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A GENERALIZATION OF A COMPLETENESS LEMMA IN MINIMAL SURFACE THEORY

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Abstract

We settle a question posed by Umehara and Yamada, which generalizes a completeness lemma useful in differential geometry.

The following answers affirmatively a question posed by Umehara and Yamada [7, Question C].

Theorem. *Let* f *be a holomorphic function on* $\{|\zeta| > 1\} \subset \mathbf{C}$ *such that* $f(\{|\zeta| > 1\}) \subset \mathbf{C} \backslash \{0\}$ *and let* n *be a non-negative integer. If every real-analytic curve* $\gamma : [0, 1) \to \{|\zeta| > 1\}$ *tending to* ∞ *satisfies*

$$\int_{\gamma} |\log \zeta|^n |f(\zeta)|\, |d\zeta| = \infty, \tag{1}$$

then f *is meromorphic at* ∞.

In the special case of $n = 0$, this Theorem reduces to the completeness lemma due to MacLane and Voss (cf. Osserman [5, p. 89]), which plays an important role in minimal surface theory. A new insight by Umehara and Yamada is the possibility to take into account the variation of the argument of the curve γ, namely the imaginary part of $\log \gamma$, motivated by their investigation of parabolic ends of constant mean curvature one surfaces in de Sitter 3-space. A notable consequence of Theorem is an affirmative answer to [8, Question 2]. This implication is due to Umehara and Yamada [8]. For more details and backgrounds, we refer [7] and [8]. Our proof is based on the theory of entire functions, i.e., holomorphic functions on $\mathbf{C}$, and of harmonic measures, while the problem has its origin in differential geometry.

After having written this article, we learned that the Theorem, except for the real-analyticity of the path γ, could be shown using Huber's result [3]. Our

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proof of the Theorem is an improvement of the argument in Osserman's book [5].

Proof. Since the integral in (1) is non-decreasing as $n \geq 0$ increases, the Theorem for $n = 0$ is a consequence of that for $n > 0$. We assume that n is a positive integer.

We reduce the problem to the case where f is defined over all of $\mathbf{C}$ and satisfies $f(\mathbf{C}\backslash\{0\}) \subset \mathbf{C}\backslash\{0\}$ (cf. Osserman [5, p. 89]). Let us consider possibly multivalued holomorphic functions $\log f(\zeta)$ and $\log \zeta$ on $\{|\zeta| > 1\}$, and choose $k \in \mathbf{Z}$ such that $\log f(\zeta) - k \log \zeta$ is single-valued and holomorphic on $\{|\zeta| > 1\}$. Then its Laurent expansion is written as

$$\log f(\zeta) - k \log \zeta = \sum_{j=-\infty}^{\infty} c_j \zeta^j = H(\zeta) + h(\zeta),$$

where $H(\zeta) = \sum_{j=0}^{\infty} c_j \zeta^j$ is an entire function and $h(\zeta) = \sum_{j=-\infty}^{-1} c_j \zeta^j$ is a holomorphic function on $\{|\zeta| > 1\} \cup \{\infty\} \subset \hat{\mathbf{C}}$ such that $h(\infty) = 0$. Hence

$$f(\zeta) = e^{h(\zeta)} \zeta^k e^{H(\zeta)}.$$

Set $g(\zeta) = \zeta^k e^{H(\zeta)}$ if $k \geq 0$, and $g(\zeta) = e^{H(\zeta)}$ if $k < 0$. Then g is entire and $f(\zeta)/g(\zeta)$ is holomorphic near ∞. The condition (1) implies

$$\int_\gamma |\log \zeta|^n |g(\zeta)|\, |d\zeta| = \infty,$$

and if g is meromorphic at ∞, then so is f. Hence replacing f with g if necessary, we may assume that f is an entire function on $\mathbf{C}$ and that $f(\mathbf{C}\backslash\{0\}) \subset \mathbf{C}\backslash\{0\}$.

Consider an indefinite integral

$$G(\zeta) = \int_0^\zeta f(\zeta)\, d\zeta$$

of f. Since the zeros of G are isolated, there is $a > 0$ such that

$$\min_{t \in \mathbf{R}} |G(e^{a+it})| > 0.$$

We fix $a > 0$ with this property throughout. Set

$$F(z) := \int_a^{a+z} z^n f(e^z) e^z\, dz.$$

Then we note from our earlier discussion that $-a$ is the only critical point of F, that is,

$$\{z \in \mathbf{C}; F'(z) = 0\} = \{-a\}.$$

Let $\zeta = \pi(z) := e^{z+a}$ be a covering map from $\mathbf{C}$ to $\mathbf{C}\backslash\{0\}$. For any real-analytic curve Γ in z-plane, we have

$$\int_{\pi\circ\Gamma} |\log \zeta|^n |f(\zeta)|\,|d\zeta| = (\text{the Euclidean length of } F\circ\Gamma). \tag{2}$$

LEMMA 1. $\lim_{t\in\mathbf{R},\,|t|\to\infty} |F(it)| = \infty$.

