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CORRECTIONS TO "EXTREMAL DISKS AND EXTREMAL SURFACES OF GENUS THREE"

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Abstract

We correct a result in "Extremal disks and extremal surfaces of genus three", Kodai Math. J. 28, no. 1 (2005), 111–130. In the paper we have shown that there exist 16 compact Riemann surfaces of genus three up to conformal equivalence in which two extremal disks are isometrically embedded. However we have three more of them up to conformal equivalence. In the present paper we give these three surfaces and show that they are hyperelliptic. We also determine the groups of automorphisms of them.

1. Introduction

A compact Riemann surface S of genus $g \geq 2$ is equipped with the metric induced by the hyperbolic metric $ds = 2|dz|/(1-|z|^2)$ of the unit disk $\Delta = \{z \in \mathbf{C}; |z| < 1\}$. Let $D(r)$ be a disk of hyperbolic radius $r > 0$ isometrically embedded in S. By Bavard [1] we know that the radius r satisfies the inequality

$$\cosh r \leq \frac{1}{2 \sin \beta_g}, \tag{1}$$

where $\beta_g = \pi/(12g-6)$. Let R_g be the radius which satisfies the equality in (1), that is, $R_g = \cosh^{-1}(1/(2\sin\beta_g))$. Then S is called an extremal surface if it admits a disk $D(R_g)$ (called an extremal disk). In the previous paper ([3]), we have shown that there exist 16 extremal surfaces of genus three up to conformal equivalence. However there are three more of them. The author found them in a study of non-orientable extremal surfaces of genus 6.

We have obtained all extremal surfaces of genus three in [3]. There are 1726 extremal surfaces up to conformal equivalence; 927 extremal surfaces up to

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conformal or anti-conformal equivalence, which are described by side-pairing patterns of the hyperbolic regular 30-gon. We denote by P_j $(j = 1, 2, \dots, 927)$ the regular 30-gon with a side-pairing pattern appeared in [2] and by P_j' the mirror image of P_j (P_j' is omitted if it gives the same side-pairings as those of P_j). We denote by S_j (resp. S_j') the surface obtained from P_j (resp. P_j') by identifying each pair of sides. Then the set of the 1726 extremal surfaces consists of S_j and S_j'. In [3] surfaces S_{382}, S_{631}, and S_{631}' were considered to admit a unique extremal disk, but it is false. In what follows, we write $\beta := \beta_3$ and $R := R_3$ for simplicity of notation.

Theorem 1.1. *The surfaces S_{382}, S_{631}, and S_{631}' are hyperelliptic ones admitting exactly two extremal disks. The centers of extremal disks embedded in each surface are described as in Table* 1, *where π denotes the natural projection from Δ onto the surface. For each surface the group of automorphisms is isomorphic to $\mathbf{Z}_2$, the cyclic group of order two.*

S	The centers of extremal disks	Aut S
S_{382}	$\pi(0),\ \pi\left(\dfrac{2\sin 4\beta}{\tanh R}i\right)$	$\mathbf{Z}_2$
$S_{631},\ S_{631}'$	$\pi(0),\ \pi\left(\dfrac{2\sin 4\beta}{\tanh R}i\right)$	$\mathbf{Z}_2$

Table 1. Three extremal surfaces

Consequently, part of Theorem 8 in [3] is corrected as follows:

1. Extremal surfaces with two extremal disks: there are *19* surfaces (*14* surfaces up to conformal or anti-conformal equivalence).
2. (2) Surfaces only with the trivial automorphism: there are *1605* surfaces (*857* surfaces up to conformal or anti-conformal equivalence).

2. Preliminaries

We shall describe our methods and the notation used in [3]. An extremal surface S of genus three is represented by a Fuchsian group of which fundamental region is a regular 30-gon P in the unit disk Δ. Then S is obtained by identifying each pair of sides suitably. We may assume that P lies in the unit disk Δ such that the vertices v_n of P satisfy $\arg v_n = (2n-1)\pi/30$, $n = 1, 2, \dots, 30$. We denote by C_n the side of P with vertices v_n and v_{n+1} ([3, Figure 1]) and by w_n the middle point of C_n. We denote by K_n the hyperbolic pentagon with vertices w_{n-1}, v_n, v_{n+1}, w_{n+1}, and the origin. Here subscripts are taken as modulo 30.

