S. MATSUMOTO
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NEW PROOFS OF THEOREMS OF KATHRYN MANN

Shigenori Matsumoto

To the memory of Akio Hattori

Abstract

We give a shorter proof of the following theorem of Kathryn Mann [**M**]: the identity component of the group of the compactly supported C^r diffeomorphisms of $\mathbf{R}^n$ cannot admit a nontrivial C^p-action on S^1, provided $n \geq 2$, $r \neq n+1$ and $p \geq 2$. We also give a new proof of another theorem of Mann [**M**]: any nontrivial homomorphism from the group of the orientation preserving C^r diffeomorphisms of the circle to the group of C^p diffeomorphisms of the circle is the conjugation of the standard inclusion by a C^p diffeomorphism, if $r \geq p$, $r \neq 2$ and $p \neq 1$.

1. Introduction

É. Ghys [**G**] asked if the group of diffeomorphisms of a manifold admits a nontrivial action on a lower dimensional manifold. A break through towards this problem was obtained by Kathryn Mann [**M**] in the case where the target manifold is one dimensional. Let us denote by $\mathrm{Diff}_c^r(\mathbf{R}^n)_0$ the identitiy component of the group of the compactly supported C^r diffeomorphisms of $\mathbf{R}^n$, $r = 0, 1, \ldots, \infty$. She showed the following theorem.

Theorem 1. *Assume $n \geq 2$, $r \neq n+1$ and $p \geq 2$. Then any (abstract) homomorphism from $\mathrm{Diff}_c^r(\mathbf{R}^n)_0$ to $\mathrm{Diff}^p(S^1)$ is trivial.*

The condition $r \neq n+1$ is for the simplicity of the source group. The condition $p \geq 2$ is necessary since the proof is built upon a theorem of Kopell and Szekeres. Notice that by the fragmentation lemma, the same statement holds true if we replace $\mathbf{R}^n$ by any n-dimensional manifold, compact or not. One aim of this notes is to give a short proof of the above theorem. We also show the following result.

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THEOREM 2. *Assume* $n \geq 2$, $r \neq n+1$ *and* $p \geq 2$. *Then any homomorphism from* $\mathrm{Diff}_c^r(\mathbf{R}^n)_0$ *to* $\mathrm{Diff}^p(\mathbf{R})$ *is trivial.*

This is a generalization of a result of [**M**] for the target group $\mathrm{Diff}^p([0,1))$.

Next we consider the case where the source manifold is one dimensional. We provide a shorter proof of the following theorem, also contained in [**M**]. Denote by $\mathrm{Diff}_+^r(S^1)$ the group of the orientation preserving C^r diffeomorphisms of S^1.

THEOREM 3. *Assume* $r \geq p$, $r \neq 2$ *and* $p \neq 1$. *Then any nontrivial homomorphism from* $\mathrm{Diff}_+^r(S^1)$ *to* $\mathrm{Diff}^p(S^1)$ *is the conjugation of the standard inclusion by a* C^p *diffeomorphism.*

In the above theorem, the case where $p = 0$ is new. We also have the following.

THEOREM 4. *Assume* $p \neq 1$. *Then any nontrivial homomorphism from* $\mathrm{PSL}(2,\mathbf{R})$ *to* $\mathrm{Diff}^p(S^1)$ *is the conjugation of the standard inclusion by a* C^p *diffeomorphism.*

As for Ghys's question for target manifolds of dimension > 1, a satisfactory answer is obtained by S. Hurtado [**H**]. Some part of his argument is an induction on the dimension of the target manifold. It is based upon Theorems 1 and 3.

In [**M**], Theorems 1 and 3 are shown using the following result.

THEOREM 1.1. *Assume* $r \geq 3$, $p \geq 2$ *and* $r \geq p$. *Any nontrivial homomorphism* Φ *from* $\mathrm{Diff}_c^r((0,1))$ *to* $\mathrm{Diff}^p([0,1))$ *without interior global fixed point of the* Φ*-action is the conjugation of the standard inclusion by a* C^p *diffeomorphism of* $(0,1)$.

Our proofs of Theorems 1 and 3 do not use Theorem 1.1. On the other hand, we would like to stress that Theorem 1.1 is more involved, and cannot be shown by the techniques of the present paper.

