K. WATANABE
KODAI MATH. J.
37 (2014), 453–474

PLANAR p-ELASTIC CURVES AND RELATED GENERALIZED COMPLETE ELLIPTIC INTEGRALS

Kohtaro Watanabe

Abstract

Planar elastica problem is a classical but has broad connections with various fields, such as elliptic functions, differential geometry, soliton theory, material mechanics, etc. This paper regards classical elastica as a theory corresponding to Lebesgue L^2 case, and extends it to L^p cases. For the sake of the effect of p-Laplacian, novel curious solutions appear especially for cases $p > 2$. These solutions never appear in $1 < p \leq 2$ cases and we call them flat-core solutions according to Takeuchi [6, 7].

1. Introduction

Let C be a plane curve with length L, s an arc-length parameter and $\kappa(s)$ its curvature. Further, we fix an orthogonal coordinate system (x, y) in the plane, and $\theta(s)$ be an angle between a tangent $(dx(s)/ds, dy(s)/ds)$ at the point $(x(s), y(s)) \in C$ and the positive x-axis. It is well-known that classical variational problem called "elastica", see for example Antman [1], Truesdell [8], minimizes the total squared curvature (elastic energy) of C:

$$(1) \qquad J(\theta) = \frac{1}{2}\int_0^L (\theta_s(s))^2 \, ds = \frac{1}{2}\int_0^L \kappa^2(s) \, ds,$$

subject to

$$\theta \in \left\{ \theta \in C^2[0, L] \,\middle|\, \int_0^L \cos\theta(s) \, ds = a, \int_0^L \sin\theta(s) \, ds = 0, \right.$$
$$\left. \theta(0) = 0, \, \theta(L) = 2n\pi \;\; (n \in \mathbf{N} \cup \{0\}) \right\},$$

2000 *Mathematics Subject Classification*. Primary 15A15, Secondary 15A09, 15A23.

Key words and phrases. elastica, p-Laplacian, generalized complete elliptic integrals, flat-core solution.

This work is partially supported by the Grant-in-Aid for Scientific Research (C) (No. 24540199) from Japan Society for the Promotion of Science.

Received November 5, 2013; revised December 10, 2013.

where a runs over the range $(-L \le a \le L)$. To show the existence of the minimizer, it is convenient to extend the definition domain of θ to some subset of Sobolev space $W^{1,2}(0,L)$:

$$W(2,a,n) := \left\{ \theta \in W^{1,2}(0,L) \,\middle|\, \int_0^L \cos\theta(s)\, ds = a, \int_0^L \sin\theta(s)\, ds = 0, \right.$$
$$\left. \theta(0) = 0,\ \theta(L) = 2n\pi \right\}.$$

The stationary curve, i.e. the solution of the Euler-Lagrange equation for the functional J is called elastic curves and their structures are well known: see Fig. 1 (Fig. 1 is the same one as Fig. 1.7 of Koiso [3]). As we see form this figure, elastic curves are intrinsically periodic. For this reason, we have imposed the condition $\theta(0) = 0,\ \theta(L) = 2n\pi\ (n \in \mathbf{N} \cup \{0\})$. As a natural extension, one may think of area constraint (if curves are restricted to be closed) as an additional constraint condition, for this aspect see; [10, 4]. This paper regards classical elastica problem as L^2 case and extends it to L^p problem, that is:

Minimize

$$J(\theta) = \frac{1}{p}\int_0^L |\theta_s(s)|^p\, ds = \frac{1}{p}\int_0^L |\kappa(s)|^p\, ds. \tag{2}$$

Subject to $\theta \in W(p,a,n)$:

$$W(p,a,n) := \left\{ \theta \in W^{1,p}(0,L) \,\middle|\, \int_0^L \cos\theta(s)\, ds = a, \int_0^L \sin\theta(s)\, ds = 0, \right.$$
$$\left. \theta(0) = 0,\ \theta(L) = 2n\pi \right\},$$

where a and n are as before. Although the existence of the minimizer can be readily shown with this setting; see Appendix, we are much more concerned with the detailed relation between the solutions of the Euler-Lagrange equation and the shapes of stationary curves. So, let us introduce the Euler-Lagrange equation for this case concretely. Since

$$x(L) = \int_0^L \cos\theta(s)\, ds = a, \quad y(L) = \int_0^L \sin\theta(s)\, ds = 0, \tag{3}$$

if θ is stationary, there exist Lagrange multipliers $\lambda_0, \lambda_1, \lambda_2 \in \mathbf{R}$, such that for arbitrary $\varphi \in W^{1,p}(0,L)$ satisfying $\varphi(0) = \varphi(L) = 0$ (since the values of $\theta(0)$ and $\theta(L)$ are fixed), it holds that

$$\lambda_0 \int_0^L |\theta_s|^{p-2}\theta_s\varphi_s(s)\, ds - \lambda_1 \int_0^L \sin\theta(s)\varphi(s)\, ds - \lambda_2 \int_0^L \cos\theta(s)\varphi(s)\, ds = 0.$$

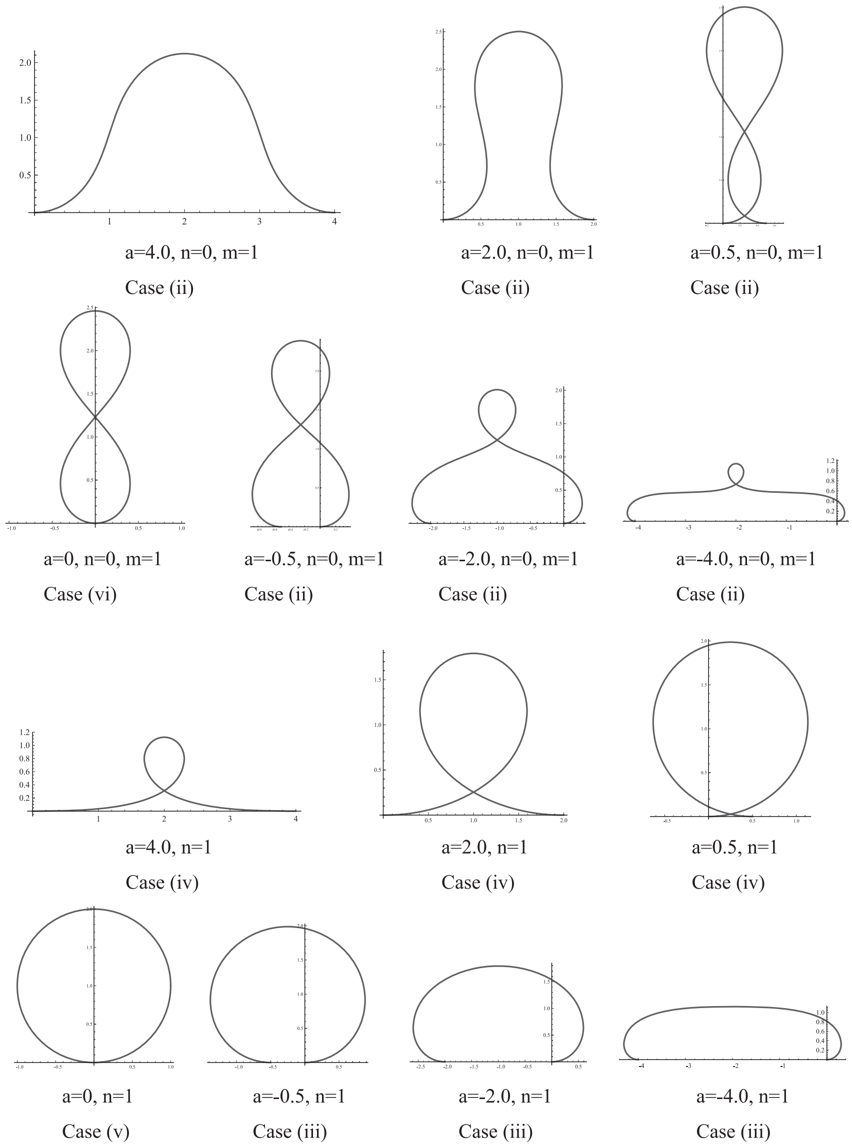


FIGURE 1. Elastic curves of various a, n and m for $L = 2\pi$, where the parameter m means the number of shape repetition (see Theorem 1).

