F. DELLA PIETRA
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SOME REMARKS ON A SHAPE OPTIMIZATION PROBLEM

Francesco Della Pietra

Abstract

Given a bounded open set Ω of $\mathbf{R}^n$, $n \geq 2$, and $\alpha \in \mathbf{R}$, let us consider

$$\mu(\Omega, \alpha) = \min_{\substack{v \in W_0^{1,2}(\Omega) \\ v \not\equiv 0}} \frac{\displaystyle\int_\Omega |\nabla v|^2\, dx + \alpha \left| \int_\Omega |v| v\, dx \right|}{\displaystyle\int_\Omega |v|^2\, dx}.$$

We study some properties of $\mu(\Omega, \alpha)$ and of its minimizers, and, depending on α, we determine the sets Ω_α among those of fixed measure such that $\mu(\Omega_\alpha, \alpha)$ is the smallest possible.

1. Statement of the problem and main result

Let Ω be a bounded open set of $\mathbf{R}^n$, $n \geq 2$, and consider the following minimum problem

$$\mu(\Omega, \alpha) = \min_{\substack{v \in W_0^{1,2}(\Omega) \\ v \not\equiv 0}} \mathscr{Q}(v, \alpha) \tag{1.1}$$

where α is a fixed real number and

$$\mathscr{Q}(v, \alpha) = \frac{\displaystyle\int_\Omega |\nabla v|^2\, dx + \alpha \left| \int_\Omega |v| v\, dx \right|}{\displaystyle\int_\Omega |v|^2\, dx}.$$

The objective of this paper is to study some properties of $\mu(\Omega, \alpha)$ and of its minimizers. Moreover, we aim to determine and to characterize the sets $\tilde{\Omega}$ among those of fixed measure such that $\mu(\tilde{\Omega}, \alpha)$ is the smallest possible. As we will show, the shape of $\tilde{\Omega}$ depends on α. More precisely if we denote, as usual, by ω_n the measure of the unit ball in $\mathbf{R}^n$, and by $j_{n/2-1,1}$ the first zero of the Bessel function of first kind of order $n/2 - 1$, the main result of the paper is the following.

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THEOREM 1.1. *Let $n \geq 2$. There exists a positive number*

$$\alpha_c = j_{n/2-1,1}^2 \omega_n^{2/n}[2^{2/n} - 1]$$

such that, for every bounded open set $\Omega \subset \mathbf{R}^n$ and for every $\alpha \in \mathbf{R}$, it holds

$$\mu(\Omega, \alpha) \geq \begin{cases} \mu(\Omega^{\#}, \alpha) & \text{if } \alpha|\Omega|^{2/n} \leq \alpha_c, \\ \dfrac{2^{2/n}\omega_n^{2/n} j_{n/2-1,1}^2}{|\Omega|^{2/n}} & \text{if } \alpha|\Omega|^{2/n} \geq \alpha_c, \end{cases} \tag{1.2}$$

where $\Omega^{\#}$ is the ball centered at the origin with Lebesgue measure $|\Omega^{\#}| = |\Omega|$. If the equality sign holds when $\alpha|\Omega|^{2/n} < \alpha_c$, then Ω is a ball. If the equality sign holds when $\alpha|\Omega|^{2/n} > \alpha_c$, then Ω is the union of two disjoint balls of equal measure. If $\alpha|\Omega|^{2/n} = \alpha_c$ and the equality sign holds, Ω is a ball or the union of two disjoint balls of equal measure.

On the other hand, the above result provides the best constant $\mu(\tilde{\Omega}, \alpha)$ in the corresponding Sobolev-Poincaré inequality:

$$\mu(\tilde{\Omega}, \alpha) \int_\Omega |v|^2 \, dx \leq \int_\Omega |\nabla v|^2 \, dx + \alpha \left| \int_\Omega |v| v \, dx \right|, \quad v \in W_0^{1,2}(\Omega),$$

among all the open bounded sets Ω with fixed measure.

Let us observe that when $\alpha = 0$, the above inequality reduces to the classical Poincaré inequality. Moreover, $\mathscr{Q}(\Omega, 0)$ is the Rayleigh quotient associated to the Dirichlet Laplacian eigenvalue problem, and $\mu(\Omega, 0)$ corresponds to its first eigenvalue in Ω. Then, it is well known the Faber-Krahn inequality:

$$\mu(\Omega, 0) \geq \mu(\Omega^{\#}, 0).$$

Moreover if the equality replaces the inequality, then Ω is a ball.