If $\pi(\zeta)$ $(\zeta \in P)$ is the center of an extremal disk in S, then ζ is on the following curves L_n or M_n provided that $\zeta \in K_n$ for some n:

$$L_n = L_{n,m} : \left| z - \frac{\tanh R}{2\cos(n-m)\beta} e^{i(n+m)\beta} \right| = \frac{\tanh R}{2|\cos(n-m)\beta|}$$

$$(m \not\equiv n + 15 \pmod{30}),$$

$$M_n = M_{n,m} : z = \frac{e^{2in\beta}}{\tanh R} - te^{i(n+m+15)\beta} \quad (t \in \mathbf{R}),$$

where m is determined by n according as C_n is paired with C_m. Drawing L_n and M_n in K_n for every n, we select points ζ of the intersection $(L_n \cup M_n) \cap (L_{n+1} \cup M_{n+1})$ in $K_n \cap K_{n+1}$ which satisfy a certain distance condition, namely, the hyperbolic distance between ζ and $A_{k,l}(\zeta)$ must be one of the restricted values for every side-pairing mappings $A_{k,l} : C_k \to C_l$ of P ([3, Lemma 1]). To verify that $\pi(\zeta)$ is the center of an extremal disk, it is sufficient to construct a Möbius transformation γ which is compatible with the side-pairing mappings of P and $\gamma(0) = \zeta$ (we note that $\pi(0)$ is the center of an extremal disk).

3. Three surfaces

In this section we shall prove Theorem 1.1. For a vertex v_{n+1} of the regular 30-gon, $\pi(v_{n+1})$ can be a candidate for the center of an extremal disk, namely, v_{n+1} can be in the intersection of $L_n \cup M_n$ and $L_{n+1} \cup M_{n+1}$. We see that there are 4 cases for v_{n+1} to be in the intersection. For example, we depict figures when $n = 15$ (Figures 1 and 2).

The author missed the pair of $M_{15,26}$ and $M_{16,5}$, precisely the pair of $M_{n,n+11}$ and $M_{n+1,n+20}$ for every n, in the previous paper, so that S_{382}, S_{631}, and S'_{631} were dropped from the set of surfaces with two extremal disks.

Since P'_{631} is the mirror image of P_{631}, we discuss only S_{382} and S_{631}. The points obtained from our methods are v_8, v_{18}, v_{28}, and the origin, where the first three are equivalent points. We put $\zeta = v_8$, which is equal to $(2i \sin 4\beta)/\tanh R$, and put $\gamma(z) = (\zeta - z)/(1 - \bar{\zeta}z)$. Then the compatibility of γ with the side-pairing mappings of P_{382} and P_{631} are described as follows.

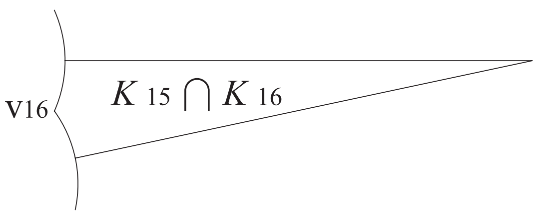

FIGURE 1. The intersection of K_{15} and K_{16}

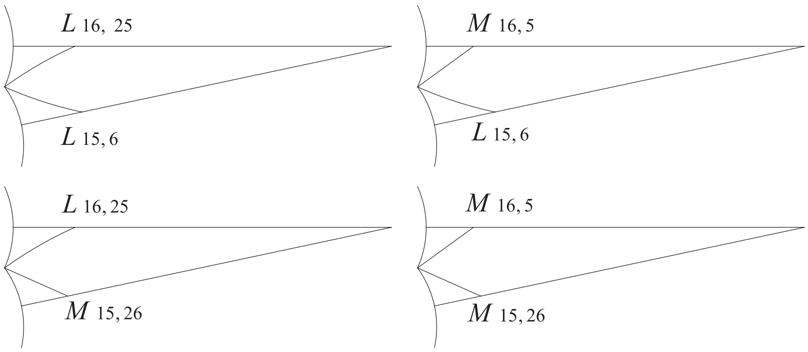

FIGURE 2. 4 pairs of curves in $K_{15} \cap K_{16}$ which touch the vertex v_{16}

