\rm out}(r,t)$ *for all $t \ge 0$ and $r \in [0, R_0]$.*

Proof. For the proof of (i)–(iii), see [3, Lemma 3.2] and [7, Lemma 3.2]. We will show (iv). Put

$$C_B = \inf\{\eta^{m+\lambda_1}F(\eta) : 0 < \eta \le B\}.$$

From (3.1) we have $C_B > 0$. For all $t \ge 0$ and $r \in [0, B(t+\tau)^{1/2}]$, we have

$$\begin{aligned} C_0 U_{\rm out}(r,t) &= C_0(t+\tau)^{-(\ell-m-\lambda_1)/2} r^{-m-\lambda_1}\eta^{m+\lambda_1}F(\eta) \\ &\ge C_0 C_B (t+\tau)^{-(\ell-m-\lambda_1)/2} r^{-m-\lambda_1}. \end{aligned}$$

Choose $R_0 \in (0, B\tau^{1/2}]$ such that $C_0 C_B R_0^{-m-\lambda_1} > \psi(0)$. Then, for all $t \geq 0$ and $r \in [0, R_0]$, we have

$$C_0 U_{\text{out}}(r,t) > (t+\tau)^{-(\ell-m-\lambda_1)/2}\psi(0).$$

Since $\psi(r)$ is decreasing and $\Psi(r)$ is positive for $r > 0$ by Lemma 2.3, we obtain

$$(t+\tau)^{-(\ell-m-\lambda_1)/2}\psi(0) \geq (t+\tau)^{-(\ell-m-\lambda_1)/2}\psi(r) \geq U_{\text{in}}(r,t)$$

for $r \geq 0$. Thus (iv) holds. □

Assume that B, C_0, τ_0 and R_0 are constants given in Lemma 3.3. Let $\tau \geq \tau_0$, and define U_{out} and U_{in} by (3.3) and (3.4), respectively. Put

$$r^*(t) = \sup\{r > 0 : U_{\text{in}}(\rho,t) < C_0 U_{\text{out}}(\rho,t) \text{ for } \rho \in [0,r)\}.$$

By Lemma 3.3, the function $r^*(t)$ is well-defined and satisfies

$$r^*(t) \in (R_0, B(t+\tau)^{1/2}) \quad \text{for all } t \geq 0. \tag{3.5}$$

For each $t \geq 0$, define

$$U^+(r,t) = \begin{cases} U_{\text{in}}(r,t) & \text{for } 0 \leq r \leq r^*(t), \\ C_0 U_{\text{out}}(r,t) & \text{for } r > r^*(t). \end{cases} \tag{3.6}$$

We will show the following results.

Lemma 3.4. *Let $\ell' \in [0, \ell)$. Then the following* (i) *and* (ii) *hold.*

(i) *The function U^+ satisfies*

$$\|U^+(|\cdot|,t)\|_{\ell'} = O(t^{-\nu}) \quad \text{as } t \to \infty, \tag{3.7}$$

where ν is the constant defined by (1.7).

(ii) *For any constants $c > 0$ and $\mu \geq 1/2$, the function U^+ satisfies*

$$\limsup_{t\to\infty} t^{(\ell-\ell')\mu}\left(\sup_{|x|\geq ct^\mu}(1+|x|)^{\ell'}|U^+(|x|,t)|\right) < \infty. \tag{3.8}$$

Proof. (i) First, we consider the case $\ell' \in (0, m+\lambda_1]$. We will show that

$$\sup_{r\geq r^*(t)}(1+r)^{\ell'}U^+(r,t) = O(t^{-(\ell-m-\lambda_1)/2}) \quad \text{as } t \to \infty \tag{3.9}$$

and

$$\sup_{0<r\leq r^*(t)}(1+r)^{\ell'}U^+(r,t) = O(t^{-(\ell-m-\lambda_1)/2}) \quad \text{as } t \to \infty. \tag{3.10}$$

For each fixed $t > 0$, from (3.5) we have,

$$\sup_{r\geq r^*(t)}(1+r)^{\ell'}U^+(r,t) \leq \sup_{r\geq R_0}(1+r)^{m+\lambda_1}C_0 U_{\text{out}}(r,t) \leq \sup_{r\geq R_0} Cr^{m+\lambda_1}U_{\text{out}}(r,t)$$

with some constant $C > 0$. Observe that

$$r^{m+\lambda_1} U_{\text{out}}(r,t) = (t+\tau)^{-(\ell-m-\lambda_1)/2} \eta^{m+\lambda_1} F(\eta) \quad \text{with } \eta = \frac{r}{(t+\tau)^{1/2}}.$$

It follows from (3.1) and (3.2) that $\sup_{\eta>0} \eta^{m+\lambda_1} F(\eta) < \infty$. Then we obtain (3.9). For each $t > 0$, we have,

$$\sup_{0 \le r \le r^*(t)} (1+r)^{\ell'} U^+(r,t) \le \sup_{r \ge 0} (1+r)^{\ell'} U_{\text{in}}(r,t)$$
$$\le \sup_{r \ge 0} (t+\tau)^{-(\ell-m-\lambda_1)/2} (1+r)^{\ell'} \psi(r).$$

Since $(1+r)^{\ell'} \psi(r)$ is bounded for $r \ge 0$ by (2.10), we obtain (3.10). Combining (3.9) and (3.10), we obtain (3.7) with $v = (\ell - m - \lambda_1)/2$.