Proof. Let us define $n+2$ auxiliary entire functions $G_0, G_1, \dots, G_{n+1}$ inductively; put

$$G_0(\zeta) := f(\zeta)\cdot\zeta$$

and for each $j \in \{0, 1, \dots, n\}$,

$$G_{j+1}(\zeta) := \int_0^\zeta \frac{G_j(\zeta)}{\zeta}\,d\zeta.$$

Then $G_1(\zeta) = G(\zeta)$, and for every $j \in \{0, 1, \dots, n\}$,

$$\frac{dG_{j+1}(e^{a+it})}{dt} = iG_j(e^{a+it}).$$

Hence for $j \in \{0, 1, \dots, n-1\}$,

$$\begin{aligned}
&i\int_0^t (a+it)^{n-j} G_j(e^{a+it})\,dt \\
&\quad = \int_0^t (a+it)^{n-j}\cdot\frac{dG_{j+1}(e^{a+it})}{dt}\,dt \\
&\quad = [(a+it)^{n-j} G_{j+1}(e^{a+it})]_0^t - (n-j)i\int_0^t (a+it)^{n-j-1} G_{j+1}(e^{a+it})\,dt \\
&\quad = (a+it)^{n-j} G_{j+1}(e^{a+it}) - a^{n-j} G_{j+1}(e^a) - (n-j)i\int_0^t (a+it)^{n-(j+1)} G_{j+1}(e^{a+it})\,dt.
\end{aligned}$$

Similarly,

$$i\int_0^t G_n(e^{a+it})\,dt = G_{n+1}(e^{a+it}) - G_{n+1}(e^a).$$

Hence there are constants $C_j \in \mathbf{C}$ $(j = 2, \dots, n+1)$ and $C \in \mathbf{C}$ such that for every $t \in \mathbf{R}$,

$$\begin{aligned}
F(it) &= \int_a^{a+it} z^n G_0(e^z)\,dz = \int_0^t (a+it)^n G_0(e^{a+it})i\,dt \\
&= G_1(e^{a+it})(a+it)^n + \sum_{j=1}^n C_{j+1} G_{j+1}(e^{a+it})(a+it)^{n-j} + C.
\end{aligned}$$

Now the assumption $\min_{t \in \mathbf{R}} |G_1(e^{a+it})| > 0$ together with $\max_{t \in \mathbf{R}} |G_j(e^{a+it})| < \infty$ $(j = 2, \dots, n+1)$ completes the proof. $\square$

Next we consider asymptotic curves of F. A curve $\Gamma : [0,1) \to \mathbf{C}$ is called an asymptotic curve of an entire function g with a finite asymptotic value $b \in \mathbf{C}$ if Γ tends to ∞ and $\lim_{t \to 1} g \circ \Gamma(t) = b$. We recall the following well-known

IVERSEN'S THEOREM (cf. [4]). *Let g be a non-constant entire function. Suppose that $z_0 \in \mathbf{C}$ is not a critical point of g, and put $w_0 := g(z_0)$. Let ϕ be a single-valued analytic branch of g^{-1} at w_0 such that $\phi(w_0) = z_0$, and $\gamma : [0,1] \to \mathbf{C}$ be a curve with $\gamma(0) = w_0$. If the analytic continuation of ϕ along $\gamma \mid [0,t]$ is possible for any $t \in [0,1)$, but impossible for $t = 1$, then either*
- *$\lim_{t \to 1} \phi \circ \gamma(t) \in \mathbf{C}$ exists and is a critical point of g, or*
- *$\phi \circ \gamma$ tends to ∞. In this case, $\phi \circ \gamma$ is an asymptotic curve of g with the finite asymptotic value $\gamma(1)$.*

For completeness, we include a proof.

Proof. We claim that the cluster set $C := \bigcap_{t \in [0,1)} \overline{\phi \circ \gamma([t,1))}$, where the closure is taken in $\hat{\mathbf{C}}$, is non-empty and connected: indeed, from the compactness of $\hat{\mathbf{C}}$, $C \neq \emptyset$. If C is not connected, then there are distinct open subsets U_1 and U_2 in $\hat{\mathbf{C}}$ intersecting C such that $(C \cap U_1) \cup (C \cap U_2) = C$. There are (t_j^1) and (t_j^2) in $[0,1)$ tending to 1 such that $\lim_{j \to 1} \phi(\gamma(t_j^i))$ exists in $C \cap U_i$ for each $i \in \{1,2\}$ and that for every $j \in \mathbf{N}$, $t_j^1 < t_j^2 < t_{j+1}^1$. For every $j \in \mathbf{N}$, since $\phi \circ \gamma$ is continuous, $\phi \circ \gamma([t_j^1, t_j^2])$ is connected. Hence there is $t_j \in [t_j^1, t_j^2]$ such that $\phi(\gamma(t_j)) \in \hat{\mathbf{C}} \backslash (U_1 \cup U_2)$. From the compactness of $\hat{\mathbf{C}}$, there is a subsequence (s_j) of (t_j) tending to 1 such that $\lim_{j \to 1} \phi(\gamma(s_j))$ exists in $C \backslash (U_1 \cup U_2)$. This is a contradiction. Thus C is connected.