P_{382}:

$$
\begin{aligned}
&\gamma A_{1,25} = A_{27,8}A_{25,1}A_{8,27}\gamma, && \gamma A_{2,9} = A_{9,2}\gamma,\\
&\gamma A_{3,26} = A_{27,8}A_{9,2}\gamma, && \gamma A_{4,30} = A_{27,8}A_{10,24}\gamma,\\
&\gamma A_{5,21} = A_{27,8}A_{22,29}A_{10,24}\gamma, && \gamma A_{6,13} = A_{13,6}\gamma,\\
&\gamma A_{7,18} = A_{18,7}\gamma, && \gamma A_{8,27} = A_{27,8}\gamma,\\
&\gamma A_{10,24} = A_{27,8}A_{4,30}\gamma, && \gamma A_{11,15} = A_{18,7}A_{5,21}\gamma,\\
&\gamma A_{12,19} = A_{18,7}A_{6,13}\gamma, && \gamma A_{14,20} = A_{18,7}A_{11,15}A_{6,13}\gamma,\\
&\gamma A_{16,23} = A_{18,7}A_{23,16}A_{7,18}\gamma, && \gamma A_{17,28} = A_{27,8}A_{7,18}\gamma,\\
&\gamma A_{22,29} = A_{27,8}A_{16,23}A_{7,18}\gamma.
\end{aligned}
$$

P_{631}:

$$
\begin{aligned}
&\gamma A_{1,15} = A_{3,6}A_{15,1}A_{9,12}\gamma, && \gamma A_{2,19} = A_{18,7}A_{9,12}\gamma,\\
&\gamma A_{3,6} = A_{9,12}\gamma, && \gamma A_{4,10} = A_{10,4}\gamma,\\
&\gamma A_{5,11} = A_{11,5}\gamma, && \gamma A_{7,18} = A_{18,7}\gamma,\\
&\gamma A_{8,27} = A_{27,8}\gamma, && \gamma A_{9,12} = A_{3,6}\gamma,\\
&\gamma A_{13,26} = A_{18,7}A_{28,17}A_{6,3}\gamma, && \gamma A_{14,20} = A_{18,7}A_{1,15}A_{6,3}\gamma,\\
&\gamma A_{16,23} = A_{18,7}A_{23,16}A_{7,18}\gamma, && \gamma A_{17,28} = A_{27,8}A_{7,18}\gamma,\\
&\gamma A_{21,25} = A_{27,8}A_{15,1}A_{7,18}\gamma, && \gamma A_{22,29} = A_{27,8}A_{16,23}A_{7,18}\gamma,\\
&\gamma A_{24,30} = A_{27,8}A_{30,24}A_{8,27}\gamma.
\end{aligned}
$$

Therefore $\pi(\zeta)$ is the center of an extremal disk.

The hyperellipticity of the surface is proved by the existence of the hyperelliptic involution. The automorphism induced from γ is the hyperelliptic