Next, we consider the case $\ell' \in (m + \lambda_1, \ell)$. For each fixed $t > 0$, we have

$$\sup_{r \ge r^*(t)} (1+r)^{\ell'} U^+(r,t) \le \sup_{r \ge R_0} (1+r)^{\ell'} C_0 U_{\text{out}}(r,t) \le \sup_{r \ge R_0} C r^{\ell'} U_{\text{out}}(r,t)$$

with some constant $C > 0$. Observe that

$$r^{\ell'} U_{\text{out}}(r,t) = (t+\tau)^{-(\ell-\ell')/2} \eta^{\ell'} F(\eta) \quad \text{with } \eta = \frac{r}{(t+\tau)^{1/2}}.$$

It follows from (3.2) that $\sup_{\eta>0} \eta^{\ell'} F(\eta) < \infty$. Then

(3.11) $$\sup_{r \ge r^*(t)} (1+r)^{\ell'} U^+(r,t) = O(t^{-(\ell-\ell')/2}) \quad \text{as } t \to \infty.$$

On the other hand, for each $t > 0$, we have

(3.12) $$\sup_{0 \le r \le r^*(t)} (1+r)^{\ell'} U^+(r,t) \le \sup_{0 \le r \le B(t+\tau)^{1/2}} (1+r)^{\ell'} U_{\text{in}}(r,t)$$
$$\le \sup_{0 \le r \le B(t+\tau)^{1/2}} (t+\tau)^{-(\ell-m-\lambda_1)/2} (1+r)^{\ell'} \psi(r).$$

By virtue of (2.10), there exist a positive constant C satisfying

$$(1+r)^{\ell'} \psi(r) \le C(1 + r^{\ell'-m-\lambda_1}) \quad \text{for } r \ge 0.$$

Then it follows that

$$\sup_{0 \le r \le B(t+\tau)^{1/2}} (1+r)^{\ell'} \psi(r) \le C \sup_{0 \le r \le B(t+\tau)^{1/2}} (1 + r^{\ell'-m-\lambda_1}) = O((t+\tau)^{(\ell'-m-\lambda_1)/2})$$

as $t \to \infty$. Thus, from (3.12), we obtain

(3.13) $$\sup_{0 < r \le r^*(t)} (1+r)^{\ell'} U^+(r,t) = O(t^{-(\ell-\ell')/2}) \quad \text{as } t \to \infty.$$

Combining (3.11) and (3.13), we obtain (3.7) with $v = (\ell - \ell')/2$.

(ii) First we show that

$$\sup_{r \geq ct^\mu} (1+r)^{\ell'} U_{\mathrm{out}}(r,t) = O(t^{-(\ell-\ell')\mu}) \quad \text{as } t \to \infty \tag{3.14}$$

for any constants $c > 0$ and $\mu \geq 1/2$. For $r \geq 1$, we have

$$(1+r)^{\ell'} U_{\mathrm{out}}(r,t) \leq C r^{\ell'} (t+\tau)^{-\ell/2} F(\eta) = C r^{-(\ell-\ell')} \eta^{\ell} F(\eta)$$

with some constant $C > 1$. Note here that $r^{-(\ell-\ell')} \leq c^{-(\ell-\ell')} t^{-(\ell-\ell')\mu}$ for $r \geq ct^\mu$. Thus, from (3.2), we obtain (3.14).

In the case $\mu > 1/2$, from (3.5), there exists $t_0 \geq 0$ such that $ct^\mu > r^*(t)$ for $t \geq t_0$. Thus, from (3.14), we obtain

$$\sup_{r \geq ct^\mu} (1+r)^{\ell'} U^+(r,t) = \sup_{r \geq ct^\mu} (1+r)^{\ell'} C_0 U_{\mathrm{out}}(r,t) = O(t^{-(\ell-\ell')\mu}) \quad \text{as } t \to \infty.$$