If $\lambda_0 = 0$, $\theta \equiv 0$ holds, and hence the stationary curve is $(x, y) = (s, 0)$ $(0 \le s \le L)$. Thus, in the following we assume $\lambda_0 = 1$, and consider the equation:

$$(|\theta_s|^{p-2}\theta_s)_s = -\lambda_1 \sin\theta - \lambda_2 \cos\theta, \tag{4}$$

$$\theta(0) = 0, \quad \theta(L) = 2n\pi, \quad (n \in \mathbf{N} \cup \{0\}), \tag{5}$$

$$\int_0^L \cos\theta(s)\, ds = a, \quad (-L \le a \le L), \tag{6}$$

$$\int_0^L \sin\theta(s)\, ds = 0. \tag{7}$$

We call a solution which satisfies (4)–(7), p-elastic curve. Clearly, p-elastic curves are stationary because admissible functions φ satisfy

$$\int_0^L \sin\theta(s)\varphi(s)\, ds = 0, \quad \int_0^L \cos\theta(s)\varphi(s)\, ds = 0. \tag{8}$$

Since the equation (4) includes p-Laplacian term, as is seen in Takeuchi [6, 7], the value $p = 2$ divides qualitative behavior of the solution. Now, putting $\lambda_1 = R\cos\alpha$, $\lambda_2 = R\sin\alpha$ where $R = \sqrt{\lambda_1^2 + \lambda_2^2}$, we have from (4),

$$(|\theta_s|^{p-2}\theta_s)_s = -R\sin(\theta + \alpha). \tag{9}$$

Further, multiplying θ_s both sides of (9), we obtain L^p extension of energy conservation law:

$$\frac{p-1}{p}|\theta_s|^p = E + R\cos(\theta + \alpha), \tag{10}$$

where E is a constant (corresponding to a total energy). Here, E, R and α must be taken to satisfy (5), (6) and (7). We can assume without loss of generality, $\alpha = 0$. Since if solutions of (10) satisfying (5), (6) and (7) exists for $E = E_0$, $R = R_0$, $\alpha = \alpha_0$, it can be obtained by suitable parallel translation and rotating $-\alpha_0$ (rad) the curve generated for $E = E_0$, $R = R_0$, $\alpha = 0$. Thus, we analyze

$$\frac{p-1}{p}|\theta_s|^p = E + R\cos\theta, \tag{11}$$

for the following five cases:

$$\begin{cases} \text{(I)} & R = 0 \\ \text{(II)} & E = -R \\ \text{(III)} & -R < E < R \\ \text{(IV)} & E = R \\ \text{(V)} & R < E \end{cases} \tag{12}$$

where, for cases (II) to (V), we assume $R > 0$. In the next section, main results are stated, and in Section 3, we prove the results of Section 2, in accordance with above cases.

2. Main results

To state the results, we introduce some notations which are generalization of complete elliptic integrals:

DEFINITION 1.

$$K_{1,p}(q) = \int_0^{\pi/2} \frac{(\cos\phi)^{1-2/p}}{\sqrt{1-q^2\sin^2\phi}}\,d\phi \tag{13}$$

$$E_{1,p}(q) = \int_0^{\pi/2} \sqrt{1-q^2\sin^2\phi}(\cos\phi)^{1-2/p}\,d\phi \tag{14}$$

$$K_{2,p}(q) = \int_0^{\pi/2} \frac{1}{\sqrt[p]{1-q^2\sin^2\phi}}\,d\phi \tag{15}$$

$$E_{2,p}(q) = \int_0^{\pi/2} \sqrt[p]{1-q^2\sin^2\phi}\,d\phi \tag{16}$$

When $p = 2$, we see that these integrals coincide with complete elliptic integrals of first and second kinds.

Remark 1. In [7, p. 89], a generalization of complete elliptic integral is given by

$$K_{p,m,r}(q) = \int_0^1 \frac{ds}{\sqrt[p]{1-s^m}\sqrt[r]{1-q^m s^m}} = \int_0^{\pi/2} \frac{\cos\phi}{\sqrt[p]{1-\sin^m\phi}\sqrt[r]{1-q^m\sin^m\phi}}\,d\phi, \tag{17}$$

where $m > 1$ is a parameter. So, if we take $m = 2$, $r = 2$ or $p = 2$, $m = 2$, $r = p$ in (17), we obtain $K_{1,p}(q)$, $K_{2,p}(q)$ respectively. Nevertheless, we would like to use the notation in Definition 1 for the simplicity.

For $1 < p \le 2$, we have the following result:

THEOREM 1. *Let* $1 < p \le 2$. *Then, the solutions of* (4)–(7) *i.e. p-elastic curves* (x, y) *exist such that followings hold*:

(i) *The case* $a = \pm L$, *and* $n = 0$.

$$x(s) = \pm s, \quad y(s) = 0, \quad (0 \le s \le L).$$

(ii) *The case* $-L < a < L$ $(a \ne 0)$ *and* $n = 0$.

$$(18)\quad x(\phi) = \frac{L}{4m} K_{1,p}(q)^{-1} \left(2 \int_0^{\phi} \sqrt{1 - q^2 \sin^2 \varphi} |\cos \varphi|^{1-2/p} \, d\varphi - \int_0^{\phi} \frac{|\cos \varphi|^{1-2/p}}{\sqrt{1 - q^2 \sin^2 \varphi}} \, d\varphi \right)$$

$$y(\phi) = \pm \frac{L}{4m} K_{1,p}(q)^{-1} \int_0^{\phi} \frac{\sin\{2 \sin^{-1}(q \sin \varphi)\} |\cos \varphi|^{1-2/p}}{\sqrt{1 - q^2 \sin^2 \varphi}} \, d\varphi$$

where $m \in \mathbf{N}$, $0 \le \phi \le 2m\pi$ *and* q *is a unique solution of the equation*

$$(19)\qquad LK_{1,p}(q)^{-1}(2E_{1,p}(q) - K_{1,p}(q)) = a.$$

We note that the p-elastic curves generated by q correspond to case (III) *of* (12).

(iii) *The case* $-L < a < 0$ *and* $n \in \mathbf{N}$.

$$(20)\qquad x(\phi) = \frac{L}{2n} (K_{2,p}(q))^{-1} \int_0^{\phi} \frac{\cos 2\varphi}{\sqrt[p]{1 - q^2 \sin^2 \varphi}} \, d\varphi$$

$$y(\phi) = \frac{L}{2n} (K_{2,p}(q))^{-1} \int_0^{\phi} \frac{\sin 2\varphi}{\sqrt[p]{1 - q^2 \sin^2 \varphi}} \, d\varphi,$$

where $0 \le \phi \le n\pi$, *and* q *is a unique solution of the equation*

$$(21)\qquad L(K_{2,p}(q))^{-1} \left(\frac{q^2 - 2}{q^2} K_{2,p}(q) + \frac{2}{q^2} E_{2,\, p/(p-1)}(q) \right) = a.$$

(iv) *The case* $0 < a < L$ *and* $n \in \mathbf{N}$.

$$(22)\qquad x(\phi) = \frac{L}{2n} (K_{2,p}(q))^{-1} \int_0^{\phi} \frac{\cos 2\varphi}{\sqrt[p]{1 - q^2 \cos^2 \varphi}} \, d\varphi$$

$$y(\phi) = \frac{L}{2n} (K_{2,p}(q))^{-1} \int_0^{\phi} \frac{\sin 2\varphi}{\sqrt[p]{1 - q^2 \cos^2 \varphi}} \, d\varphi,$$

where $0 \le \phi \le n\pi$, *and* q *is a unique solution of the equation*

$$(23)\qquad -L(K_{2,p}(q))^{-1} \left(\frac{q^2 - 2}{q^2} K_{2,p}(q) + \frac{2}{q^2} E_{2,\, p/(p-1)}(q) \right) = a.$$

For cases (iii) *and* (iv), *we note that the p-elastic curves generated by q correspond to case* (V) *of* (12).

(v) *The case* $a = 0$ *and* $n \in \mathbf{N}$.
For this case, locus of (x, y) *is a circle of radius* $L/2n\pi$.