The problem of finding the optimal shape of set-dependent functionals is largely studied in many settings. Several results can be found for example in [14], related to eigenvalue problems, or in [16]. Recent results are contained for example in [1–3, 5–13, 17]. Moreover, we recall that in [2] a result analogous to Theorem 1.1 is given for the functional

$$\tilde{\lambda}(\Omega, \alpha) = \min_{\substack{v \in W_0^{1,2}(\Omega) \\ v \not\equiv 0}} \frac{\displaystyle\int_\Omega |\nabla v|^2 \, dx + \alpha \left(\int_\Omega v \, dx \right)^2}{\displaystyle\int_\Omega |v|^2 \, dx},$$

which is related to a nonlocal eigenvalue problem. It has been proved that there exists a positive threshold value $\tilde{\alpha}$ such that if $\alpha < \tilde{\alpha}$, the minimum of $\tilde{\lambda}(\Omega, \alpha)$ among the sets with fixed measure is attained at one ball, while for α greater than $\tilde{\alpha}$, such minimum is given at two balls of equal measure.

The paper is organized as follows. In Section 2, we recall some basic results on Schwarz symmetrization and on the Dirichlet Laplacian. Moreover, depending on α, we give some properties of $\mu(\Omega, \alpha)$ and of its minimizers. Finally, in Section 3 we give the proof of the main result.

2. Notation and preliminary results

2.1. Schwarz symmetrization. Let Ω be a bounded open set of $\mathbf{R}^n$, $n \geq 2$ and let $u : \Omega \to \mathbf{R}$ be a measurable function. We denote by $\Omega^\#$ the ball centered at the origin with the same Lebesgue measure of Ω. The Schwarz rearrangement of u is the spherically symmetric decreasing function

$$u^\# : \Omega^\# \to [0, +\infty[$$

whose level sets are balls having the same measure of the level sets of $|u|$, that is

$$|\{u^\# > t\}| = |\{|u| > t\}|, \quad t \geq 0.$$

The Schwarz symmetrization enjoys the following properties.

a) By definition, $u^\#$ preserves the L^p-norm of u:

$$\|u\|_{L^p(\Omega)} = \|u^\#\|_{L^p(\Omega^\#)}, \quad 1 \leq p \leq +\infty. \tag{2.1}$$

b) The Pólya-Szegö inequality holds: if $u \in W_0^{1,2}(\Omega)$ is a nonnegative function, then

$$\int_\Omega |\nabla u|^2 \, dx \geq \int_{\Omega^\#} |\nabla u^\#|^2 \, dx. \tag{2.2}$$

Moreover, if the above inequality becomes an equality, and

$$|\{|\nabla u^\#| = 0\} \cap (u^\#)^{-1}(0, \operatorname{ess\,sup} u)| = 0,$$

then, up to translations, $\Omega = \Omega^\#$ and $u = u^\#$ almost everywhere (see [4]).

For an exhaustive treatment on rearrangements and symmetrization, we refer the reader, for example, to [15].

2.2. Some basic facts about the Dirichlet Laplacian. Given G a bounded open set of $\mathbf{R}^n$, $n \geq 2$, throughout the paper we will denote by $\lambda_\Delta(G)$ the first Dirichlet-Laplace eigenvalue relative to G:

$$\lambda_\Delta(G) = \min_{v \in W_0^{1,2}(G) \setminus \{0\}} \frac{\displaystyle\int_G |\nabla v|^2 \, dx}{\displaystyle\int_G |v|^2 \, dx}, \tag{2.3}$$

and by $\lambda_T(G)$ the minimum of the constrained problem

$$\lambda_T(G) = \min_{\substack{v \in W_0^{1,2}(G) \setminus \{0\} \\ \int_G |v| v \, dx = 0}} \frac{\displaystyle\int_G |\nabla v|^2 \, dx}{\displaystyle\int_G |v|^2 \, dx}. \tag{2.4}$$

As regards (2.3), we recall the following basic properties.

(1) The Faber-Krahn inequality: for any bounded open set in $\Omega \subset \mathbf{R}^n$, $n \geq 2$, it holds that

$$\lambda(\Omega) \geq \lambda(\Omega^{\#}) = \frac{\omega_n^{2/n}}{|\Omega|^{2/n}} j_{n/2-1,1}^2,$$

where $j_{n/2-1,1}$ denotes, as usual, the first zero of the Bessel function of first kind of order $n/2-1$. If equality sign occurs, then Ω is a ball.