In the case $\mu = 1/2$, we observe that

$$\begin{aligned} &\sup_{r \geq ct^\mu} (1+r)^{\ell'} U^+(r,t) \\ &\qquad = \max\left\{ \sup_{ct^\mu \leq r \leq r^*(t)} (1+r)^{\ell'} U_{\mathrm{in}}(r,t),\ \sup_{r \geq r^*(t)} (1+r)^{\ell'} C_0 U_{\mathrm{out}}(r,t) \right\}, \end{aligned} \tag{3.15}$$

if $ct^\mu < r^*(t)$, and that

$$\sup_{r \geq ct^\mu} (1+r)^{\ell'} U^+(r,t) \leq \sup_{r \geq r^*(t)} (1+r)^{\ell'} C_0 U_{\mathrm{out}}(r,t),$$

if $ct^\mu \geq r^*(t)$. Thus, in the case $\mu = 1/2$, we may assume that $ct^\mu < r^*(t)$ for $t \geq t_0$ with some $t_0 \geq 0$. From $ct^\mu < r^*(t)$, we have

$$\sup_{r \geq r^*(t)} (1+r)^{\ell'} U_{\mathrm{out}}(r,t) \leq \sup_{r \geq ct^\mu} (1+r)^{\ell'} U_{\mathrm{out}}(r,t) \quad \text{for } t \geq t_0.$$

Thus, from (3.14), we obtain

$$\sup_{r \geq r^*(t)} (1+r)^{\ell'} U_{\mathrm{out}}(r,t) = O(t^{-(\ell-\ell')\mu}) \quad \text{as } t \to \infty. \tag{3.16}$$

From (3.5) it follows that

$$\sup_{ct^\mu \leq r \leq r^*(t)} (1+r)^{\ell'} U_{\mathrm{in}}(r,t) \leq \sup_{ct^\mu \leq r \leq B(t+\tau)^{1/2}} (1+r)^{\ell'} (t+\tau)^{-(\ell-m-\lambda_1)/2} \psi(r) \tag{3.17}$$

for $t \geq t_0$. Since $\psi(r)$ is decreasing and satisfies (2.10) by Lemma 2.3, we have $\psi(r) \leq \psi(ct^\mu)$ for $r \geq ct^\mu$ and

$$\psi(ct^\mu) = O(t^{-(m+\lambda_1)\mu}) \quad \text{as } t \to \infty.$$

Recall that $\mu = 1/2$. Then it follows that

$$\sup_{ct^\mu \leq r \leq B(t+\tau)^{1/2}} (1+r)^{\ell'} (t+\tau)^{-(\ell-m-\lambda_1)/2} \psi(r) = O(t^{-(\ell-\ell')\mu}) \quad \text{as } t \to \infty.$$

Thus, from (3.17), we obtain

$$\sup_{ct^{\mu}\le r\le r^*(t)} (1+r)^{\ell'} U_{\rm in}(r,t) = O(t^{-(\ell-\ell')\mu}) \quad \text{as } t\to\infty. \tag{3.18}$$

From (3.15) with (3.16) and (3.18), we obtain

$$\sup_{r\ge ct^{\mu}} (1+r)^{\ell'} U^{+}(r,t) = O(t^{-(\ell-\ell')\mu}) \quad \text{as } t\to\infty.$$

Thus we obtain (3.8). □

Proof of Theorem 1.1. Put $U_0(r) = \max_{r=|x|} |u_0(x) - \tilde{u}_0(x)|$ for $r\ge 0$, and let U be a solution of (2.7). Then, by Lemma 2.4 (i), we obtain (2.11).

Define U^+ by (3.6). By the definition of $r^*(t)$, we see that $U_{\rm in} = C_0 U_{\rm out}$ and $\partial U_{\rm in}/\partial r \ge \partial(C_0 U_{\rm out})/\partial r$ at $(r,t) = (r^*(t),t)$ for all $t\ge 0$. Thus, if there exists a constant $C>0$ such that $CU^+(|x|,0)\ge U_0(x)$ for $x\in\mathbf{R}^N$, then CU^+ is a continuous weak supersolution to (2.7) by Lemma 2.2 (i).

Observe that $r^{\ell} U_{\rm out}(r,0) = \eta^{\ell} F(\eta)$ with $\eta = r/\tau^{1/2}$. Then, by (3.2), we see that

$$\liminf_{r\to\infty} r^{\ell} U^{+}(r,0) = C_0 \liminf_{\eta\to\infty} \eta^{\ell} F(\eta) > 0. \tag{3.19}$$

From (1.5) the function U_0 satisfies $\limsup_{r\to\infty} r^{\ell} U_0(r) < \infty$. Since $U^+(r,0)>0$ for $r\ge 0$, there exists a constant $C_1>0$ such that $U_0(r)\le C_1 U^+(r,0)$ for $r\ge 0$. Thus $C_1U^+(r,t)$ is a continuous weak supersolution to (2.7). Then, by applying Lemma 2.1 with $f(r,u) = p\phi_{\alpha}^{p-1}(r)u$, we obtain $0\le U(|x|,t)\le C_1U^+(|x|,t)$ for $x\in\mathbf{R}$ and $t>0$. Thus, from (2.11), we obtain

$$|u(x,t)-\tilde{u}(x,t)| \le U(|x|,t) \le C_1 U^+(|x|,t) \quad \text{for all } x\in\mathbf{R}^N,\ t\ge 0. \tag{3.20}$$

By Lemma 3.4 (i) and (ii), we obtain (1.6) and (1.8), respectively. □

4. Proof of Theorem 1.2

In the proof of Theorem 1.2, we construct a subsolution of (2.7) by connecting inner and outer solutions. We give an inner solution $U_{\rm in}$ in the same way as [3, Lemma 4.1].