(vi) *The case* $a = 0$ *and* $n = 0$.

$$(24)\qquad x(\phi) = \frac{L}{4m} K_{1,p}(q)^{-1}\left(2\int_{\alpha}^{\phi+\alpha} \sqrt{1 - q^2 \sin^2 \varphi}|\cos \varphi|^{1-2/p}\, d\varphi - \int_{\alpha}^{\phi+\alpha} \frac{|\cos \varphi|^{1-2/p}}{\sqrt{1 - q^2 \sin^2 \varphi}}\, d\varphi\right)$$

$$y(\phi) = \pm\frac{L}{4m} K_{1,p}(q)^{-1} \int_{\alpha}^{\phi+\alpha} \frac{\sin\{2 \sin^{-1}(q \sin \varphi)\}|\cos \varphi|^{1-2/p}}{\sqrt{1 - q^2 \sin^2 \varphi}}\, d\varphi$$

where $m \in \mathbf{N}$, $0 \le \phi \le 2m\pi$, α *is an arbitrary angle satisfying* $0 \le \alpha < 2\pi$ *and* q *is a unique solution of the equation*

$$(25)\qquad LK_{1,p}(q)^{-1}(2E_{1,p}(q) - K_{1,p}(q)) = 0.$$

Figure 2 shows examples of *p*-elastic curves for $p = 1.2$ and $L = 2\pi$. For the case $p > 2$, we have the following theorem. In this case, thanks to the existence of *p*-Laplacian term, novel curious solutions called flat-core solutions appear. We note our *flat-core* solutions are essentially the same as the one introduced by Takeuchi [6, 7].

THEOREM 2. *Let* $p > 2$. *Then, the solutions of* (4)–(7) *i.e. p-elastic curves* (x, y) *exist such that followings hold*:

(i) *The case* $a = \pm L$, *and* $n = 0$.

$$x(s) = \pm s, \quad y(s) = 0, \quad (0 \le s \le L).$$

(ii) *The case* $-L/(p-1) < a < L$ $(a \neq 0)$ *and* $n = 0$.
For this case, (x, y) *satisfies* (18), *where* q *is the unique solution of* (19). *As Theorem* 1, *the p-elastic curves generated by* q *correspond to case* (III) *of* (12).

(iii) *The case* $-L/(p-1) < a < 0$ *and* $n \in \mathbf{N}$.
For this case, (x, y) *satisfies* (20), *where* q *is the unique solution of* (21). *As Theorem* 1, *the p-elastic curves generated by* q *correspond to case* (V) *of* (12).

(iv) *The case* $0 < a < L/(p-1)$ *and* $n \in \mathbf{N}$.
For this case, (x, y) *satisfies* (22), *where* q *is the unique solution of* (23). *As Theorem* 1, *the p-elastic curves generated by* q *correspond to case* (V) *of* (12).

(v) *The case* $a = 0$ *and* $n \in \mathbf{N}$.
For this case, locus of (x, y) *is a circle of radius* $L/2n\pi$.

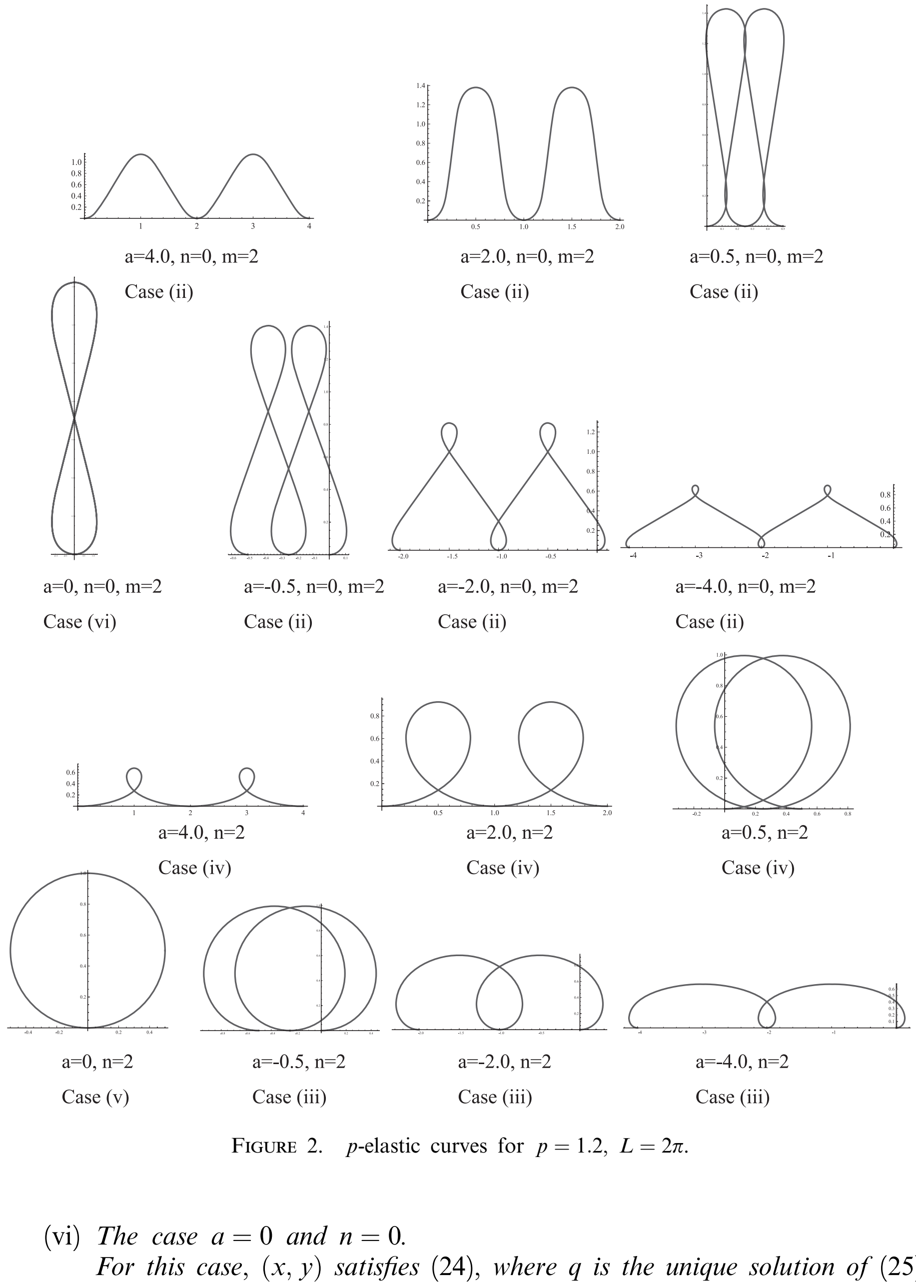


FIGURE 2. p-elastic curves for $p = 1.2$, $L = 2\pi$.

(vi) *The case* $a = 0$ *and* $n = 0$.
For this case, (x, y) *satisfies* (24), *where* q *is the unique solution of* (25).

To describe the existence result of flat core solutions, we introduce following definition.

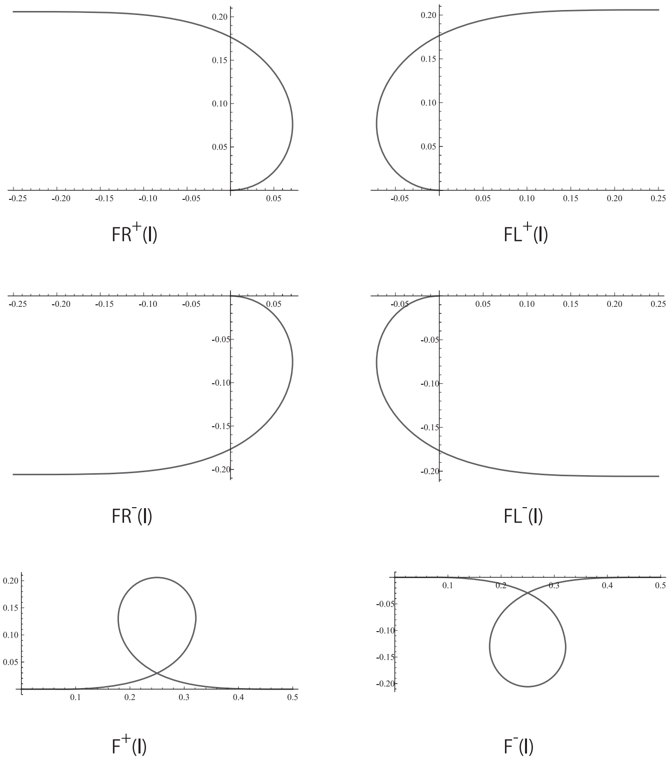


FIGURE 3. $FR^{+}(l)$, $FL^{+}(l)$, $FR^{-}(l)$, $FL^{-}(l)$, $F^{+}(l)$, and $F^{-}(l)$, when $l = 1$.