Unless C is a singleton, C is a continuum. From $g(\phi \circ \gamma(t)) = \gamma(t)$ for every $t \in [0,1)$ and the continuity of g, $g(C) = \{\gamma(1)\}$. Then by the identity theorem, g must be constant. This is a contradiction. Hence C is a singleton, so $z_1 := \lim_{t \to 1} \phi(\gamma(t)) \in \hat{\mathbf{C}}$ exists. If $z_1 \in \mathbf{C}$, then z_1 is a critical point of g since ϕ cannot be continued analytically along all over the γ. If $z_1 = \infty$, then $\phi \circ \gamma$ tends to ∞. $\square$

For each $w_0 \in \mathbf{C}$ and each $r > 0$, put $\mathbf{D}_r(w_0) := \{w \in \mathbf{C}; |w - w_0| < r\}$. Put $\mathbf{H}_+ := \{z \in \mathbf{C}; \Re z > 0\}$, $I := \{z \in \mathbf{C}; \Re z = 0\}$ and $\mathbf{H}_- := \{z \in \mathbf{C}; \Re z < 0\}$.

LEMMA 2. *Let $\Gamma : [0,1) \to \mathbf{C}$ be an asymptotic curve of F with a finite asymptotic value. Then for every $t \in [0,1)$ close enough to 1, we have $\Gamma(t) \in \mathbf{H}_-$.*

Proof. We begin with

CLAIM 1. *For any real-analytic curve $C : [0,1) \to \mathbf{H}_+$ tending to ∞, the length of $F \circ C$ is infinite.*

Proof. If the real part $\Re C$ of C tends to ∞, then the curve $\pi \circ C$ also tends to ∞. Thus by (2), assumption (1) implies that the length of $F \circ C$ is infinite. If $M := \sup \Re C < \infty$, then $\pi \circ C \subset \{e^a < |\zeta| < e^{M+a}\}$ and $\lim_{t \to 1} |\arg(\pi \circ C(t))| = \infty$. Hence

$$\int_{\pi \circ C} |f(\zeta)|\, |d\zeta| = \infty. \tag{3}$$

Since $|\log \zeta| \geq a$ on the curve $\pi \circ C$, (3) with equality (2) implies that the length of $F \circ C$ is infinite.

In the remaining case, C should transverse some vertical strip $\{b_1 \leq \Re z \leq b_2\}$ infinitely often. Then $\pi \circ C$ transverses a round annulus $\{e^{b_1+a} \leq |\zeta| \leq e^{b_2+a}\}$ infinitely many times. Hence again we get (3), and the same argument as the above implies that the length of $F \circ C$ is infinite. $\square$

Let $w_1 \in \mathbf{C}$ be the finite asymptotic value of F along $\Gamma : [0,1) \to \mathbf{C}$, that is, $\lim_{t \to 1} F \circ \Gamma(t) = w_1$.

Claim 2. *There exists $r > 0$ such that any component of $F^{-1}(\mathbf{D}_r(w_1))$ which intersects I is bounded.*

Proof. By Lemma 1, there is $R > 0$ such that $\min_{s \in \mathbf{R}, |s| \geq R} |F(is)| \geq |w_1| + 1$. Increasing $R > 0$ if necessary, we assume that $w_1 \notin F(\{|z| = R\})$, so there is $r \in (0,1)$ such that $\mathbf{D}_r(w_1) \cap F(\{|z| = R\}) = \emptyset$. Then $F^{-1}(\mathbf{D}_r(w_1))$ intersects neither $I \cap \{|z| \geq R\}$ nor $\{|z| = R\}$, so any component of $F^{-1}(\mathbf{D}_r(w_1))$ intersecting with I is contained in $\{|z| < R\}$. $\square$

Fix $r > 0$ with the property claimed above. Fix $t_0 \in [0,1)$ such that

$$F \circ \Gamma([t_0, 1)) \subset \mathbf{D}_{r/4}(w_1). \tag{4}$$

Let Ω be a component of $F^{-1}(\mathbf{D}_r(w_1))$ which contains $\Gamma(t_0)$. Then Ω contains the whole $\Gamma([t_0, 1))$, so Ω is unbounded. Hence by Claim 2, Ω does not intersect with I, so is contained in either $\mathbf{H}_-$ or $\mathbf{H}_+$.

Assume contrary that the conclusion of the lemma does not hold. Then $\Omega \subset \mathbf{H}_+$. Let ϕ be a germ of a single-valued analytic branch of F^{-1} at $F(\Gamma(t_0))$ such that $\phi(F(\Gamma(t_0))) = \Gamma(t_0)$. Then ϕ is holomorphic on the disc $\mathbf{D}_{r/2}(F(\Gamma(t_0)))$, or else there exists a largest disk $\mathbf{D}_\rho(F(\Gamma(t_0)))$ with $\rho \in (0, r/2)$ to which ϕ can be extended analytically.