LEMMA 4.1. *Let $\ell > m+\lambda_1$, and put*

$$U_{\rm in}(r,t) = (t+\tau)^{-(\ell-m-\lambda_1)/2}\psi(r). \tag{4.1}$$

Then $U_{\rm in}$ satisfies $(U_{\rm in})_t \le P_{\alpha} U_{\rm in}$ for all $t>0$ and $r>0$.

We construct an outer solution by following the idea presented in [3, Lemma 4.2].

LEMMA 4.2. *Let* $0 < \ell < N-2$, *and define* $U_{\rm in}$ *by* (4.1). *Assume that* k *is a constant satisfying*

$$0 < k < \min\left\{1, \frac{N-2-\ell}{2}\right\}. \tag{4.2}$$

Put

$$U_{\rm out}(r,t) = \max\{0, r^{-\ell} - b^{2k}(t+\tau)^k r^{-\ell-2k}\} \quad \text{with } b = \left(\frac{\ell(N-2-\ell)}{k}\right)^{1/2}. \tag{4.3}$$

Then the following (i) *and* (ii) *hold.*

(i) *In the range* $r > b(t+\tau)^{1/2}$, $U_{\rm out}(r,t)$ *is positive and satisfies* $(U_{\rm out})_t \le P_\alpha U_{\rm out}$.

(ii) *Let* $B > b$. *Then there exist positive constants* C_0 *and* τ_0 *such that, for* $\tau \ge \tau_0$, $U_{\rm in}(r,t) < C_0 U_{\rm out}(r,t)$ *at* $r = B(t+\tau)^{1/2}$ *for all* $t \ge 0$.

Remark 4.1. We see that $m + \lambda_2 + 2 < N-2$ if $p \ge p_c$. In fact, we have

$$m + \lambda_2 + 2 = \frac{N+2+\sqrt{(N-2-m)^2 - 8(N-2-m)}}{2}.$$

Recall that $N \ge 11$, $m + \lambda_1 \le (N-2)/2$ and $\lambda_1 > 0$. Then we have $m < (N-2)/2$ and $N-2-m > (N-2)/2 > 4$. Since the function $x^2 - 8x$ is increasing for $x > 4$, it follows that

$$(N-2-m)^2 - 8(N-2-m) < (N-2)^2 - 8(N-2) < (N-6)^2.$$

Thus we obtain $m + \lambda_2 + 2 < N-2$.

Proof. (i) In the range $r > b(t+\tau)^{1/2}$, we compute

$$\begin{aligned}(U_{\rm out})_t - P_\alpha U_{\rm out} &\le (U_{\rm out})_t - (U_{\rm out})_{rr} - \frac{N-1}{r}(U_{\rm out})_r \\ &= -kb^{2k}(t+\tau)^{k-1}r^{-\ell-2k} + \ell(N-2-\ell)r^{-\ell-2} \\ &\quad - b^{2k}(\ell+2k)(N-2-\ell-2k)(t+\tau)^k r^{-\ell-2-2k}.\end{aligned}$$

From (4.2) and $r/(t+\tau)^{1/2} > b$, it follows that

$$\begin{aligned}(U_{\rm out})_t - P_\alpha U_{\rm out} &\le -kb^{2k}(t+\tau)^{k-1}r^{-\ell-2k} + \ell(N-2-\ell)r^{-\ell-2} \\ &= (-kb^{2k}(r/(t+\tau)^{1/2})^{2-2k} + \ell(N-2-\ell))r^{-\ell-2} \\ &\le (-kb^2 + \ell(N-2-\ell))r^{-\ell-2} = 0.\end{aligned}$$

Thus we obtain $(U_{\rm out})_t \le P_\alpha U_{\rm out}$.

(ii) At $r = B(t+\tau)^{1/2}$, we have

$$U_{\rm out}(r,t) = B^{-\ell-2k}(B^{2k} - b^{2k})(t+\tau)^{-\ell/2} > 0 \tag{4.4}$$

and

$$U_{\text{in}}(r,t) = (t+\tau)^{-(\ell-m-\lambda_1)/2}\psi(B(t+\tau)^{1/2})$$

for all $t \geq 0$. From (2.10) there exist $r_0 > 0$ and $c > 0$ such that

$$\psi(r) \leq cr^{-m-\lambda_1} \quad \text{for } r \geq r_0.$$

Then there exists $\tau_0 > 0$ such that, for $\tau \geq \tau_0$, we have $B\tau^{1/2} > r_0$ and

$$U_{\text{in}}(r,t) \leq cB^{-m-\lambda_1}(t+\tau)^{-\ell/2} \quad \text{at } r = B(t+\tau)^{1/2} \quad \text{for all } t \geq 0. \tag{4.5}$$