DEFINITION 2. We denote by $L(l)$ a line segment of length l parallel to x axis. Moreover, $FR^{+}(l)$, $FL^{+}(l)$, $FR^{-}(l)$, $FL^{-}(l)$, $F^{+}(l)$, $F^{-}(l)$ are curves defined by the following expressions. Here, $(x(\phi), y(\phi))$ $(0 \le \phi \le \pi/2)$ denote the locus of the curves; see Figure 3.

$$FR^{\pm}(l) : \begin{cases} x(\phi) = \dfrac{l\Gamma\left(\dfrac{p-1}{p}\right)}{\sqrt{\pi}\Gamma\left(\dfrac{p-2}{2p}\right)} \displaystyle\int_0^{\phi} \frac{\cos 2\varphi}{(\cos\varphi)^{2/p}}\, d\varphi \\ y(\phi) = \pm \dfrac{l\Gamma\left(\dfrac{p-1}{p}\right)}{\sqrt{\pi}\Gamma\left(\dfrac{p-2}{2p}\right)} \displaystyle\int_0^{\phi} \frac{\sin 2\varphi}{(\cos\varphi)^{2/p}}\, d\varphi \end{cases} \tag{26}$$

$$(27)\qquad FL^{\pm}(l):\begin{cases} x(\phi)=-\dfrac{l\Gamma\left(\dfrac{p-1}{p}\right)}{\sqrt{\pi}\Gamma\left(\dfrac{p-2}{2p}\right)}\displaystyle\int_0^{\phi}\frac{\cos 2\varphi}{(\cos\varphi)^{2/p}}\,d\varphi \\ y(\phi)=\pm\dfrac{l\Gamma\left(\dfrac{p-1}{p}\right)}{\sqrt{\pi}\Gamma\left(\dfrac{p-2}{2p}\right)}\displaystyle\int_0^{\phi}\frac{\sin 2\varphi}{(\cos\varphi)^{2/p}}\,d\varphi \end{cases}$$

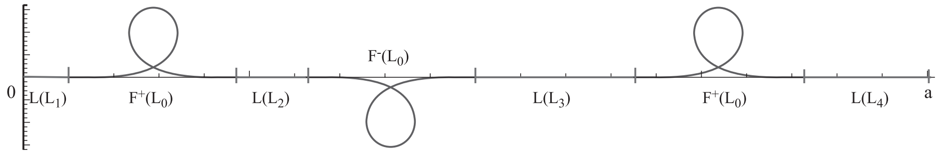
$$(28)\qquad F^{\pm}(l):\begin{cases} x(\phi)=\dfrac{l\Gamma\left(\dfrac{p-1}{p}\right)}{\sqrt{\pi}\Gamma\left(\dfrac{p-2}{2p}\right)}\displaystyle\int_0^{\phi}\frac{\cos 2\varphi}{(\sin\varphi)^{2/p}}\,d\varphi \\ y(\phi)=\pm\dfrac{l\Gamma\left(\dfrac{p-1}{p}\right)}{\sqrt{\pi}\Gamma\left(\dfrac{p-2}{2p}\right)}\displaystyle\int_0^{\phi}\frac{\sin 2\varphi}{(\sin\varphi)^{2/p}}\,d\varphi \end{cases}$$

In addition, we introduce following numbers: $n_+ \in \mathbf{N}$, $n_-, n_l \in \mathbf{N} \cup \{0\}$, $L_0 > 0$, $L_i > 0$ $(1 \le i \le n_l)$. With these notions, we obtain the structure theorem of flat core solutions.

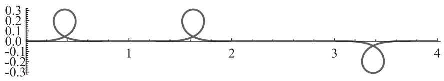
THEOREM 3. *Let $p > 2$. Then, the solutions of* (4)–(7) *i.e. p-elastic curves (x, y) are as cases* (i)–(vi) *of Theorem* 2. *In addition, we have the following case.*

(vii) (*Flat Core Solution Case* I) *The case $L/(p-1) \le a < L$ and $n \in \mathbf{N} \cup \{0\}$.*
Let n_+, n_-, n_0, be non-negative integers, $L_0 > 0$, $L_i \ge 0$ $(1 \le i \le n_0)$ satisfying $n = n_+ - n_-$ and $L = L_0(n_+ + n_-) + \sum_{i=1}^{n_0} L_i$ (if $n_0 = 0$, we assume $L = L_0(n_+ + n_-)$). Then, p-elastic curves consists of n_+ piece of $F^+(L_0)$, n_- piece of $F^-(L_0)$ and n_0 piece of line segments $L(L_i)$ $(1 \le i \le n_0)$ (if $n_0 = 0$, it means there is no line segments) joined each other in arbitrary order, but satisfying both end points are on $(0, 0)$ and $(a, 0)$; see Figure 4. *We express these solutions with $F^+(L_0)$, $F^-(L_0)$ and $\{L(L_i)\}$ $(1 \le i \le n_0)$, in accordance with joined order from left to right.*

(viii) (*Flat Core Solution Case* II-1) *The case $-L < a \le -L/(p-1)$ and $n \in \mathbf{N} \cup \{0\}$.*
Let n_+, n_-, n_0, $L_0 > 0$, $L_i \ge 0$ $(1 \le i \le n_0)$ are as case (vii) *and satisfying $n = n_+ - n_- + 1$, $L = L_0(n_+ + n_- + 1) + \sum_{i=1}^{n_0} L_i$ (if $n_0 = 0$, we assume $L = L_0(n_+ + n_- + 1)$). Then, p-elastic curves consists of $FL^+(L_0)$, $FR^+(L_0)$, n_+ piece of $F^-(L_0)$, n_- piece of $F^+(L_0)$ and n_0 piece of line segments $L(L_i)$ $(1 \le i \le n_0)$ joined each other in arbitrary*

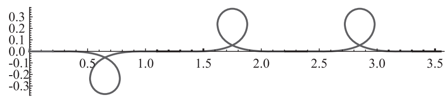


$L(L_1)F^+(L_0)L(L_2)F^-(L_0)L(L_3)F^+(L_0)L(L_4)$



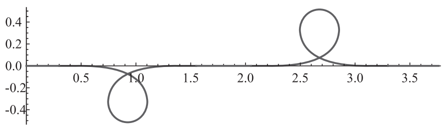
(i) $F^+(1.5)L(0.5)F^+(1.5)L(1)F^-(1.5)L(0.28)$

a=4.033, p=3, n=1



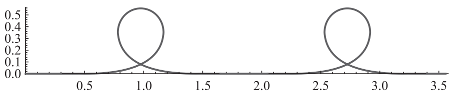
(ii) $L(0.2)$ $F^-(1.8)L(0.2)F^+(1.8)L(0.2)F^+(1.8)L(0.28)$

a=3.583, p=3, n=1



(iii) $L(0.3)F^-(2.5)L(0.5)F^+(2.5)L(0.48)$

a=3.783, p=3, n=0



(iv) $L(0.3)F^+(2.7)L(0.4)F^+(2.7)L(0.18)$

a=3.583, p=3, n=2

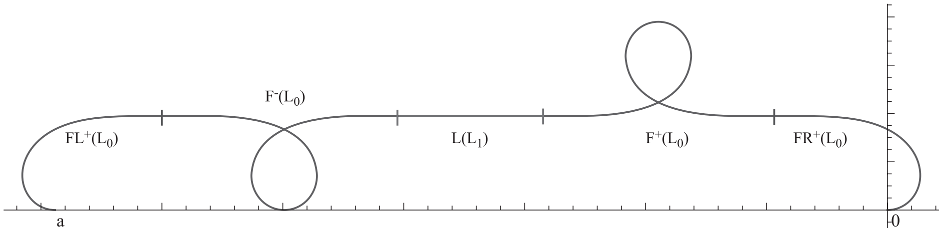
FIGURE 4. Flat core solutions of the case (vii) of Theorem 3.

order, but satisfying both ends are $FL^+(L_0)$ *and* $FR^+(L_0)$ *and end points of them are on* $(0,0)$ *and* $(a,0)$ *respectively; see Figure* 5-(i). *We express these solutions as in the case* (vii).