(2) If $\Omega = B_1 \cup B_2$ is the union of two disjoint balls B_1, B_2 with different radii $R_1 > R_2 > 0$, then

$$\lambda_\Delta(\Omega) = \lambda_\Delta(B_1) = \frac{j_{n/2-1,1}^2}{R_1^2}.$$

Hence $\lambda_\Delta(\Omega)$ is simple, any associated eigenfuction does not change sign in the largest ball B_1, and it is identically zero in B_2.

(3) If $\Omega = B_1 \cup B_2$ is the union of two disjoint balls B_1, B_2 with equal radii $0 < R_1 = R_2$, the first eigenvalue is not simple, and there exists an eigenfunction u positive in B_1, negative in B_2 and such that $\int_{B_1 \cup B_2} |u| u \, dx = 0$. In particular, this eigenfunction coincides with the positive first eigenfunction of $\lambda_\Delta(B_1)$, and to its opposite (up to a translation) in B_2.

2.3. Some properties of $\mu(\Omega, \alpha)$. In what follows, for a given function $u : \Omega \to \mathbf{R}$, the functions $u_+ = \max\{u, 0\}$ and $u_- = \max\{-u, 0\}$ will denote its positive and negative part, and

$$\Omega_+ = \{u_+ > 0\}, \quad \Omega_- = \{u_- > 0\}.$$

Proposition 2.1. *Let Ω be a bounded open set of $\mathbf{R}^n$. The following properties for $\mu(\Omega, \alpha)$ hold.*

(a) *The minimum $\mu(\Omega, \alpha)$ is 1-Lipschitz continuous and non-decreasing with respect to $\alpha \in \mathbf{R}$.*

(b) *For $\alpha < 0$,*

$$\mu(\Omega, \alpha) = \lambda_\Delta(\Omega) + \alpha.$$

(c) *For $\alpha \geq 0$,*

$$\lambda_\Delta(\Omega) \leq \mu(\Omega, \alpha) \leq \min\{\lambda_T(\Omega), \lambda_\Delta(\Omega) + \alpha\}. \tag{2.5}$$

(d) *As $\alpha \to +\infty$, we have that*

$$\lim_{\alpha \to +\infty} \mu(\Omega, \alpha) = \lambda_T(\Omega).$$

Proof. (a) For any $\varepsilon > 0$,

$$\mathcal{Q}(v, \alpha) \leq \mathcal{Q}(v, \alpha + \varepsilon) \leq \mathcal{Q}(v, \alpha) + \varepsilon.$$

Taking the minimum over $W_0^{1,2}(\Omega)\backslash\{0\}$, we have

$$0 \leq \mu(\Omega, \alpha+\varepsilon) - \mu(\Omega, \alpha) \leq \varepsilon,$$

and the proof of (a) is concluded.

(b) Being $\alpha < 0$, we have that $\mathcal{Q}(v,\alpha) \geq \mathcal{Q}(|v|,\alpha) = \mathcal{Q}(v,0) + \alpha \geq \lambda_\Delta(\Omega) + \alpha$, for any $v \in W_0^{1,2}(\Omega)$. On the other hand, if $u \in W_0^{1,2}(\Omega)$ is a nonnegative minimizer for (2.3), $\mathcal{Q}(u,\alpha) = \lambda_\Delta(\Omega) + \alpha$, and then necessarily $\lambda_\Delta(\Omega) + \alpha = \mu(\Omega,\alpha)$.

(c) It follows immediately from the definitions of μ, λ_Δ and λ_T.

(d) Let $0 \leq \alpha_k$, $k \in \mathbf{N}$, be a positively divergent sequence. For any k, consider a minimizer $u_k \in W_0^{1,2}(\Omega)$ of (1.1) such that $\|u_k\|_2 = 1$. We have that

$$\mu(\Omega, \alpha_k) = \int_\Omega |\nabla u_k|^2 \, dx + \alpha_k \left| \int_\Omega |u_k| u_k \, dx \right| \leq \lambda_T(\Omega).$$

Then u_k converges (up to a subsequence) to a function $U \in W_0^{1,2}(\Omega)$ strongly in $L^2(\Omega)$ and weakly in $W_0^{1,2}(\Omega)$. Moreover, $\|U\|_{L^2(\Omega)} = 1$ and

$$\left| \int_\Omega |u_k| u_k \right| \leq \frac{\lambda_T(\Omega)}{\alpha_k} \to 0 \quad \text{as } k \to +\infty,$$

which gives that $\int_\Omega |U| U \, dx = 0$. On the other hand, the weak convergence in $W_0^{1,2}(\Omega)$ implies that

$$\int_\Omega |\nabla U|^2 \, dx \leq \liminf \int_\Omega |\nabla u_k|^2 \, dx.$$