But the latter cannot occur: for there would then be a point $\xi \in \partial \mathbf{D}_\rho(F(\Gamma(t_0)))$ over which ϕ cannot extend analytically. Let α be the radial segment $[0,1] \ni s \mapsto F(\Gamma(t_0)) + s(\xi - F(\Gamma(t_0))) \in \mathbf{D}_{r/2}(F(\Gamma(t_0)))$ joining $F(\Gamma(t_0))$ and ξ. Since $\mathbf{D}_{r/2}(F(\Gamma(t_0))) \subset \mathbf{D}_{3r/4}(w_1)$, the curve $\phi \circ \alpha \,|\, [0,1)$ is contained in Ω, so in $\mathbf{H}_+$. Since the unique critical point $-a$ of F is in $\mathbf{H}_-$, Iversen's theorem yields that the curve $\phi \circ \alpha \,|\, [0,1)$ tends to ∞. On the other hand, $\phi \circ \alpha \,|\, [0,1)$ is real-analytic and the length of $F \circ (\phi \circ \alpha \,|\, [0,1)) = \alpha \,|\, [0,1)$ is finite, so by Claim 1, $\phi \circ \alpha \,|\, [0,1)$ cannot tend to ∞. This is a contradiction.

Thus ϕ is holomorphic on $\mathbf{D}_{r/2}(F(\Gamma(t_0)))$, which contains $F \circ \Gamma([t_0, 1))$ by (4). Hence $\lim_{t\to 1} \Gamma(t) = \phi(\lim_{t\to 1} F \circ \Gamma(t)) = \phi(w_1)$, which contradicts that Γ tends to ∞.

Now the proof is complete. $\square$

For a domain D in $\mathbf{C}$, a subset c in D is called a crosscut (or a transverse arc) of D if c is homeomorphic to $(0, 1)$, the closure $\bar{c}$ in $\mathbf{C}$ is homeomorphic to $[0, 1]$ and $\bar{c} \cap \partial D$ consists of two points.

For each $r > 0$, put $\mathbf{D}_r := \mathbf{D}_r(0) = \{w \in \mathbf{C}; |w| < r\}$.

Lemma 3. *For every $R > 0$, $F^{-1}(\mathbf{D}_R) \cap \mathbf{H}_+$ has no unbounded components.*

Proof. Let Ω be a component of $F^{-1}(\mathbf{D}_R) \cap \mathbf{H}_+$. From Lemma 1, $(\partial\Omega) \cap I$ has at most finitely many components, which are closed intervals. The image of each component of $(\partial\Omega) \cap I$ under F is a real-analytic curve in $\overline{\mathbf{D}_R}$, and $\overline{\mathbf{D}_R} \backslash F((\partial\Omega) \cap I)$ has at most finitely many components. Fix a triangulation of $\overline{\mathbf{D}_R}$ such that the interior of any triangle is contained in $\mathbf{D}_R \backslash F((\partial\Omega) \cap I)$.

As convention, we call the interior of each triangle an *open triangle*.

Claim 1. *For every open triangle V and every component U of $F^{-1}(V) \cap \Omega$, U is bounded and the restriction $F_{\overline{U}}$ of F on $\overline{U}$ is a homeomorphism from $\overline{U}$ onto $\overline{V}$.*

Proof. Fix $z_0 \in U$. By $F'(z_0) \neq 0$, there is a germ ϕ of a single-valued branch of F^{-1} with $\phi(F(z_0)) = z_0$. Assume that there is a curve $\gamma : [0, 1] \to \overline{V}$ with $\gamma(0) = F(z_0)$ such that the analytic continuation of ϕ along $\gamma \mid [0, t]$ is possible for any $t \in [0, 1)$, but impossible for $t = 1$.

Since the unique critical point $-a$ of F is in $\mathbf{H}_-$, by Iversen's theorem, the curve $\phi \circ \gamma$ is an asymptotic curve of F with the finite asymptotic value $\gamma(1) \in \mathbf{C}$. Then by Lemma 2, there is $t_0 \in [0, 1)$ such that $\phi \circ \gamma(t_0) \in \mathbf{H}_-$. On the other hand, from $F(I) \cap V = \emptyset$, U is a component of $F^{-1}(V)$. Moreover, $\overline{U}$ is a component of $F^{-1}(\overline{V})$ since there is no critical point of F on I. Thus the curve $\phi \circ \gamma$ is in $\overline{U}$, so in $\mathbf{H}_+ \cup I$. This contradicts that $\phi \circ \gamma(t_0) \in \mathbf{H}_-$.

We have shown that ϕ extends analytically along all curves in $\overline{V}$. Now by the monodromy theorem, a single-valued continuous branch $F^{-1} : \overline{V} \to \overline{U}$ exists. Hence U is bounded and $F_{\overline{U}} : \overline{U} \to \overline{V}$ is homeomorphic. $\square$

Let N be the number of triangles in $\overline{\mathbf{D}_R}$.