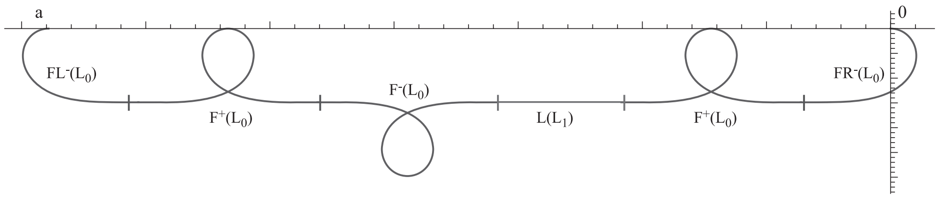
(ix) (*Flat Core Solution Case* II-2) *In the case* $-L < a \le -L/(p-1)$ *and* $n \in \mathbf{N} \cup \{0\}$.
Let n_+, n_-, n_0, L_0, L_i $(1 \le i \le n_0)$ *are as case* (vii) *and satisfying* $n = n_+ - n_- - 1$, $L = L_0(n_+ + n_- + 1) + \sum_{i=1}^{n_0} L_i$ (*if* $n_0 = 0$, *we fix* $L = L_0(n_+ + n_- + 1)$). *Then, p-elastic curves consists of* $FL^-(L_0)$, $FR^-(L_0)$, n_+ *piece of* $F^-(L_0)$, n_- *piece of* $F^+(L_0)$ *and* n_0 *piece of line segments* $L(L_i)$ $(1 \le i \le n_0)$ *joined each other in arbitrary order, but satisfying both ends are* $FL^-(L_0)$ *and* $FR^-(L_0)$ *and end points of them are on* $(0,0)$ *and* $(a,0)$ *respectively; see Figure* 5-(ii). *We express these solutions as in the case* (vii).

3. Proof of theorems

We investigate cases (I) to (V) of (12) in detail. We note that for the proof of theorems, the argument of Yanamoto [9] was helpful.



(i) $FL^{+}(L_0)F^{-}(L_0)L(L_1)F^{+}(L_0)FR^{+}(L_0)$



(ii) $FL^{-}(L_0)F^{+}(L_0)F^{-}(L_0)L(L_1)F^{+}(L_0)FR^{-}(L_0)$

FIGURE 5. Flat core solutions of the case (viii) and the case (ix) of Theorem 3.

The case (I) of (12).
In this case, the solution of (11) is $\theta_s \equiv$ Constant. This applies to cases (i) and (v) of Theorem 1 and 2.

The case (II) of (12).
In this case, the solution of (11) is $\theta \equiv 0$. This applies to cases (i) of Theorem 1 and 2.

The case (III) of (12).
From (11), we have

$$\frac{p-1}{p}|\theta_s|^p = E + R\cos\theta = 2R\left(\frac{E+R}{2R} - \sin^2\frac{\theta}{2}\right). \tag{29}$$

Putting $q^2 = (E+R)/2R$, and noting it must be $\theta(0) = \theta(L) = 0$ in this case, we have

$$\theta_s(s) = \left(\frac{2pR}{p-1}\right)^{1/p} \sqrt[p]{q^2 - \sin^2\frac{\theta}{2}}, \quad \left(0 \le s \le \frac{L}{4m}\right), \tag{30}$$

where m is a positive integer. Thus, we obtain

$$s = \left(\frac{p-1}{2pR}\right)^{1/p} \int_0^\theta \frac{d\theta}{\sqrt[p]{q^2 - \sin^2\dfrac{\theta}{2}}}, \quad (0 \le \theta \le 2\sin^{-1} q).$$

It is convenient to introduce the new variable ϕ satisfying

$$\sin\frac{\theta}{2} = q\sin\phi. \tag{31}$$

We note that as θ varies from 0 to $2\sin^{-1} q$, ϕ does from 0 to $\pi/2$. Using ϕ, we can see that s can be expressed as

$$s = 2q^{1-2/p}\left(\frac{p-1}{2pR}\right)^{1/p}\int_0^{\phi}\frac{(\cos\phi)^{1-2/p}}{\sqrt{1-q^2\sin^2\phi}}\,d\phi, \quad \left(0 \le \phi \le \frac{\pi}{2}\right).$$

Moreover, from the symmetry of θ we can extend the definition domain of $s = s(\phi)$ from $[0,\pi/2]$ to $[0,2\pi]$ as

$$s = 2q^{1-2/p}\left(\frac{p-1}{2pR}\right)^{1/p}\int_0^{\phi}\frac{|\cos\phi|^{1-2/p}}{\sqrt{1-q^2\sin^2\phi}}\,d\phi, \quad (0 \le \phi \le 2\pi). \tag{32}$$

From (32) and the relation $\phi(L/4m) = \pi/2$, it holds that

$$\frac{L}{4m} = 2q^{1-2/p}\left(\frac{p-1}{2pR}\right)^{1/p}\int_0^{\pi/2}\frac{(\cos\phi)^{1-2/p}}{\sqrt{1-q^2\sin^2\phi}}\,d\phi. \tag{33}$$

Thus R can be expressed as a function of q:

$$R(q) = R = \frac{2^{3p-1}m^p q^{p-2}(p-1)}{pL^p}\left(\int_0^{\pi/2}\frac{(\cos\phi)^{1-2/p}}{\sqrt{1-q^2\sin^2\phi}}\,d\phi\right)^p. \tag{34}$$

Applying (33), we can express $(x(L), y(L))$ as followings:

LEMMA 1.

$$x(L) = \int_0^L \cos\theta(s)\,ds = LK_{1,p}(q)^{-1}(2E_{1,p}(q) - K_{1,p}(q)),$$

$$y(L) = \int_0^L \sin\theta(s)\,ds = 0.$$

Proof. Changing the variable from s to ϕ with (32), we have

$$\begin{aligned}
x(L) &= \int_0^L \cos\theta(s)\,ds = m\int_0^{L/m}\cos\theta(s)\,ds \\
&= 2mq^{1-2/p}\left(\frac{p-1}{2pR}\right)^{1/p}\int_0^{2\pi}\frac{\cos\{2\sin^{-1}(q\sin\phi)\}|\cos\phi|^{1-2/p}}{\sqrt{1-q^2\sin^2\phi}}\,d\phi \\
&= 2mq^{1-2/p}\left(\frac{p-1}{2pR}\right)^{1/p}\int_0^{2\pi}\frac{1-2q^2\sin^2\phi}{\sqrt{1-q^2\sin^2\phi}}|\cos\phi|^{1-2/p}\,d\phi,
\end{aligned}$$

where in the last equality, the relation

$$\cos\{2\sin^{-1}(q\sin\phi)\} = 1 - 2\sin^2(\sin^{-1}(q\sin\phi)) = 1 - 2q^2\sin^2\phi.$$

was applied. Hence, using (33), we see that the right-hand-side of above equation becomes

$$L\left(\int_0^{\pi/2}\frac{(\cos\phi)^{1-2/p}}{\sqrt{1-q^2\sin^2\phi}}\,d\phi\right)^{-1}\int_0^{\pi/2}\frac{1-2q^2\sin^2\phi}{\sqrt{1-q^2\sin^2\phi}}|\cos\phi|^{1-2/p}\,d\phi$$

$$= L\left(\int_0^{\pi/2}\frac{(\cos\phi)^{1-2/p}}{\sqrt{1-q^2\sin^2\phi}}\,d\phi\right)^{-1}\cdot$$

$$\left(2\int_0^{\pi/2}\sqrt{1-q^2\sin^2\phi}(\cos\phi)^{1-2/p}\,d\phi - \int_0^{\pi/2}\frac{(\cos\phi)^{1-2/p}}{\sqrt{1-q^2\sin^2\phi}}\,d\phi\right)$$

$$= LK_{1,p}(q)^{-1}(2E_{1,p}(q) - K_{1,p}(q)).$$

Next, we compute $y(L)$. By definition,

$$y(L) = \int_0^L \sin\theta(s)\,ds = m\int_0^{L/m}\sin\theta(s)\,ds$$

$$= 2mq^{1-2/p}\left(\frac{p-1}{2pR}\right)^{1/p}\int_0^{2\pi}\frac{\sin\{2\sin^{-1}(q\sin\phi)\}|\cos\phi|^{1-2/p}}{\sqrt{1-q^2\sin^2\phi}}\,d\phi.$$