From (4.4) and (4.5), there exists $C_0 > 0$ such that $U_{\text{in}}(r,t) < C_0 U_{\text{out}}(r,t)$ at $r = B(t+\tau)^{1/2}$ for all $t \geq 0$. □

Assume that B, C_0 and τ_0 are constants given in Lemma 4.2. Let $\tau \geq \tau_0$, and define U_{in} and U_{out} by (4.1) and (4.3), respectively. Put

$$r^*(t) = \sup\{r > 0 : U_{\text{in}}(\rho,t) < C_0 U_{\text{out}}(\rho,t) \text{ for } \rho \in [0,r)\}.$$

By Lemma 4.2, the function $r^*(t)$ is well-defined and satisfies

$$r^*(t) \in (b(t+\tau)^{1/2}, B(t+\tau)^{1/2}) \quad \text{for all } t \geq 0. \tag{4.6}$$

For each $t \geq 0$, define

$$U^-(r,t) = \begin{cases} U_{\text{in}}(r,t) & \text{for } 0 \leq r \leq r^*(t), \\ C_0 U_{\text{out}}(r,t) & \text{for } r > r^*(t). \end{cases} \tag{4.7}$$

We will show the following results.

Lemma 4.3. *Let $\ell' \in [0,\ell)$. Then the following* (i)–(iii) *hold.*

(i) *There exists some constant $c > 0$ such that*

$$\|U^-(|\cdot|,t)\|_{\ell'} \geq ct^{-\nu} \quad \textit{for all } t > 0, \tag{4.8}$$

where ν is the constant defined by (1.7).

(ii) *For any constants $c > 0$ and $\mu \geq 1/2$, the function U^- satisfies*

$$\liminf_{t\to\infty} t^{(\ell-\ell')\mu}\left(\sup_{r \geq ct^\mu}(1+r)^{\ell'}U^-(r,t)\right) > 0. \tag{4.9}$$

(iii) *For any constant $c > 0$, the function U^- satisfies*

$$\liminf_{t\to\infty} t^{(\ell-m-\lambda_1)/2}\left(\inf_{r \leq ct^{1/2}}(1+r)^{m+\lambda_1}U^-(r,t)\right) > 0. \tag{4.10}$$

Proof. (i) We see that

$$\sup_{0\leq r\leq r^*(t)}(1+r)^{\ell'}U^-(r,t) = \sup_{0\leq r\leq r^*(t)}(1+r)^{\ell'}U_{\text{in}}(r,t) \geq (t+\tau)^{-(\ell-m-\lambda_1)/2}\psi(0).$$

From (4.6) we have, for each $t > 0$,

$$\sup_{r \geq r^*(t)} (1+r)^{\ell'} U^-(r,t) \geq C_0 \sup_{r \geq B(t+\tau)^{1/2}} r^{\ell'} U_{\text{out}}(r,t).$$

Recall that (4.4) holds at $r = B(t+\tau)^{1/2}$. Then it follows that

$$\sup_{r \geq r^*(t)} (1+r)^{\ell'} U^-(r,t) \geq C_0 B^{\ell'-\ell-2k}(B^{2k} - b^{2k})(t+\tau)^{-(\ell-\ell')/2}.$$

Thus, if $\ell' \in [0, m+\lambda_1]$, we have

$$\sup_{r>0} (1+r)^{\ell'} U^-(r,t) \geq \sup_{0<r\leq r^*(t)} (1+r)^{\ell'} U^-(r,t) \geq (t+\tau)^{-(\ell-m-\lambda_1)/2}\psi(0),$$

and, if $\ell' \in (m+\lambda_1, \ell)$, we have

$$\sup_{r>0} (1+r)^{\ell'} U^-(r,t) \geq \sup_{r\geq r^*(t)} (1+r)^{\ell'} U^-(r,t) \geq c(t+\tau)^{-(\ell-\ell')/2}$$

with $c = C_0 B^{\ell'-\ell-2k}(B^{2k} - b^{2k})$. We therefore obtain (4.8) with the constant v defined by (1.7).

(ii) Let $\mu \geq 1/2$. From (4.6), we see that

$$\sup_{r \geq ct^\mu} (1+r)^{\ell'} U^-(r,t) \geq \sup_{r \geq B(t+\tau)^{1/2}} (1+r)^{\ell'} C_0 U_{\text{out}}(r,t),$$

if $ct^\mu < B(t+\tau)^{1/2}$, and that

$$\sup_{r \geq ct^\mu} (1+r)^{\ell'} U^-(r,t) = \sup_{r \geq ct^\mu} (1+r)^{\ell'} C_0 U_{\text{out}}(r,t), \tag{4.11}$$

if $ct^\mu \geq B(t+\tau)^{1/2}$. Thus, we may assume that there exists $t_0 \geq 0$ such that $ct^\mu \geq B(t+\tau)^{1/2}$ for $t \geq t_0$. Observe that U_{out} can be written by

$$U_{\text{out}}(r,t) = r^{-\ell}\left(1 - b^{2k}\left(\frac{t+\tau}{r^2}\right)^k\right) \tag{4.12}$$

for $r > b(t+\tau)^{1/2}$. Then, for $r \geq B(t+\tau)^{1/2}$, we have $U_{\text{out}}(r,t) \geq c_1 r^{-\ell}$ with $c_1 = (1-(b/B))^{2k} > 0$. Thus we obtain

$$\sup_{r \geq ct^\mu} (1+r)^{\ell'} U_{\text{out}}(r,t) \geq \sup_{r \geq ct^\mu} c_1 r^{-(\ell-\ell')} = c_1 c^{-(\ell-\ell')} t^{-(\ell-\ell')\mu}.$$

From (4.11) we obtain (4.9).