Claim 2. *There is an increasing sequence of closed sets*

$$D_1 \subset D_2 \subset \cdots \subset D_N = \overline{\mathbf{D}_R}$$

such that for each $j \in \{1, \ldots, N\}$, D_j consists of j triangles and int D_j *is connected and simply connected.*

Proof. This is clear if $N = 1$, so we assume that $N \geq 2$. The construction is decreasingly inductive. For $j = N$, $D_N = \overline{\mathbf{D}_R}$ consists of N triangles and $\operatorname{int} D_N = \mathbf{D}_R$ is connected and simply connected. Fix $j \in \{1, \dots, N-1\}$, and suppose that we obtain a closed set D_{j+1} consisting of $j+1$ triangles such that $\operatorname{int} D_{j+1}$ is connected and simply connected.

Let $\mathcal{S}_j$ be the set of all triangles Δ in D_{j+1} having an edge in ∂D_{j+1} such that $\operatorname{int}(D_{j+1} \setminus \Delta)$ is not connected. Let us find a triangle Δ_j in D_{j+1} which has an edge in ∂D_{j+1} and does not belong to $\mathcal{S}_j$. We can certainly do this when $\mathcal{S}_j = \emptyset$. Suppose that $\mathcal{S}_j \neq \emptyset$. For each $\Delta \in \mathcal{S}_j$, there are two components P and P' of $\operatorname{int}(D_{j+1} \setminus \Delta)$ and put $N(\Delta)$ be the minimum of the number of triangles in $\bar{P}$ and that of $\overline{P'}$. Fix a triangle $\Delta \in \mathcal{S}_j$ satisfying

$$N(\Delta) = \min_{\Delta' \in \mathcal{S}_j} N(\Delta'), \tag{5}$$

and a component P of $\operatorname{int}(D_{j+1} \setminus \Delta)$ such that $\bar{P}$ consists of $N(\Delta)$ triangles. Then any triangle Δ_j in $\bar{P}$ having an edge in $(\partial D_{j+1}) \cap \bar{P}$ will not belong to $\mathcal{S}_j$: for, if $\Delta_j \in \mathcal{S}_j$, then there is a component of $\operatorname{int}(D_{j+1} \setminus \Delta_j)$, which is a subset of $\operatorname{int}(P \setminus \Delta_j)$, so $N(\Delta_j) < N(\Delta)$. This contradicts (5).

With such Δ_j, set $D_j := \overline{D_{j+1} \setminus \Delta_j}$. Then $\operatorname{int} D_j$ is connected, and moreover $(\partial \Delta_j) \cap (\operatorname{int} D_{j+1})$ is a crosscut of $\operatorname{int} D_{j+1}$, so $\operatorname{int} D_j$ is simply connected. □

Let (D_j) be the increasing sequence of closed sets obtained in Claim 2. We show by induction that for each $j \in \{1, \dots, N\}$, $F^{-1}(\operatorname{int} D_j) \cap \mathbf{H}_+$ has no unbounded components.

For $j = 1$, D_1 is a single triangle. This case is covered by Claim 1.

Fix $j \in \{1, \dots, N-1\}$, and suppose the assertion holds for D_j. Put $C := (\partial D_j) \cap \operatorname{int} D_{j+1}$, which is a crosscut of $\operatorname{int} D_{j+1}$. Assume that a component Ω_0 of $F^{-1}(\operatorname{int} D_{j+1}) \cap \mathbf{H}_+$ is unbounded. Let c_1 be a component of $F^{-1}(C) \cap \Omega_0$. Since F has no critical point in Ω_0, c_1 is a crosscut of Ω_0, and $\Omega_0 \setminus c_1$, which is possibly still connected, has an unbounded component Ω_1.

Let U_1 be a component of $\Omega_1 \setminus F^{-1}(C)$ such that $c_1 \subset \partial U_1$. Then U_1 is a component of either $F^{-1}(\operatorname{int} D_j) \cap \Omega$ or $F^{-1}(\operatorname{int} \Delta_j) \cap \Omega$. In either case, by the assumption for j and the assertion for $j = 1$, U_1 is bounded. Hence $\Omega_1 \setminus \overline{U_1}$ has an unbounded component Ω_2. Let c_2 be a component of $\partial \Omega_2 \cap \Omega_1$. Then c_2 is a crosscut of Ω_1 and $c_2 \subset \partial U_1$.

Let U_2 be a component of $\Omega_2 \setminus F^{-1}(C)$ such that $c_2 \subset \partial U_2$. Then U_2 is bounded by the same reason. Let c_3 be a component of $\partial U_2 \cap \Omega_2$, which is a crosscut of Ω_2.

Now $c_2 \subset \partial U_1 \cap \partial U_2$. Hence at least one of U_1 and U_2, say U_*, is a component of $F^{-1}(\operatorname{int} \Delta_j) \cap \Omega$. By Claim 1, F restricts to a homeomorphism from $\partial(U_*)$ to $\partial \Delta_j$. But $F^{-1}(C) \cap \partial(U_*)$ contains not only c_2 but also either c_1 or c_3, which is a contradiction.

Hence $F^{-1}(\operatorname{int} D_{j+1}) \cap \mathbf{H}_+$ also has no unbounded components. This completes the induction.