Noting the relation

$$\int_\pi^{2\pi}\frac{\sin\{2\sin^{-1}(q\sin\phi)\}|\cos\phi|^{1-2/p}}{\sqrt{1-q^2\sin^2\phi}}\,d\phi = -\int_0^{\pi}\frac{\sin\{2\sin^{-1}(q\sin\phi)\}|\cos\phi|^{1-2/p}}{\sqrt{1-q^2\sin^2\phi}}\,d\phi,$$

we obtain the second assertion. □

Here, we define the function X_1 as:

(35) $$X_1(q) = LK_{1,p}(q)^{-1}(2E_{1,p}(q) - K_{1,p}(q)).$$

Lemma 2. *For X_1, following properties hold.*
(i) *$X_1(q)$ is monotone decreasing on $(0,1)$.*
(ii) *$X_1(0) = L$.*
(iii)

(36) $$X_1(1) = \begin{cases} -L & (1 < p \le 2) \\ -\dfrac{L}{p-1} & (2 < p). \end{cases}$$

Proof. Differentiating $K_{1,p}$ and $E_{1,p}$, with q we have

$$\frac{dK_{1,p}}{dq}(q) = q\int_0^{\pi/2} \frac{(\sin\phi)^2(\cos\phi)^{1-2/p}}{(1-q^2\sin^2\phi)^{3/2}}\,d\phi > 0$$

$$\frac{dE_{1,p}}{dq}(q) = -q\int_0^{\pi/2} \frac{(\sin\phi)^2(\cos\phi)^{1-2/p}}{\sqrt{1-q^2\sin^2\phi}}\,d\phi < 0.$$

So, $K_{1,p}(q)$ and $E_{1,p}(q)$ are monotone increasing and decreasing on $(0,1)$ respectively. Thus $X_1(q) = L(2E_{1,p}(q)/K_{1,p}(q) - 1)$ is monotone decreasing on $(0,1)$. For (ii), we note that

$$E_{1,p}(0) = K_{1,p}(0) = \int_0^{\pi/2} (\cos\phi)^{1-2/p}\,d\phi = \frac{\sqrt{\pi}\,\Gamma\left(1-\frac{1}{p}\right)}{2\Gamma\left(\frac{3}{2}-\frac{1}{p}\right)} < \infty,$$

where $\Gamma(\cdot)$ is a Gamma function. From this, we obtain (ii). To see (iii), we note the relation

$$E_{1,p}(1) = \int_0^{\pi/2} (\cos\phi)^{2-2/p}\,d\phi = \frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}-\frac{1}{p}\right)}{2\Gamma\left(2-\frac{1}{p}\right)} \tag{37}$$

$$\lim_{q\to 1} K_{1,p}(q) = \int_0^{\pi/2} (\cos\phi)^{-2/p}\,d\phi = \begin{cases} \infty, & (1 < p \le 2) \\ \dfrac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\frac{1}{p}\right)}{2\Gamma\left(1-\frac{1}{p}\right)}, & (2 < p). \end{cases}$$

Thus, for the case $1 < p \le 2$ we have $X_1(1) = -L$. For the case $p > 2$, we have from (37) and the formula $z\Gamma(z) = \Gamma(z+1)$ ([5, Formula 5.51]),

$$X_1(1) = L\left(\frac{2\Gamma\left(1-\frac{1}{p}\right)\Gamma\left(\frac{3}{2}-\frac{1}{p}\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{p}\right)\Gamma\left(2-\frac{1}{p}\right)} - 1\right) = L\left(\frac{2\left(\frac{1}{2}-\frac{1}{p}\right)}{\left(1-\frac{1}{p}\right)} - 1\right) = -\frac{L}{p-1}. \quad \square$$

From Lemma 2, it holds that $X_1(q) = a$ has a unique root for $(-L < a < L)$ if $(1 < p \le 2)$ and for $(-L/(p-1) < a < L)$ if $(2 < p)$. Using this q and (32), (33), we can express locus (x, y) of p-elastic curve as

$$x(s) = \int_0^s \cos\theta(s)\, ds$$
$$= \frac{L}{4m}(K_{1,p}(q))^{-1}\int_0^{\phi} \cos 2(\sin^{-1}(q\sin\varphi))\frac{|\cos\varphi|^{1-2/p}}{\sqrt{1-q^2\sin^2\varphi}}\,d\varphi$$
$$= \frac{L}{4m}(K_{1,p}(q))^{-1}\int_0^{\phi}(1-2q^2\sin^2\varphi)\frac{|\cos\varphi|^{1-2/p}}{\sqrt{1-q^2\sin^2\varphi}}\,d\varphi = x(\phi),$$
$$y(s) = \pm\int_0^s \sin\theta(s)\, ds$$
$$= \pm\frac{L}{4m}(K_{1,p}(q))^{-1}\int_0^{\phi} \sin 2(\sin^{-1}(q\sin\varphi))\frac{|\cos\varphi|^{1-2/p}}{\sqrt{1-q^2\sin^2\varphi}}\,d\varphi = y(\phi).$$

Therefore, we have shown the case (ii) of Theorem 1 and 2. Especially, when $a = 0$, locus (x, y) makes a closed curve. This becomes like figure eight; see Figure 2-(vi). Thus we have proved the case (vi) of Theorem 1 and 2.

The case (V) of (12).
As in the case (III) of (12), from (29), putting $q^2 = (E+R)/2R$ and noting $\theta(0) = 0$, $\theta(L) = 2n\pi$, we have

$$\theta_s(s) = \left(\frac{2pR}{p-1}\right)^{1/p}\sqrt[p]{q^2 - \sin^2\frac{\theta}{2}}, \quad (0 \le s \le L), \tag{38}$$

and hence

$$s = \left(\frac{p-1}{2pR}\right)^{1/p}\int_0^{\theta}\frac{d\theta}{\sqrt[p]{q^2-\sin^2\frac{\theta}{2}}}, \quad (0 \le \theta \le 2n\pi) \tag{39}$$
$$= \left(\frac{p-1}{p(E+R)}\right)^{1/p}\int_0^{\theta}\frac{d\theta}{\sqrt[p]{1-r^2\sin^2\frac{\theta}{2}}}$$
$$= 2\left(\frac{p-1}{p(E+R)}\right)^{1/p}\int_0^{\phi}\frac{d\phi}{\sqrt[p]{1-r^2\sin^2\phi}}, \quad (0 \le \phi \le n\pi),$$

where $r = 1/q$ and $\phi = \theta/2$. Thus it holds

$$\frac{L}{2n} = 2\left(\frac{p-1}{p(E+R)}\right)^{1/p}\int_0^{\pi/2}\frac{d\phi}{\sqrt[p]{1-r^2\sin^2\phi}}, \tag{40}$$

so, R can be expressed as a function of r:

$$R(r) = R = \frac{2^{2p-1} r^2 n^p (p-1)}{pL^p} \left(\int_0^{\pi/2} \frac{d\phi}{\sqrt[p]{1 - r^2 \sin^2 \phi}} \right)^p. \tag{41}$$

Applying (40), we can express $(x(L), y(L))$ as followings:

LEMMA 3.