2. Theorem of Kopell and Szekeres

Our main tool for the proof of Theorems 1 and 2 is the following theorem due to Kopell [**K**] and Szekeres. (See 4.1.11 in [**N2**].) This forces us to assume $p \geq 2$ in these theorems.

THEOREM 2.1. *Let* $p \geq 2$. *Assume that* $\psi \in \mathrm{Diff}^p([0,1))$ *admits no interior fixed point. Then there is a unique* C^1 *flow* $\{\psi^t\}$ *on* $[0,1)$ *such that* $\psi = \psi^1$. *Moreover any element* ϕ *of the centralizer* $C(\psi)$ *of* ψ *in* $\mathrm{Diff}^p([0,1))$ *can be written as* $\phi = \psi^t$ *for some* $t \in \mathbf{R}$.

Corollary 2.2. *Let* $p \geq 2$ *and* $\psi \in \mathrm{Diff}^p([0,1))$. *Then* $C(\psi) = C(\psi^2)$.

Proof. Choose any element $g \in C(\psi^2)$. Let J be the closure of any component of $[0,1)\backslash\mathrm{Fix}(\psi)$. (Notice that $\mathrm{Fix}(\psi^2) = \mathrm{Fix}(\psi)$.) Then by Theorem 2.1, g commutes with ψ on J. Since J is arbitrary, g commutes with ψ everywhere. □

We also have the following result, whose proof is the same as above.

Corollary 2.3. *Assume* $p \geq 2$ *and* $\psi \in \mathrm{Diff}^p_+(\mathbf{R})$ *admits fixed points. Then the centralizers in* $\mathrm{Diff}^p_+(\mathbf{R})$ *satisfy* $C(\psi) = C(\psi^2)$.

3. Commuting subgroups of $\mathrm{Diff}^0_+(S^1)$

Another basic result needed for the proof is the following.

Proposition 3.1. *Let* G_1 *and* G_2 *be simple nonabelian subgroups of* $\mathrm{Diff}^0_+(S^1)$. *Assume that* G_2 *is conjugate to* G_1 *in* $\mathrm{Diff}^0_+(S^1)$ *and that any element of* G_1 *commutes with any element of* G_2. *Then there is a global fixed point of* $G_1 : \mathrm{Fix}(G_1) \neq \emptyset$.

Proof. First of all, let us show that there is an element $g \in G_1\backslash\{\mathrm{id}\}$ such that $\mathrm{Fix}(g)$ is nonempty. Assume the contrary. Consider the group $\tilde{G}_1$ formed by any lift of any element of G_1 to the universal covering space $\mathbf{R} \to S^1$. The canonical projection $\pi : \tilde{G}_1 \to G_1$ is a group homomorphism. Now $\tilde{G}_1$ acts freely on $\mathbf{R}$. A theorem of Hölder asserts that $\tilde{G}_1$ is abelian. See for example [**N1**]. Therefore $G_1 = \pi(\tilde{G}_1)$ would be abelian, contrary to the assumption of the proposition.

Let $X_2 \subset S^1$ be a minimal set of G_2. The set X_2 is either a finite set, a Cantor set or the whole of S^1. If X_2 is a singleton, then G_2 admits a fixed point. Since G_1 is conjugate to G_2, we have $\mathrm{Fix}(G_1) \neq \emptyset$, as is required. If X_2 is a finite set which is not a singleton, we get a nontrivial homomorphism from G_2 to a finite abelian group, contrary to the assumption. In the remaining case, it is well known, easy to show, that the minimal set is unique. That is, X_2 is contained in any nonempty G_2 invariant closed subset.