Finally, by definition of $\lambda_T(\Omega)$, and (2.5) we have

$$\begin{aligned} \lambda_T(\Omega) &\leq \int_\Omega |\nabla U|^2 \, dx \\ &\leq \liminf_{k\to+\infty} \left(\int_\Omega |\nabla u_k|^2 \, dx + \alpha_k \left| \int_\Omega |u_k| u_k \, dx \right| \right) \\ &= \lim_{k\to+\infty} \mu(\Omega, \alpha_k) \leq \lambda_T(\Omega), \end{aligned}$$

and the proof is completed. □

Remark 2.1. Let us observe that from the above proposition, (b) gives that $\mu(\Omega,\cdot)$ is unbounded from below. Moreover, $\mu(\Omega,\alpha) = 0$ corresponds to $-\alpha = \lambda(\Omega)$.

Remark 2.2. Among the properties of $\mu(\Omega,\alpha)$, we observe also that it does not have the same behavior of the usual Dirichlet Laplacian with respect to the rescaling of the domain, being also the term α affected of the rescaling. Indeed, while $\lambda_\Delta(t\Omega) = t^{-2}\lambda(\Omega)$, it holds that $\mu(t\Omega;\alpha) = t^{-2}\mu(\Omega; t^2\alpha)$.

In the lemma below, we describe some features of $\mu(\Omega, \alpha)$ by computing the associated Euler equation. Without loss of generality we may assume that a minimizer u satisfies $\int_\Omega |u|u\,dx \geq 0$.

Lemma 2.1. *Consider $\alpha \geq 0$ and Ω bounded open set of $\mathbf{R}^n$, $n \geq 2$, and suppose that $u \in W_0^{1,2}(\Omega)$ is a minimizer for* (1.1). *Then $u_+ \in W_0^{1,2}(\Omega_+)$ and $u_- \in W_0^{1,2}(\Omega_-)$ are first eigenfunctions of the Dirichlet Laplacian associated to Ω_+ and Ω_- respectively. Moreover*:

(1) *suppose that $\int_\Omega |u|u\,dx > 0$.*

(a) *If $u_- \equiv 0$ in Ω, then*

$$\mu(\Omega, \alpha) = \lambda_\Delta(\Omega_+) + \alpha. \tag{2.6}$$

(b) *If $u_- \not\equiv 0$ in Ω, then*

$$\mu(\Omega, \alpha) = \lambda_\Delta(\Omega_+) + \alpha = \frac{\lambda_\Delta(\Omega_+) + \lambda_\Delta(\Omega_-)}{2}, \tag{2.7}$$

and then the parameter α corresponds to

$$\alpha = \frac{\lambda_\Delta(\Omega_-) - \lambda_\Delta(\Omega_+)}{2}. \tag{2.8}$$

In both cases (a) *and* (b),

$$\lambda_\Delta(\Omega_+) = \lambda_\Delta(\Omega), \tag{2.9}$$

and

$$\mu(\Omega, \alpha) = \lambda_\Delta(\Omega) + \alpha = \lambda_\Delta(\Omega_+) + \alpha. \tag{2.10}$$

(2) *Suppose that $\int_\Omega |u|u\,dx = 0$. Then*

$$\mu(\Omega, \alpha) = \lambda_T(\Omega) = \frac{\lambda_\Delta(\Omega_+) + \lambda_\Delta(\Omega_-)}{2}. \tag{2.11}$$

More precisely, if there exists $\bar{\alpha}$ such that a minimizer $\bar{u}$ of $\mu(\Omega, \bar{\alpha})$ satisfies $\int_\Omega |\bar{u}|\bar{u}\,dx = 0$, then for any $\alpha > \bar{\alpha}$, $\bar{u}$ is a minimizer for $\mu(\Omega, \alpha)$, the equality in (2.11) *holds, and $\bar{u}$ is a minimizer also for $\lambda_T(\Omega)$.*

Proof. For sake of simplicity, here we write $\mu = \mu(\Omega, \alpha)$, and distinguish two cases.