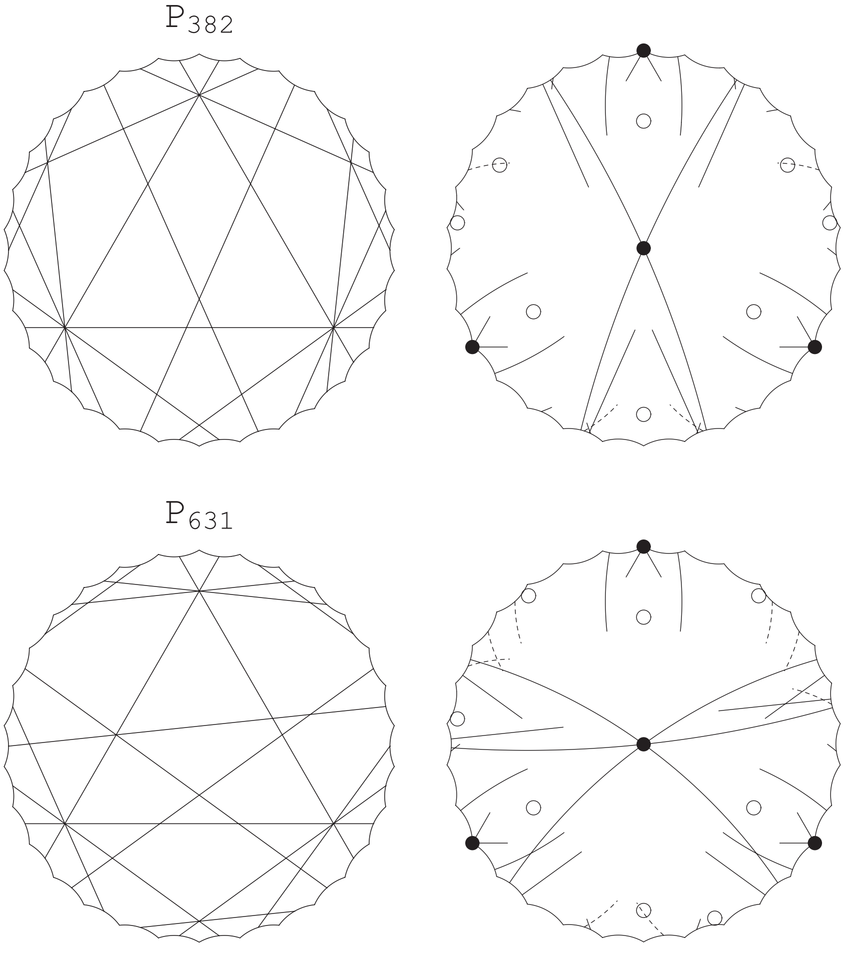

FIGURE 3. Left: Side-pairing patterns, Right: The centers of extremal disks (•) and the Weierstrass points (○)

involution of each surface because it is an involution with $8(=2g+2)$ fixed points. Every fixed point (the Weierstrass point) is derived from a fixed point of $B\gamma$, where B denotes an element of the Fuchsian group generated by every side-pairing mappings. Table 2 shows the fixed points of $B\gamma$, which are given by approximate values. Figure 3 shows the side-pairing patterns, the pre-images of the centers of extremal disks, the pre-images of the Weierstrass points, and the curves L_n and M_n (dotted ones intersect only with the boundary of the polygon). Since an automorphism fixes or interchanges the centers of extremal disks, it is verified that the group of automorphisms of each surface is generated only by the

B	Fixed points of $B\gamma$ in P_{382}	B	Fixed points of $B\gamma$ in P_{631}
id	$0.5349i$	id	$0.5349i$
$A_{6,13}$	$-0.6057+0.3497i$	$A_{5,11}$	$-0.4841+0.6249i$
$A_{7,18}$	$-0.4632-0.2674i$	$A_{7,18}$	$-0.4632-0.2674i$
$A_{8,27}$	$0.4632-0.2674i$	$A_{8,27}$	$0.4632-0.2674i$
$A_{9,2}$	$0.6057+0.3497i$	$A_{10,4}$	$0.4841+0.6249i$
$A_{11,15}A_{6,13}$	$-0.7833+0.1067i$	$A_{1,15}A_{6,3}$	$-0.7833+0.1067i$
$A_{16,23}A_{7,18}$	$-0.6994i$	$A_{16,23}A_{7,18}$	$-0.6994i$
$A_{25,1}A_{8,27}$	$0.7833+0.1067i$	$A_{30,24}A_{8,27}$	$0.2992-0.7317i$

Table 2. Weierstrass points

hyperelliptic involution. In fact, for the case of S_{382}, if $F \in \mathrm{Aut}\, S_{382}$ fixes $\pi(0)$, then we can take a lift $f : \Delta \to \Delta$ of F as a rotation around the origin for some angle $2n\beta$ ($n \in \mathbf{Z}$). Since a non-trivial rotation does not induce an automorphism (e.g., $\pi(v_5)$, $\pi(v_{22})$, and $\pi(v_{30})$ are the same point in S_{382}, but $\pi(f(v_5))$, $\pi(f(v_{22}))$, and $\pi(f(v_{30}))$ are not), F must be the identity. If F interchanges the two centers of extremal disks, then the composition TF^{-1} with the hyperelliptic involution T is the identity. Thus $F = T$. Hence $\mathrm{Aut}\, S_{382} = \langle T \rangle$. The similar proof works for the case of S_{631}.

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