This applies in particular to $\operatorname{int} D_N = \mathbf{D}_R$. Hence $F^{-1}(\mathbf{D}_R) \cap \mathbf{H}_+$ has no unbounded components, which completes the proof. □

Lemma 4. *For every*

$$z \in \mathbf{H}_- \cup I, \quad |F(z)| \le \max\{|z|, |a|\}^{n+1} \cdot 2^n \max_{|\zeta| \le e^a} |\zeta f(\zeta)|.$$

Proof. For each $z \in \mathbf{H}_- \cup I$,

$$|F(z)| = \left| \int_{[a, a+z]} z^n f(e^z) e^z \, dz \right| \le (|z| + |a|)^n |z| \max_{|\zeta| \le e^a} |\zeta f(\zeta)|,$$

where $[a, a+z]$ is the closed segment joining a and $a+z$. □

Lemma 5. *For each* $r > 0$, *put*

$$\mu_+(r) := \min\{|F(z)|; z \in \mathbf{H}_+ \cup I, |z| = r\}.$$

If f *is transcendental, then* $\liminf_{r \to \infty} \mu_+(r) \le 1$.

Proof. For every $r > 0$, put

$$D_r := \{r/4 < |z| < 2r\} \cap \mathbf{H}_+.$$

Then $D_1 \cong D_r$ under the similarity $z \mapsto rz$. Let $\varphi : \mathbf{D}_1 \to D_1$ be a (inverse of) Riemann mapping such that $\varphi(0) = 1 \in D_1$. For every $r > 0$, the conformal map

$$\varphi_r := r \cdot \varphi : \mathbf{D}_1 \to D_r$$

satisfies that $\varphi_r(0) = r \in D_r$ and extends to a homeomorphism from $\overline{\mathbf{D}_1}$ onto $\overline{D_r}$. The Poisson kernel on $\mathbf{D}_1$ is

$$P(w, \xi) := \Re\left(\frac{\xi + w}{\xi - w}\right) = \frac{1 - |w|^2}{|\xi - w|^2}$$

for $w \in \mathbf{D}_1$ and $\xi \in \partial \mathbf{D}_1$. For each $w \in \mathbf{D}_1$, $P(w, \xi)|d\xi|/(2\pi)$ is a probability measure on $\partial \mathbf{D}_1$, and more specifically, the harmonic measure for $\mathbf{D}_1$ with pole at w (for the details, see, e.g., [6, §1.2]).

Assume that

$$\liminf_{r \to \infty} \mu_+(r) > 1.$$

Then there is $r_0 > 0$ such that $\log|F|$ is positive and harmonic on $\{|z| > r_0\} \cap (\mathbf{H}_+ \cup I)$.

Let us compare $\log|F(r)|$ and $\log|F(r/2)|$ for each $r > 4r_0$. Since $\log|F \circ \varphi_r|$ is positive and harmonic on $\mathbf{D}_1$, Harnack's inequality (cf. [6, Theorem 1.3.1]) yields

$$\frac{1 - |\varphi^{-1}(1/2)|}{1 + |\varphi^{-1}(1/2)|} \log|F(r)| \le \log|F(r/2)|$$

(we note that $\log|F(r)| = \log|F \circ \varphi_r(0)|$ and that $\log|F(r/2)| = \log|F \circ \varphi_r(\varphi^{-1}(1/2))|$). Hence we have

$$\log|F(r)| \le C_0 \log|F(r/2)|,$$

where we put $C_0 := \dfrac{1 + |\varphi^{-1}(1/2)|}{1 - |\varphi^{-1}(1/2)|} > 1$.

A repeated use of this estimate implies that

$$\log|F(r)| \le C_0^{\max\{j \in \mathbf{N};\, r/2^j > 2r_0\}} \cdot \max_{s \in [2r_0, 4r_0]} \log|F(s)| \le C_1 r^{\alpha}, \tag{6}$$

where we put $\alpha := \log_2 C_0 > 0$ and $C_1 := (2r_0)^{-\alpha} \max_{s \in [2r_0, 4r_0]} \log|F(s)| > 0$.

Let us next compare $\log|F(z)|$ and $\log|F(|z|)|$ for each $z \in \mathbf{H}_+$ with $|z| > 4r_0$. Fix $z \in \mathbf{H}_+$ with $|z| > 4r_0$ and put $r = |z|$. Then $z \in D_r$. Let us decompose ∂D_r into the disjoint subsets $I_r := (\partial D_r) \cap I$ and $S_r := (\partial D_r) \setminus I$. Then $\varphi_r^{-1}(I_r) = \varphi^{-1}(I_1)$ and $\varphi_r^{-1}(S_r) = \varphi^{-1}(S_1)$, and

$$\begin{aligned} \log|F(z)| = &\int_{\varphi^{-1}(I_1)} (\log|F(\varphi_r(\xi))|) P(\varphi_r^{-1}(z), \xi) \frac{|d\xi|}{2\pi} \\ &+ \int_{\varphi^{-1}(S_1)} (\log|F(\varphi_r(\xi))|) P(\varphi_r^{-1}(z), \xi) \frac{|d\xi|}{2\pi}. \end{aligned} \tag{7}$$