Case 1. $\int_\Omega |u|u\,dx > 0$. We have that u solves

$$\begin{cases} -\Delta u = \mu u - \alpha |u| & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

If $u \geq 0$ in Ω, then u_+ satisfies

$$\begin{cases} -\Delta u_+ = (\mu - \alpha) u_+ & \text{in } \Omega_+, \\ u_+ = 0 & \text{on } \partial\Omega_+. \end{cases}$$

The positivity of the eigenfunction u_+ in Ω_+ guarantees that $\mu - \alpha$ coincides with the first eigenvalue $\lambda_+(\Omega)$ on Ω_+, and then (2.6) holds. Moreover, (2.9) follows from the inequalities

$$\lambda_\Delta(\Omega_+) + \alpha \le \lambda_\Delta(\Omega) + \alpha \le \lambda_\Delta(\Omega_+) + \alpha,$$

obtained by substituting (2.6) in (2.5), and recalling the monotonicity of the Dirichlet-Laplace eigenvalues with respect to the inclusion of sets.

If u changes sign in Ω, then u_+ and u_- satisfy

$$\begin{cases} -\Delta u_+ = (\mu - \alpha)u_+ & \text{in } \Omega_+, \\ u = 0 & \text{on } \partial\Omega_+, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_- = (\mu + \alpha)u_- & \text{in } \Omega_-, \\ u_- = 0 & \text{on } \partial\Omega_-. \end{cases}$$

Hence

$$\lambda_\Delta(\Omega_+) = \mu - \alpha, \quad \lambda_\Delta(\Omega_-) = \mu + \alpha,$$

that give (2.7) and (2.8). Similarly as before, substituting (2.7) and (2.8) in (2.5) and using the monotonicity of $\lambda_\Delta(\cdot)$, the equality (2.9) holds. By (2.6), (2.7) and (2.9) we get also (2.10).

Case 2. $\int_\Omega |u|u\,dx = 0$. First, we observe that in this case

$$\mu(\Omega, \alpha) = \lambda_T(\Omega).$$

Indeed, by definition of μ and λ_T, and being u an admissible function for (2.4), we have

$$\lambda_T(\Omega) \ge \mu(\Omega, \alpha) = \mathcal{Q}(u, \alpha) \ge \lambda_T(\Omega).$$

Computing the Euler equation with the constraint $\int_\Omega |u|u\,dx = 0$, the functions $u_+ \in W_0^{1,2}(\Omega_+)$ and $u_- \in W_0^{1,2}(\Omega_-)$ satisfy

$$\text{(2.12)} \quad \begin{cases} -\Delta u_+ = \lambda_+ u_+ & \text{in } \Omega_+, \\ u_+ = 0 & \text{on } \partial\Omega_+, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_- = \lambda_- u_- & \text{in } \Omega_-, \\ u_- = 0 & \text{on } \partial\Omega_-, \end{cases}$$

for some positive values λ_+ and λ_- (see also [17]). Moreover, being u_+ and u_- positive functions in Ω_+ and Ω_- respectively, it follows that

$$\lambda_+ = \lambda_\Delta(\Omega_+), \quad \lambda_- = \lambda_\Delta(\Omega_-).$$

Hence, in this case we have that $\int_{\Omega_+} u_+^2\,dx = \int_{\Omega_-} u_-^2\,dx$, and from the minimality of u and using (2.12) it follows that

$$\mu(\Omega, \alpha) = \mathcal{Q}(u, \alpha) = \frac{\displaystyle\int_{\Omega_+} |\nabla u_+|^2\,dx + \int_{\Omega_-} |\nabla u_-|^2\,dx}{\displaystyle\int_{\Omega_+} u_+^2\,dx + \int_{\Omega_-} u_-^2\,dx} = \frac{\lambda_\Delta(\Omega_+) + \lambda_\Delta(\Omega_-)}{2},$$

and (2.11) is proved. The proof of (2) is completed by recalling that the function $\mu(\Omega, \cdot)$ is nondecreasing and bounded from above by $\lambda_T(\Omega)$. $\square$

Using the above lemma, the minimum $\mu(\Omega, \alpha)$ can be characterized as follows.

PROPOSITION 2.2. *If Ω is a bounded open set of $\mathbf{R}^n$, $n \geq 2$, then*

$$\mu(\Omega, \alpha) = \min\{\lambda_\Delta(\Omega) + \alpha, \lambda_T(\Omega)\} = \begin{cases} \lambda_\Delta(\Omega) + \alpha, & \text{if } \alpha \leq \lambda_T(\Omega) - \lambda_\Delta(\Omega), \\ \lambda_T(\Omega), & \text{if } \alpha \geq \lambda_T(\Omega) - \lambda_\Delta(\Omega). \end{cases}$$

Proof. Let $\alpha \geq 0$ be fixed. We have to show that $\mu(\Omega, \alpha) = \min\{\lambda_\Delta(\Omega) + \alpha, \lambda_T(\Omega)\}$.