$$\begin{aligned} x(L) = \int_0^L \cos\theta(s)\, ds &= L \left(\int_0^{\pi/2} \frac{d\phi}{\sqrt[p]{1 - r^2 \sin^2 \phi}} \right)^{-1} \left(\int_0^{\pi/2} \frac{\cos 2\phi}{\sqrt[p]{1 - r^2 \sin^2 \phi}}\, d\phi \right) \\ &= LK_{2,p}(r)^{-1} \left(\frac{r^2 - 2}{r^2} K_{2,p}(r) + \frac{2}{r^2} E_{2,\, p/(p-1)}(r) \right), \\ y(L) = \int_0^L \sin\theta(s)\, ds &= 0. \end{aligned}$$

Proof.

$$\begin{aligned} x(L) &= \int_0^L \cos\theta(s)\, ds = n \int_0^{L/n} \cos\theta(s)\, ds \\ &= 4n \left(\frac{p-1}{p(E+R)} \right)^{1/p} \int_0^{\pi/2} \frac{\cos 2\phi}{\sqrt[p]{1 - r^2 \sin^2 \phi}}\, d\phi \\ &= L \left(\int_0^{\pi/2} \frac{d\phi}{\sqrt[p]{1 - r^2 \sin^2 \phi}} \right)^{-1} \left(\int_0^{\pi/2} \frac{\cos 2\phi}{\sqrt[p]{1 - r^2 \sin^2 \phi}}\, d\phi \right) \\ &= LK_{2,p}(r)^{-1} \left(\frac{r^2 - 2}{r^2} K_{2,p}(r) + \frac{2}{r^2} E_{2,\, p/(p-1)}(r) \right). \end{aligned}$$

For $y(L)$, we have

$$\begin{aligned} y(L) &= \int_0^L \sin\theta(s)\, ds = n \int_0^{L/n} \sin\theta(s)\, ds \\ &= 2n \left(\frac{p-1}{p(E+R)} \right)^{1/p} \int_0^{\pi} \frac{\sin 2\phi}{\sqrt[p]{1 - r^2 \sin^2 \phi}}\, d\phi = 0. \end{aligned}$$ □

Here, we define the function X_2 as:

$$X_2(r) = L\left(\int_0^{\pi/2} \frac{d\phi}{\sqrt[p]{1 - r^2 \sin^2 \phi}}\right)^{-1} \left(\int_0^{\pi/2} \frac{\cos 2\phi}{\sqrt[p]{1 - r^2 \sin^2 \phi}} d\phi\right) \tag{42}$$

$$= LK_{2,p}(r)^{-1}\left(\frac{r^2 - 2}{r^2} K_{2,p}(r) + \frac{2}{r^2} E_{2,\, p/(p-1)}(r)\right).$$

For X_2, following properties hold.

LEMMA 4.
(i) $X_2(r)$ *is monotone decreasing on* $(0,1)$.
(ii) $X_2(0) = 0$.
(iii)

$$X_2(1) = \begin{cases} -L & (1 < p \leq 2) \\ -\dfrac{L}{p-1} & (2 < p). \end{cases} \tag{43}$$

Proof. Differentiating $K_{2,p}$ with respect to r we have

$$\frac{dK_{2,p}(r)}{dr}(r) = \frac{2r}{p}\int_0^{\pi/2} (1 - r^2 \sin^2 \phi)^{-1/p-1} \sin^2 \phi \, d\phi > 0,$$

so $K_{2,p}(r)^{-1}$ is monotone decreasing. Moreover

$$\frac{d}{dr}\left(\int_0^{\pi/2} \frac{\cos 2\phi}{\sqrt[p]{1 - r^2 \sin^2 \phi}} d\phi\right) = \frac{2r}{p}\int_0^{\pi/2} \cos 2\phi(\sin \phi)^2(1 - r^2 \sin^2 \phi)^{-1/p-1} \, d\phi$$

$$= \frac{r}{p}\int_{-\pi/2}^{\pi/2} -\sin \varphi \sin^2\left(\frac{\varphi}{2} + \frac{\pi}{4}\right)\left(1 - r^2 \sin^2\left(\frac{\varphi}{2} + \frac{\pi}{4}\right)\right)^{-1/p-1} d\varphi$$

$$= \frac{r}{p}\int_0^{\pi/2} -\sin \varphi\left\{\sin^2\left(\frac{\varphi}{2} + \frac{\pi}{4}\right)\left(1 - r^2 \sin^2\left(\frac{\varphi}{2} + \frac{\pi}{4}\right)\right)^{-1/p-1}\right.$$

$$\left. - \sin^2\left(-\frac{\varphi}{2} + \frac{\pi}{4}\right)\left(1 - r^2 \sin^2\left(-\frac{\varphi}{2} + \frac{\pi}{4}\right)\right)^{-1/p-1}\right\} d\varphi < 0.$$

Thus, (i) is proved. (ii) is obvious. To see (iii), we note $E_{2,p/(p-1)}(1) = E_{1,p}(1)$ and $\lim_{r\to 1} K_{2,p}(r) = \lim_{q\to 1} K_{1,p}(q)$. From these relations, as in the case of $X_1(1)$, using (37) and formula $z\Gamma(z) = \Gamma(z+1)$, we obtain the result. □

From Lemma 4, it holds that $X_2(r) = a$ has a unique root for $(-L < a < 0)$ if $(1 < p \leq 2)$ and for $(-L/(p-1) < a < 0)$ if $(2 < p)$. Using this r and (39), (40), we can express locus (x, y) of p-elastic curve as in the case (iii). Concrete expressions of (x, y) are case (iii) of Theorem 1 and 2.

Next, from Figure 2-(iv), we can guess that the locus (x, y) corresponding to $\alpha = \pi$ in (10), also becomes to a p-elastic curve. Indeed, if we take $\alpha = \pi$ in (10), instead of (38), we obtain

$$\theta_s(s) = \left(\frac{2pR}{p-1}\right)^{1/p} \sqrt[p]{q^2 - \cos^2 \frac{\theta}{2}}, \quad (0 \le s \le L). \tag{44}$$

Therefore locus (x, y) generated by (44) becomes to case (iv) of Theorem 1 and 2. Similarly $x(L)$ and $y(L)$ can be expressed as

$$\begin{aligned} x(L) &= L\left(\int_0^{\pi/2} \frac{d\phi}{\sqrt[p]{1 - r^2 \sin^2 \phi}}\right)^{-1} \left(\int_0^{\pi/2} \frac{\cos 2\phi}{\sqrt[p]{1 - r^2 \cos^2 \phi}}\, d\phi\right) \\ &= -LK_{2,p}(r)^{-1}\left(\int_0^{\pi/2} \frac{\cos 2\phi}{\sqrt[p]{1 - r^2 \sin^2 \phi}}\, d\phi\right) \\ &= -LK_{2,p}(r)^{-1}\left(\frac{r^2 - 2}{r^2} K_{2,p}(r) + \frac{2}{r^2} E_{2,\, p/(p-1)}(r)\right). \end{aligned}$$

So, by Lemma 4, $x(s)$ is monotone increasing and end values are $x(0) = 0$, $x(L) = L$ if $(1 < p \le 2)$, $(x(L) = L/(p-1)$ if $(p > 2))$. For $y(L)$, we have

$$y(L) = 2n\left(\frac{p-1}{p(E+R)}\right)^{1/p} \int_0^{\pi} \frac{\sin 2\phi}{\sqrt[p]{1 - r^2 \cos^2 \phi}}\, d\phi = 0.$$

Finally, we consider the case (IV) of (12).

The case (IV) of (12).

This case corresponds to the case $q = 1$ in the case (iii) and (iv) of Theorem 1 and 2. Assume $p \le 2$, then we have from (39)

$$\lim_{\phi \to \pi/2} s(\phi) = 2\left(\frac{p-1}{2pR}\right)^{1/p} \int_0^{\pi/2} (\cos \phi)^{-2/p}\, d\phi = \infty.$$

Hence we do not have any p-elastic curve of finite arc-length. But, in the case $p > 2$ from (37), above limit is finite. So, we can expect the existence of p-elastic curve in this case. We see that from (22), $F^{\pm}(l)$ and from (20), $FL^{\pm}(l)$, $FR^{\pm}(l)$ are obtained. Indeed, appropriate combinations of $F^{\pm}(l)$, $FL^{\pm}(l)$, $FR^{\pm}(l)$ and line segments are as Theorem 3 stationary curves (p-elastic curves). We will show this by an example.