Increasing $r_0 > 0$ if necessary, Lemma 4 implies that

$$\log|F(it)| \le 2 \log(|t|^{n+1}) = 2(n+1) \log|t|$$

for every $t \in \mathbf{R}$ with $|t| > r_0$. Since $I_r \subset \{it \in \mathbf{R}; |t| < 2r\}$,

$$\int_{\varphi^{-1}(I_1)} (\log|F(\varphi_r(\xi))|) P(\varphi_r^{-1}(z), \xi) \frac{|d\xi|}{2\pi} \le 2(n+1) \log(2r). \tag{8}$$

Put $c := \varphi_r^{-1}(\{z \in D_r; |z| = r\})$, which is a crosscut of $\mathbf{D}_1$. Note that

$$\{\eta\xi^{-1} \in \overline{\mathbf{D}_1}; \eta \in \bar{c}, \xi \in \varphi^{-1}(\overline{S_1})\}$$

is compact in $\overline{\mathbf{D}_1}$ and does not contain 1. Put

$$C_2 := \max\{P(\eta\xi^{-1}, 1); \eta \in \bar{c}, \xi \in \varphi^{-1}(\overline{S_1})\} < \infty.$$

We note that $P(\eta, \xi) = P(\eta\xi^{-1}, 1)$ for every $\xi \in \partial\mathbf{D}_1$. Since $\varphi_r^{-1}(z) \in c$ and $\log|F| \ge 0$ on ∂D_r, we have

$$\int_{\varphi^{-1}(S_1)} (\log|F(\varphi_r(\xi))|) P(\varphi_r^{-1}(z), \xi) \frac{|d\xi|}{2\pi} \le C_2 \int_{\partial\mathbf{D}_1} \log|F(\varphi_r(\xi))| \frac{|d\xi|}{2\pi} = C_2 \log|F(|z|)|,$$

where the final equality follows from the mean value property of harmonic functions (we note that $\log|F(\varphi_r(0))| = \log|F(r)|$ and that $r = |z|$). This with (7) and (8) concludes

$$\log|F(z)| \le 2(n+1) \log|2z| + C_2 \log|F(|z|)|.$$

From this estimate with (6), on $\mathbf{H}_+$,

$$\log^+|F(z)| = O(\log|z|) + O(|z|^\alpha)$$

as $|z| \to \infty$. This with Lemma 4 implies that the order of F is finite. Thus by the definition of F, the order of $f(e^{z+a})$ is also finite.

On the other hand, we can show that the order of $f(e^{z+a})$ is infinite, which will prove our lemma by contradiction. Since $f(\mathbf{C}\backslash\{0\}) \subset \mathbf{C}\backslash\{0\}$, we can write as $f(\zeta) = \zeta^k e^{H(\zeta)}$ with some $k \in \mathbf{N} \cup \{0\}$ and some entire function $H(\zeta)$. By the assumption that f is transcendental, H is non-constant. Hence by Hadamard's theorem (cf. [1, p. 209]), the order of f is greater than or equal to one. Hence the order of $f(e^{z+a})$ is infinite. This is a contradiction.

Thus we have proved $\liminf_{r\to\infty} \mu_+(r) > 1$. □

Let us complete the proof of Theorem.

Assume that f is transcendental. Fix $R_1 > \max\{1, |F(-a)|\}$. Then by Lemma 5, $F^{-1}(\mathbf{D}_{R_1}) \cap \mathbf{H}_+$ is unbounded, and then by Lemma 3, there are infinitely many (bounded) components of $F^{-1}(\mathbf{D}_{R_1}) \cap \mathbf{H}_+$. By Lemma 1, the boundaries of at most finitely many components of $F^{-1}(\mathbf{D}_{R_1}) \cap \mathbf{H}_+$ intersect I, so the other (infinitely many) components of $F^{-1}(\mathbf{D}_{R_1}) \cap \mathbf{H}_+$ are all relatively compact in $\mathbf{H}_+$. Let V and W be distinct such components. Then $\overline{V} \cap \overline{W} = \emptyset$ since F has no critical point in $\mathbf{H}_+$. Join $\overline{V}$ and $\overline{W}$ by a compact line segment l, take $R_1' > \max_{z\in l}|F(z)|(\geq R_1)$ and let Ω_1 be the component of $F^{-1}(\mathbf{D}_{R_1'})$ such that $l \subset \Omega_1$. Then $\overline{V} \cup \overline{W} \subset \Omega_1 \cap F^{-1}(\overline{\mathbf{D}_{R_1}})$. Put $A_1 := \{R_1 < |w| < R_1'\}$ and

$$\Omega_1' := \Omega_1 \backslash F^{-1}(\overline{\mathbf{D}_{R_1}}).$$

Then Ω_1' is a component of $F^{-1}(A_1)$ and is at least triply-connected. The restriction