Clearly if $\lambda_\Delta(\Omega) + \alpha < \lambda_T(\Omega)$, a minimizer u of $\mu(\Omega, \alpha)$ cannot verify $\int_\Omega |u| u \, dx = 0$. Otherwise, by (2.11), and choosing a nonnegative first eigenfunction u_1 of $-\Delta$ in Ω, we have

$$Q(u, \alpha) = \mu(\Omega, \alpha) = \lambda_T(\Omega) > \lambda_\Delta(\Omega) + \alpha = \mathscr{Q}(u_1, \alpha),$$

contradicting the minimality of u. Hence $\int_\Omega |u| u \, dx > 0$, and by (2.10), $\mu(\Omega, \alpha) = \lambda_\Delta(\Omega) + \alpha$.

Analogously, if $\lambda_\Delta(\Omega) + \alpha > \lambda_T(\Omega)$, a minimizer u necessarily satisfies $\int_\Omega |u| u \, dx = 0$, and $\mu(\Omega, \alpha) = \lambda_T(\Omega)$. $\square$

Remark 2.3. We explicitly observe that, using the above proposition, if Ω is connected a minimizer u of $\mu(\Omega, \alpha)$ is either positive in Ω or $\int_\Omega |u| u \, dx = 0$.

Assuming now that Ω is the union of two disjoint balls (possibly one ball), we have the following.

COROLLARY 2.1. *If $\Omega = B_1$, with radius $R_1 > 0$, then*

$$\mu(B_1; \alpha) = \begin{cases} \dfrac{j^2_{n/2-1,1}}{R_1^2} + \alpha, & \text{if } \alpha \leq \lambda_T(B_1) - \dfrac{j^2_{n/2-1,1}}{R_1^2}, \\ \lambda_T(B_1) & \text{otherwise.} \end{cases} \tag{2.13}$$

If $\Omega = B_1 \cup B_2$, where B_1 and B_2 are disjoint balls with radii R_1, R_2 such that $R_1 \geq R_2 > 0$, then

$$\mu(B_1 \cup B_2, \alpha) = \begin{cases} \dfrac{j^2_{n/2-1,1}}{R_1^2} + \alpha, & \text{if } \alpha \leq \lambda_T(B_1 \cup B_2) - \dfrac{j^2_{n/2-1,1}}{R_1^2}, \\ \lambda_T(B_1 \cup B_2) & \text{otherwise.} \end{cases} \tag{2.14}$$

In particular, if $R_1 = R_2$, for any $\alpha \geq 0$

$$\mu(B_1 \cup B_2; \alpha) = \lambda_T(B_1 \cup B_2) = \frac{2^{2/n} \omega_n^{2/n} j^2_{n/2-1,1}}{|\Omega|^{2/n}}, \tag{2.15}$$

where the value in the right-hand side is $\lambda_\Delta(B_1 \cup B_2)$, and any minimizer of $\mu(B_1 \cup B_2, \alpha)$ is a minimizer of $\lambda_T(B_1 \cup B_2)$.

Proof. The proof of (2.13) and (2.14) follows from Proposition 2.2 by writing explicitly λ_Δ in the case of one ball or two disjoint balls. Then, we have only to show last equality in (2.15). Observe first that

$$\lambda_T(B_1 \cup B_2) \geq \lambda_\Delta(B_1 \cup B_2).$$

On the other hand, being B_1 and B_2 disjoint balls with equal measure, there exists an eigenfunction V of the Dirichlet Laplacian relative to $B_1 \cup B_2$ such that $\int_{B_1 \cup B_2} |V| V \, dx = 0$. More precisely, this eigenfunction corresponds to a first positive Dirichlet Laplacian eigenfunction on B_1, and to its opposite (up to a translation) on B_2. Then V is an admissible test function for the Rayleigh quotient of $\lambda_T(B_1 \cup B_2)$, and

$$\lambda_T(B_1 \cup B_2) \leq \frac{\displaystyle\int_{B_1 \cup B_2} |\nabla V|^2 \, dx}{\displaystyle\int_{B_1 \cup B_2} V^2 \, dx} = \lambda_\Delta(B_1 \cup B_2),$$

and then $\lambda_\Delta(B_1 \cup B_2) = \lambda_T(B_1 \cup B_2)$. $\square$

3. Proof of Theorem 1.1

The proof of Theorem 1.1 will be pursued in two main steps. First, we show that the minimum of $\mu(\Omega, \alpha)$ among all sets of fixed measure is reached at the union of two disjoint balls. Second, we minimize $\mu(\Omega, \alpha)$ among such sets.