(iii) Put

$$b^* = \liminf_{t\to\infty} r^*(t)(t+\tau)^{-1/2}. \tag{4.13}$$

Then $b^* \geq b$ by (4.6). We will verify that $b^* > b$. Assume to the contrary that $b^* = b$. Then there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ and

$$r^*(t_n)(t_n + \tau)^{-1/2} \to b \quad \text{as } n \to \infty. \tag{4.14}$$

By the definition of $r^*(t)$, we have $U_{\text{in}}(r^*(t), t) = C_0 U_{\text{out}}(r^*(t), t)$ for all $t \geq 0$. We see that

$$r^*(t_n)^\ell U_{\text{in}}(r^*(t_n), t_n) = \left(\frac{r^*(t_n)}{(t_n + \tau)^{1/2}}\right)^{\ell - m - \lambda_1} r^*(t_n)^{m+\lambda_1} \psi(r^*(t_n)).$$

Then, it follows from (2.10) and (4.14) that

$$r^*(t_n)^\ell U_{\text{in}}(r^*(t_n), t_n) \to b^{\ell - m - \lambda_1} c_\alpha > 0 \quad \text{as } n \to \infty.$$

On the other hand, it follows from (4.12) and (4.14) that

$$r^*(t_n)^\ell U_{\text{out}}(r^*(t_n), t_n) = 1 - b^{2k} \left(\frac{(t_n + \tau)^{1/2}}{r^*(t_n)}\right)^{2k} \to 0 \quad \text{as } n \to \infty.$$

This is a contradiction. Thus we obtain $b^* > b$.

We observe that

$$\begin{aligned} &\inf_{r \leq ct^{1/2}} (1 + r)^{m+\lambda_1} U^-(r, t) \\ &\quad = \min\left\{\inf_{r \leq r^*(t)} (1 + r)^{m+\lambda_1} U_{\text{in}}(r, t), \inf_{r^*(t) \leq r \leq ct^{1/2}} (1 + r)^{m+\lambda_1} C_0 U_{\text{out}}(r, t)\right\}, \end{aligned}$$

if $ct^{1/2} > r^*(t)$, and that

$$\inf_{r \leq ct^{1/2}} (1 + r)^{m+\lambda_1} U^-(r, t) \geq \inf_{r \leq r^*(t)} (1 + r)^{m+\lambda_1} U_{\text{in}}(r, t),$$

if $ct^{1/2} \leq r^*(t)$. Thus we may assume that $ct^{1/2} > r^*(t)$ for $t \geq t_0$ with some $t_0 \geq 0$. We will show that

$$\inf_{r \leq r^*(t)} (1 + r)^{m+\lambda_1} U_{\text{in}}(r, t) \geq c_0 (t + \tau)^{-(\ell - m - \lambda_1)/2} \quad \text{for } t \geq t_0 \tag{4.15}$$

and

$$\inf_{r^*(t) \leq r \leq ct^{1/2}} (1 + r)^{m+\lambda_1} U_{\text{out}}(r, t) \geq c_1 t^{-(\ell - m - \lambda_1)/2} \quad \text{for } t \geq t_1 \tag{4.16}$$

with some constants $c_0, c_1 > 0$ and $t_1 \geq t_0$. Combining (4.15) and (4.16), we obtain (4.10). For each $t > 0$, we have

$$\inf_{r \leq r^*(t)} (1 + r)^{m+\lambda_1} U_{\text{in}}(r, t) \geq \inf_{r \geq 0} (t + \tau)^{-(\ell - m - \lambda_1)/2} (1 + r)^{m+\lambda_1} \psi(r).$$

From (2.10) we obtain (4.15) with some constant $c_0 > 0$. From (4.12) we see that

$$(1 + r)^{m+\lambda_1} U_{\text{out}}(r, t) \geq r^{m+\lambda_1} U_{\text{out}}(r, t) = r^{-(\ell - m - \lambda_1)} \left(1 - b^{2k} \left(\frac{(t + \tau)^{1/2}}{r}\right)^{2k}\right).$$