$$F_{\Omega_1'} : \Omega_1' \to A_1$$

is locally homeomorphic, i.e., has no critical point since $F(-a) \notin A_1$. If F has also no asymptotic curve in Ω_1' with a finite asymptotic value in A_1, then Iversen's theorem concludes that $F_{\Omega_1'}$ have the curve lifting property, that is, any closed curve may be lifted uniquely under $F_{\Omega'}$ given any preimage of the initial point (for the details, see, e.g., [2, Definition 4.13]), and then by [2, Theorem 4.19], the local homeomorphism $F_{\Omega_1'}$ must be a covering. Since the universal covering of A_1 is topologically a disk and $\pi_1(A_1)$ is $\mathbf{Z}$, $\pi_1(\Omega_1')$ must be either $\mathbf{Z}$ or $\{1\}$, so Ω_1' must be topologically either an annulus or a disk. This contradicts that Ω_1' is at least triply connected.

Hence F has an asymptotic curve $\Gamma_1 \subset \Omega_1'$ with a finite asymptotic value a_1 in $A_1 = \{R_1 < |w| < R_1'\}$. By Lemma 2, we may assume that $\Gamma_1 \subset \Omega_1' \cap \mathbf{H}_-$.

Fix $R_2 > R_1'$. By the same argument applied to R_2, we obtain $R_2' > R_2$, a component Ω_2' of $F^{-1}(\{R_2 < |w| < R_2'\})$ and an asymptotic curve $\Gamma_2 \subset \Omega_2' \cap \mathbf{H}_-$ of F with a finite asymptotic value $a_2 \in \{R_2 < |w| < R_2'\}$. We note that $\Gamma_1 \cap \Gamma_2 = \emptyset$. Without loss of generality, we assume that both Γ_1 and Γ_2 are simple.

Let U be an unbounded domain contained in $\mathbf{H}_-$ such that

$$\partial U = \Gamma_1 \cup \Gamma_2 \cup c,$$

where c is an arc joining the endpoint of Γ_1 and that of Γ_2 in $\mathbf{H}_-$. Since $a_1 \neq a_2$, by Lindelöf's theorem (cf. [4, p. 65]),

$$\sup_U |F| = \infty.$$

On the other hand, we can show that $|F|$ is bounded on U, which will prove the Theorem by contradiction. First, we note that

$$M := \max_{z \in \partial U} \log|F(z)| < \infty.$$

For a bounded domain $D \subset \mathbf{C}$, let $(z, E) \mapsto \omega_D(z, E)$ be the harmonic measure of D, where $z \in D$ and $E \subset \partial D$ is a Borel subset. For the details, see, e.g., [6, §4.3].

Fix $z_0 \in U$. For every $r > |z_0|$, let U_r be the component of $U \cap \mathbf{D}_r$ which contains z_0. Then by the two constant theorem (cf. [6, p. 101]),

$$\begin{aligned}\log|F(z_0)| &\le M\omega_{U_r}(z_0, \partial U_r \backslash \partial \mathbf{D}_r) + \left(\sup_{z \in \mathbf{D}_r \cap \mathbf{H}_-} \log|F(z)|\right)\omega_{U_r}(z_0, \partial U_r \cap \partial \mathbf{D}_r)\\ &\le M \cdot 1 + \left(\sup_{z \in \mathbf{D}_r \cap \mathbf{H}_-} \log|F(z)|\right)\omega_{U_r}(z_0, \partial U_r \cap \partial \mathbf{D}_r),\end{aligned}$$

so we have

$$\log|F(z_0)| \le M + \limsup_{r \to \infty}\left(\left(\sup_{z \in \mathbf{D}_r \cap \mathbf{H}_-} \log|F(z)|\right)\omega_{U_r}(z_0, \partial U_r \cap \partial \mathbf{D}_r)\right). \tag{9}$$

By the monotonicity of harmonic measures (cf. [6, Corollary 4.3.9]),

$$\begin{aligned}\omega_{U_r}(z_0, \partial U_r \cap \partial \mathbf{D}_r) &\le \omega_{\mathbf{D}_r \cap \mathbf{H}_-}(z_0, \partial U_r \cap \partial \mathbf{D}_r)\\ &\le \omega_{\mathbf{D}_r \cap \mathbf{H}_-}(z_0, \partial \mathbf{D}_r \cap \mathbf{H}_-) = \frac{2}{\pi} \arg \frac{ir + z_0}{ir - z_0}\end{aligned}$$

(for the final equality, cf. [6, p. 100]). Hence as $r \to \infty$,

$$\omega_{U_r}(z_0, \partial U_r \cap \partial \mathbf{D}_r) = O(r^{-1}).$$

This with Lemma 4 implies that

$$\limsup_{r \to \infty}\left(\left(\sup_{z \in \mathbf{D}_r \cap \mathbf{H}_-} \log|F(z)|\right)\omega_{U_r}(z_0, \partial U_r \cap \partial \mathbf{D}_r)\right) \le 0.$$

Hence by (9), we conclude $\sup_U \log|F| \le M$ since $z_0 \in U$ is arbitrary. Now the proof is complete. □

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