3.1. An isoperimetric inequality for $\mu(\Omega, \alpha)$. The first step in order to prove Theorem 1.1 is to show an isoperimetric inequality for $\mu(\Omega, \alpha)$. To this aim, let

$$\mathscr{B}(|\Omega|) = \{A = B_1 \cup B_2 : B_1, B_2 \text{ open disjoint balls of } \mathbf{R}^n, |B_1 \cup B_2| = |\Omega|\}.$$

In the above definition we are implicitly assuming that $A \in \mathscr{B}(|\Omega|)$ can be a unique ball.

Proposition 3.1. *Let $\Omega \subset \mathbf{R}^n$, $n \geq 2$, be a bounded, open set such that $\Omega \notin \mathscr{B}(|\Omega|)$. Then there exists $A_\alpha = B_1 \cup B_2 \in \mathscr{B}(|\Omega|)$ such that*

$$\mu(\Omega, \alpha) > \mu(A_\alpha, \alpha) = \min_{A \in \mathscr{B}(|\Omega|)} \mu(A, \alpha).$$

Moreover,

$$\mu(A_\alpha, \alpha) = \mathscr{Q}(v_1 \chi_{B_1} - v_2 \chi_{B_2}, \alpha), \tag{3.1}$$

for some nonnegative functions v_1 and v_2, radially decreasing in B_1 and B_2 respectively. More precisely, either $v_2 \equiv 0$, and v_1 is positive in $B_1 = \Omega$, or $v_1 > 0$ in B_1 and $v_2 > 0$ in B_2.

Proof. Let $u \in W_0^{1,2}(\Omega)$ be a minimizer of (1.1). Using (2.1) and Pólya-Szegö principle (2.2), we have that

$$\mu(\Omega,\alpha)$$

$$= \frac{\int_{\Omega_+} |\nabla u_+|^2\,dx + \int_{\Omega_-} |\nabla u_-|^2\,dx + \alpha\left|\int_{\Omega_+} u_+^2\,dx - \int_{\Omega_-} u_-^2\,dx\right|}{\int_{\Omega_+} u_+^2\,dx + \int_{\Omega_-} u_-^2\,dx}$$

$$\geq \frac{\int_{\Omega_+^\#} |\nabla u_+^\#|^2\,dx + \int_{\Omega_-^\#} |\nabla u_-^\#|^2\,dx + \alpha\left|\int_{\Omega_+^\#} (u_+^\#)^2\,dx - \int_{\Omega_-^\#} (u_-^\#)^2\,dx\right|}{\int_{\Omega_+^\#} (u_+^\#)^2\,dx + \int_{\Omega_-^\#} (u_-^\#)^2\,dx} \tag{3.2}$$

$$\geq \min_{\substack{w \in W_0^{1,2}(\Omega_+^\#)\\ z \in W_0^{1,2}(\Omega_-^\#)}} \frac{\int_{\Omega_+^\#} |\nabla w|^2\,dx + \int_{\Omega_-^\#} |\nabla z|^2\,dx + \alpha\left|\int_{\Omega_+^\#} |w|w\,dx + \int_{\Omega_-^\#} |z|z\,dx\right|}{\int_{\Omega_+^\#} |w|^2\,dx + \int_{\Omega_-^\#} |z|^2\,dx}$$

$$\geq \inf_{A \in \mathscr{B}(|\Omega|)} \mu(A;\alpha). \tag{3.3}$$

If u_+ or u_- is not radially symmetric, then the inequality (3.2) is strict. Moreover, if u_+ and u_- are both radially decreasing functions, then Ω_+ and Ω_- are balls such that $|\Omega_+| + |\Omega_-| < |\Omega|$, being $\Omega \notin \mathscr{B}(|\Omega|)$. The monotonicity of $\mu(\cdot;\alpha)$ with respect to homotheties gives that in this case (3.3) is strict.