Thus, for each $t > 0$, we have

$$\inf_{r^*(t)\le r\le ct^{1/2}} (1+r)^{m+\lambda_1} U_{\rm out}(r,t) \ge \inf_{r^*(t)\le r\le ct^{1/2}} r^{-(\ell-m-\lambda_1)} \left(1 - b^{2k}\left(\frac{(t+\tau)^{1/2}}{r^*(t)}\right)^{2k}\right). \tag{4.17}$$

From (4.13) with $b^* > b$, it follows that

$$\liminf_{t\to\infty} \left(1 - b^{2k}\left(\frac{(t+\tau)^{1/2}}{r^*(t)}\right)^{2k}\right) = (1-(b/b^*)^{2k}) > 0.$$

Thus there exist constants $c_2 > 0$ and $t_1 \ge t_0$ such that

$$\left(1 - b^{2k}\left(\frac{(t+\tau)^{1/2}}{r^*(t)}\right)^{2k}\right) \ge c_2 \quad \text{for } t \ge t_1.$$

Then, from (4.17) we obtain, for $t \ge t_1$,

$$\inf_{r^*(t)\le r\le ct^{1/2}} (1+r)^{m+\lambda_1} U_{\rm out}(r,t) \ge \inf_{r^*(t)\le r\le ct^{1/2}} c_2 r^{-(\ell-m-\lambda_1)} \ge c_1 t^{-(\ell-m-\lambda_1)/2}$$

with $c_1 = c_2 c^{-(\ell-m-\lambda_1)}$. Thus (4.16) holds. □

Proof of Theorem 1.2. Put $U_0(r) = \min_{r=|x|}(u_0(x) - \tilde{u}_0(x)) > 0$ for $r \ge 0$, and let U be a solution of (2.7). Then, by Lemma 2.4 (ii), we obtain (2.12).

Define U^- by (4.7). By Lemma 2.2 (ii) we see that, if there exists a constant $C > 0$ such that $CU^-(|x|,0) \le U_0(x)$ for $x \in \mathbf{R}^N$, then CU^- is a continuous weak subsolution to (2.7). From (1.10), the function U_0 satisfies $\liminf_{r\to\infty} r^\ell U_0(r) > 0$. Since $U_0(r) > 0$ for $r \ge 0$ and

$$\limsup_{r\to\infty} r^\ell U^-(r,0) = \limsup_{r\to\infty} r^\ell U_{\rm out}(r,0) = 1,$$

there exists a constant $C_2 > 0$ such that $C_2 U^-(r,0) \le U_0(r)$ for $r \ge 0$. Then $C_2 U^-(r,t)$ is a continuous weak subsolution to (2.7). Then, by Lemma 2.1, we have $U(r,t) \ge C_2 U^-(r,t)$ for $r \ge 0$ and $t > 0$. Thus, from (2.12), we obtain

$$u(x,t) - \tilde{u}(x,t) \ge U(|x|,t) \ge C_2 U^-(|x|,t) \quad \text{for all } x \in \mathbf{R}^N,\ t \ge 0.$$

By Lemma 4.3, we obtain (1.11), (1.12) and (1.13). □

5. Proof of Theorem 1.3

Define $U^+(r,t)$ as in the proof of Theorem 1.1, that is, $U^+(r,t)$ is defined by (3.6), where $U_{\rm out}$ and $U_{\rm in}$ are given by (3.3) and (3.4), respectively. Note here that $r^*(t)$ satisfies (3.5). By the similar arguments as in the proof of Lemma 3.4, we see that (3.11) and (3.13) hold even if $\ell' = \ell$, that is,

$$\sup_{r \geq r^*(t)} (1+r)^\ell U^+(r,t) = O(1) \quad \text{and} \quad \sup_{0<r\leq r^*(t)} (1+r)^\ell U^+(r,t) = O(1) \quad \text{as } t \to \infty.$$

Then it follows that

$$\sup_{t\geq 0}\left(\sup_{r\geq 0}(1+r)^\ell U^+(r,t)\right) < \infty.$$

Put c_1 and c_2, respectively, by

$$c_1 = \sup_{t\geq 0}\left(\sup_{r\geq 0}(1+r)^\ell U^+(r,t)\right) \quad \text{and} \quad c_2 = \inf_{r\geq 0}(1+r)^\ell U^+(r,0).$$

Then, by (3.19), we have $c_2 > 0$.

Lemma 5.1. *For any* $\varepsilon > 0$, *put* $\delta = c_2\varepsilon/c > 0$ *with* $c > c_1$. *Let* U *be a solution of* (2.7). *If* U_0 *satisfies* $\|U_0(|\cdot|)\|_\ell < \delta$, *then* U *satisfies* $\|U(|\cdot|,t)\|_\ell < \varepsilon$ *for* $t \geq 0$.