Example 1. $F^+(L_0)L(L_1)F^+(L_0)L(L_2)$ is a stationary curve of $J(\theta)$ under the condition $\theta \in W(p, a, 2)$, where $a = 2L_0/(p-1) + L_1 + L_2$.

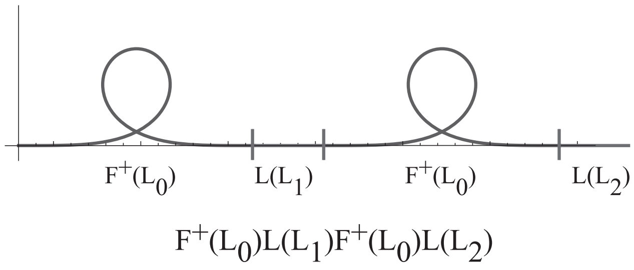


FIGURE 6. $F^+(L_0)L(L_1)F^+(L_0)L(L_2)$.

We note that θ_{ss} is discontinuous at $s = L_0, L_0 + L_1, 2L_0 + L_1$, so we divide the integration of first variation of J with these points. Noting $\theta_s(s) = 0$ at $s = L_0, L_0 + L_1, 2L_0 + L_1$, and R is given by (41), we have

$$\begin{aligned} J'(\theta)[\varphi] &= \int_0^{L_0} |\theta_s(s)|^{p-2}\theta_s(s)\varphi_s(s)\,ds + \int_{L_0}^{L_0+L_1} \cdot\,ds + \int_{L_0+L_1}^{2L_0+L_1} \cdot\,ds + \int_{2L_0+L_1}^{L} \cdot\,ds \\ &= [|\theta_s|^{p-2}\theta_s\varphi]_0^{L_0} + [\,\cdot\,]_{L_0}^{L_0+L_1} + [\,\cdot\,]_{L_0+L_1}^{2L_0+L_1} + [\,\cdot\,]_{2L_0+L_1}^{L} \\ &\quad - \int_0^{L_0} (|\theta_s(s)|^{p-2}\theta_s(s))_s\varphi(s)\,ds - \int_{L_0}^{L_0+L_1} \cdot\,ds - \int_{L_0+L_1}^{2L_0+L_1} \cdot\,ds - \int_{2L_0+L_1}^{L} \cdot\,ds \\ &= \int_0^{L_0} R\sin\theta(s)\varphi(s)\,ds + \int_{L_0}^{L_0+L_1} R\sin(2\pi)\varphi(s)\,ds \\ &\quad + \int_{L_0+L_1}^{2L_0+L_1} R\sin\theta(s)\varphi(s)\,ds + \int_{2L_0+L_1}^{L} R\sin(4\pi)\varphi(s)\,ds \\ &= \int_0^{L} R\sin\theta(s)\varphi(s)\,ds = 0, \end{aligned}$$

where the last equality holds from (8). We can see that the curve in the cases (vii), (viii) and (ix) of Theorem 3 are stationary by the same reason. Finally, we show that in this case p-elastic curve exists for a satisfying $L/(p-1) \le a < L$ (case (vii)), and $-L < a \le -L/(p-1)$ (cases (viii) and (ix)). For the case (vii), we have

$$a = \frac{L_0}{p-1}n + \sum_{i=1}^{n_0} L_i. \tag{45}$$

On the other hand, we know the relation $nL_0 + \sum_{i=1}^{n_0} L_i = L$. Eliminating L_0 from (45), we obtain

$$a = \frac{L + (p-2)\sum_{i=1}^{n_0} L_i}{p-1}. \tag{46}$$

The cases (viii) and (ix) are shown similarly. Now, we have finished the proofs of Theorem 1–3.

4. Appendix

Let us define

$$W_0(p,a,n) := \left\{ \psi \in W_0^{1,p}(0,L) \,\middle|\, \int_0^L \cos\left(\psi(s) + \frac{2n\pi}{L}s\right) ds = a, \right.$$
$$\left. \int_0^L \sin\left(\psi(s) + \frac{2n\pi}{L}s\right) ds = 0 \right\}.$$

We note, for $\psi \in W_0^{1,p}(0,L)$ it holds that (by integration by parts)

$$\left(\int_0^L |\psi(s)|^p \, ds\right)^{1/p} \le L\left(\int_0^L |\psi_s(s)|^p \, ds\right)^{1/p}$$

Thus we can assume $W_0^{1,p}(0,L)$ has a norm of the form

$$\|\psi\|_{W_0^{1,p}(0,L)} = \left(\int_0^L |\psi_s(s)|^p \, ds\right)^{1/p}$$

Lemma 5. $\inf_{\theta \in W_0(p,a,n)} J(\theta)$ *is attained.*

Proof. Applying $A = \psi_s + 2n\pi/L$, $B = -2n\pi/L$ to

$$\left(\frac{|A|^p + |B|^p}{2}\right) \ge \left(\frac{|A| + |B|}{2}\right)^p \ge \left(\frac{|A+B|}{2}\right)^p$$

and putting $\psi = \theta - 2n\pi s/L$, where $\theta \in W(p,a,n)$ we have

$$J(\theta) = J\left(\psi + \frac{2n\pi s}{L}\right) = \frac{1}{2}\int_0^L \left|\psi_s + \frac{2n\pi}{L}\right|^p ds$$
$$\ge 2^{-p}\int_0^L |\psi_s|^p \, ds - \frac{L}{2}\left(\frac{2n\pi}{L}\right)^p.$$

Thus we can assume $\|\psi\|_{W_0^{1,p}(0,L)}$ is bounded. Let us show that $W_0(p,a,n)$ is weakly closed. Let $\{\psi_m\} \subset W_0(p,a,n)$ and $\psi_m \rightharpoonup \psi_0$. Since $W_0^{1,p}(0,L)$ is compactly embedded in $C[0,L]$ ([2, p. 212]), if $\psi_m \rightharpoonup \psi_0$, then $\psi_0 \in W_0(p,a,n)$. Thus $W_0(p,a,n)$ is weakly closed, and hence $W_0(p,a,n) \cap \{\psi \in W_0^{1,p}(0,L) \mid \|\psi\|_{W_0^{1,p}(0,L)} \le C\}$ is weakly compact, where $C > 0$ is sufficientlt large. Noting that $J(\psi + 2n\pi s/L)$ is (weakly) lower-semi-continuous on $W_0^{1,p}(0,L)$, we obtain the result. □

5. Acknowledgement

The auther expresses his hearty thanks to anonymous refree for many valuable suggestions.

References

[1] S. S. Antman, Nonlinear problems of elasticity, Springer, New York, 1995.
[2] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations, Springer, New York, 2011.
[3] N. Koiso, Variational problem, Kyoritsu Syuppan, Tokyo, 1998.
[4] W. Matsumoto, M. Murai and S. Yotsutani, By which kind of sound can one hear the shape of a drum?, RIMS kôkyûroku **1315** (2003), 156–175.
[5] F. Olver, D. Lozier, R. Boisvert and C. Clark, NIST handbook of mathematical functions, Cambridge Univ. Press, New York, 2010.
[6] S. Takeuchi, Generalized Jacobian elliptic functions and their application to bifurcation problems associated with p-Laplacian, J. Math. Anal. Appl. **385** (2012), 24–35.
[7] S. Takeuchi, Generalized Jacobian elliptic functions, RIMS kôkyûroku **1838** (2012), 71–101.
[8] C. Truesdell, The influence of elasticity on analysis: the classic heritage, Bull. Amer. Math. Soc. **9** (1983), 293–310.
[9] H. Yanamoto, On the elastic closed curves, Kodai Math. J. **8** (1985), 224–235.
[10] K. Watanabe, Plane domains which are spectrally determined, Ann. Global Anal. and Geom. **18** (2000), 447–475.

Kohtaro Watanabe
Department of Computer Science
National Defense Academy
1-10-20 Hashirimizu
Yokosuka 239-8686
Japan
E-mail: wata@nda.ac.jp