The arguments just used also give (3.1). □

In order to conclude the proof of Theorem 1.1, we recall an isoperimetric inequality for $\lambda_T(\Omega)$ given in [17], which assures that if B_1, B_2 are disjoint balls with $|B_1| = |B_2| = |\Omega|/2$, then

$$\lambda_T(\Omega) \geq \lambda_T(B_1 \cup B_2).$$

Proof of Theorem 1.1. If $\alpha \leq 0$, being $\mu(\Omega,\alpha) = \lambda_\Delta(\Omega) + \alpha$ the result is given by the well-known Faber-Krahn inequality, which follows immediately from the Pólya-Szegö principle and the properties of rearrangements:

$$\mathscr{Q}(u,\alpha) \geq \mathscr{Q}(u^\#,\alpha) \geq \mu(\Omega^\#,\alpha). \tag{3.4}$$

Then we can assume that $\alpha > 0$.

Proposition 3.1 allows to restrict to the case $\Omega \in \mathscr{B}(|\Omega|)$. We denote by Ω_d the union of two disjoint balls with same measure, equal to $|\Omega|/2$.

Then the proof is completed by observing that, by Proposition 2.1, and the Faber-Krahn inequality and (3.4), each eigencurve $\alpha \mapsto \mu(\Omega,\alpha)$, $\alpha \geq 0$, is such that $\mu(\Omega,0) \geq \mu(\Omega^\#,0) = \lambda_\Delta(\Omega^\#)$, then it increases linearly until it reaches the

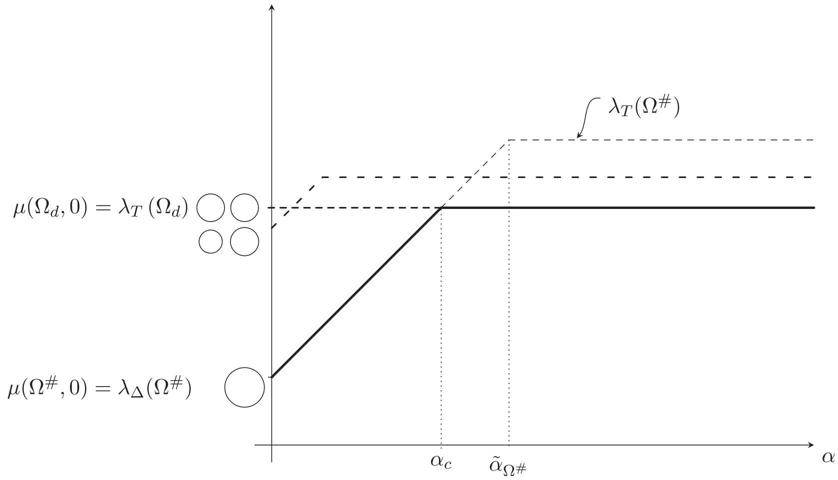

FIGURE 1. A scheme of the eigencurves $\alpha \mapsto \mu(\Omega, \alpha)$, $\alpha \geq 0$, when $|\Omega| = \kappa$ is fixed. If Ω corresponds to Ω_d union of two disjoint balls of equal measure, then $\mu(\Omega_d, \alpha)$ is constant for $\alpha \geq 0$. Otherwise, $\mu(\Omega, \alpha)$ increases until it reaches its maximum value $\mu(\Omega, \alpha_\Omega) = \lambda_T(\Omega)$ in $\tilde{\alpha}_\Omega = \lambda_T(\Omega) - \lambda_\Delta(\Omega)$, then it is constant for $\alpha \geq \tilde{\alpha}_\Omega$. The solid line represents the values of $\min\limits_{|\Omega|=\kappa} \mu(\Omega, \alpha)$.

value $\lambda_T(\Omega)$ which is greater than $\lambda_T(\Omega_d)$ (see also Figure 1). More precisely, the eigencurve $\alpha \mapsto \mu(\Omega, \alpha)$ is above the curve

$$\alpha \mapsto \begin{cases} \mu(\Omega^\#, \alpha) & \text{if } \alpha|\Omega|^{2/n} \leq \alpha_c, \\ \mu(\Omega_d, \alpha) & \text{if } \alpha|\Omega|^{2/n} \geq \alpha_c, \end{cases}$$

obtaining (1.2). □

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Francesco Della Pietra
Università degli studi di Napoli "Federico II"
Dipartimento di Matematica e Applicazioni "R. Caccioppoli"
Via Cintia, Complesso di Monte Sant'Angelo
I-80126 Napoli
Italia
E-mail: f.dellapietra@unina.it