Proof. If $\|U_0(|\cdot|)\|_\ell < \delta$, then it follow that

$$(1+r)^\ell U_0(r) < \delta \leq \frac{\varepsilon}{c}(1+r)^\ell U^+(r,0) \quad \text{for } r \geq 0.$$

This implies that $U_0(r) \leq (\varepsilon/c)U^+(r,0)$ for $r \geq 0$. Then $(\varepsilon/c)U^+(r,t)$ is a continuous weak supersolution of (2.7). By Lemma 2.1 we obtain $U(r,t) \leq (\varepsilon/c)U^+(r,t)$ for all $r \geq 0$ and $t \geq 0$. Hence,

$$(1+r)^\ell U(r,t) \leq \frac{\varepsilon}{c}(1+r)^\ell U^+(r,t) < \varepsilon \quad \text{for all } r \geq 0,\ t \geq 0.$$

Thus we obtain $\|U(|\cdot|,t)\|_\ell < \varepsilon$ for $t \geq 0$. □

Proof of Theorem 1.3. (i) Put $U_0(r) = \max_{r=|x|}(u_0(x) - \tilde{u}_0(x))$ for $r \geq 0$, and let U be a solution of (2.7). Then, by Lemma 2.4 (i), we obtain (2.11).

For any $\varepsilon > 0$, put $\delta = c_2\varepsilon/c_1 > 0$. Assume that $\|u_0(\cdot) - \tilde{u}_0(\cdot)\|_\ell < \delta$. This implies that $\|U_0(|\cdot|)\|_\ell < \delta$. By Lemma 5.1 we obtain $\|U(|\cdot|,t)\|_\ell < \varepsilon$ for $t \geq 0$. From (2.11) we conclude that $\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_\ell < \varepsilon$ for $t \geq 0$.

(ii) Define $U^-(r,t)$ as in the proof of Theorem 1.2, that is, $U^-(r,t)$ is defined by (4.7), where $U_{\rm in}$ and $U_{\rm out}$ are given by (4.1) and (4.3), respectively. For each fixed $t > 0$, observe that $\lim_{r\to\infty} r^\ell U_{\rm out}(r,t) = 1$. Then it follows that

$$\sup_{r>0}(1+r)^\ell U^-(r,t) \geq C_0 \quad \text{for all } t \geq 0.$$

Thus we obtain $\|U^-(|\cdot|,t)\|_\ell \geq C_0 > 0$ for each $t \geq 0$.

Put $U_0(r) = \min_{r=|x|}(u_0(x) - \tilde{u}_0(x)) > 0$ for $r \geq 0$, and let U be a solution of (2.7). Then, by Lemma 2.4 (ii), we obtain (2.12). By the similar argument as in the proof of Theorem 1.2, there exists a constant $C_2 > 0$ such that $C_2 U^-(r,0) \leq U_0(r)$ for $r \geq 0$. Then $C_2 U^-(r,t)$ is a continuous weak subsolution to (2.7). Then, by Lemma 2.1, we have $U(r,t) \geq C_2 U^-(r,t)$ for $r \geq 0$ and $t > 0$. Thus, from (2.11) we obtain

$$\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{\ell} \geq \|U(|\cdot|,t)\|_{\ell} \geq C_2\|U^-(|\cdot|,t)\|_{\ell} \geq C_2 C_0 > 0$$

for each fixed $t > 0$. This implies that (1.15) holds. □

References

[1] W. X. Chen and C. Li, Classification of solutions of some nonlinear elliptic equations, Duke Math. J. **63** (1991), 615–622.

[2] M. Fila, J. R. King, M. Winkler and E. Yanagida, Optimal lower bound of the grow-up rate for a supercritical parabolic equation, J. Differential Equations **228** (2006), 339–356.

[3] M. Fila, M. Winkler and E. Yanagida, Convergence rate for a parabolic equation with supercritical nonlinearity, J. Dynam. Differential Equations **17** (2005), 249–269.

[4] B. Gidas and J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, Commun. Pure Appl. Math. **34** (1981), 525–598.

[5] C. Gui, W.-M. Ni and X. Wang, On the stability and instability of positive steady states of a semilinear heat equation in R^n, Comm. Pure Appl. Math. **45** (1992), 1153–1181.

[6] C. Gui, W.-M. Ni and X. Wang, Further study on a nonlinear heat equation, J. Differential Equations **169** (2001), 588–613.

[7] M. Hoshino and E. Yanagida, Sharp estimates of the convergence rate for a semilinear parabolic equation with supercritical nonlinearity, Nonlinear Anal. TMA **69** (2008), 3136–3152.

[8] D. D. Joseph and T. S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, Arch. Rational Mech. Anal. **49** (1972/73), 241–269.

[9] Y. Naito, Global attractivity and convergence rate in the weighted norm for a supercritical semilinear heat equation, to appear in Differential Integral Equations.

[10] P. Poláčik and E. Yanagida, On bounded and unbounded global solutions of a supercritical semilinear heat equation, Math. Ann. **327** (2003), 745–771.

[11] X. Wang, On the Cauchy problem for reaction-diffusion equations, Trans. Amer. Math. Soc. **337** (1993), 549–590.

Yūki Naito
Department of Mathematics
Ehime University
Matsuyama 790-8577
Japan
E-mail: ynaito@ehime-u.ac.jp