Let F_1 be the subset of G_1 formed by the elements with nonempty fixed point set. Since G_1 and G_2 commutes, the fixed point set $\mathrm{Fix}(g)$ of any element $g \in F_1$ is G_2 invariant. Then we have:

$$X_2 \subset \mathrm{Fix}(g) \quad \text{for any } g \in F_1. \tag{3.1}$$

This shows that F_1 is in fact a subgroup. By the very definition, F_1 is normal. Since G_1 is simple and F_1 is nontrivial, $F_1 = G_1$. Finally again by (3.1), $\mathrm{Fix}(G_1) \neq \emptyset$. □

4. Proof of Theorem 1

Assume $n \geq 2$, $r \neq n+1$ and $p \geq 2$. Let $\Phi : \mathrm{Diff}_c^r(\mathbf{R}^n)_0 \to \mathrm{Diff}^p(S^1)$ be a nontrivial homomorphism. Our purpose is to deduce a contradiction. Since $\mathrm{Diff}_c^r(\mathbf{R}^n)_0$ is simple by the assumption $r \neq n+1$, the map Φ is injective and its image is contained in the group of orientation preserving diffeomorphisms.

Let B_1 and B_2 be disjoint open balls of radius 2 in $\mathbf{R}^n$. The group $\mathscr{G}_i = \mathrm{Diff}_c^r(B_i)_0$ is nonabelian and simple. Clearly $\mathscr{G}_2$ is conjugate to $\mathscr{G}_1$ in $\mathrm{Diff}_c^r(\mathbf{R}^n)_0$.

Therefore $\Phi(\mathscr{G}_i)$ satisfies all the conditions of Proposition 3.1. Thus $\Phi(\mathscr{G}_1)$ has a fixed point, and one can identify $\Phi(\mathscr{G}_1) \subset \mathrm{Diff}^p([0,1))$. In view of Corollary 2.2 and the injectivity of Φ, it is sufficent to construct an element $g \in \mathscr{G}_1$ such that $C(g) \neq C(g^2)$. Let B_1' be the concentric ball in B_1 of radius 1. Any element $g \in \mathscr{G}_1$ which is an involution on B_1' will do.

5. Proof of Theorem 2

Let $n \geq 2$, $r \neq n+1$ and $p \geq 2$. Assume there is a nontrivial homomorphism $\Phi : \mathrm{Diff}_c^r(\mathbf{R}^n)_0 \to \mathrm{Diff}^p(\mathbf{R})$. By the simplicity of $\mathrm{Diff}_c^r(\mathbf{R}^n)_0$, Φ is injective, with its image contained in the group of orientation preserving diffeomorphisms. In view of Corollary 2.3 and the last step of the previous section, it suffices to show that any element of the image of Φ has nonempty fixed point set. The rest of this section is devoted to its proof.

Assume for contradiction that there is an element $g' \in \mathrm{Diff}_c^r(\mathbf{R}^n)_0$ such that $\mathrm{Fix}(\Phi(g')) = \emptyset$. Choose open balls B_i $(i = 1,2)$ in $\mathbf{R}^n$ as in the previous section. Again let $\mathscr{G}_i = \mathrm{Diff}_c^r(B_i)_0$. There is a conjugate g of g' in $\mathscr{G}_2$. Notice that $\mathrm{Fix}(\Phi(g)) = \emptyset$. Then $\Phi(\mathscr{G}_2)$ has a cross section I in $\mathbf{R}$, that is, I is a compact interval such that any $\Phi(\mathscr{G}_2)$ orbit hits I. Now we follow the proof of Proposition 6.1 in [**DKNP**], to show that there is a unique minimal set X_2 for $\Phi(\mathscr{G}_2)$. Moreover we shall show that there is a nonempty $\Phi(\mathscr{G}_2)$ invariant closed subset X_2 in $\mathbf{R}$ which has the property that any nonempty $\Phi(\mathscr{G}_2)$ invariant closed subset contains X_2.

The proof goes as follows. Let F be the family of nonempty $\Phi(\mathscr{G}_2)$ invariant closed subsets of $\mathbf{R}$, and F_I the family of nonempty closed subsets Y in I such that $\Phi(\mathscr{G}_2)(Y) \cap I = Y$, where we denote

$$\Phi(\mathscr{G}_2)(Y) = \bigcup_{g \in \mathscr{G}_2} \Phi(g)(Y).$$

Define a map $\phi : F \to F_I$ by $\phi(X) = X \cap I$, and $\psi : F_I \to F$ by $\psi(Y) = \Phi(\mathscr{G}_2)(Y)$. They satisfiy $\psi \circ \phi = \phi \circ \psi = \mathrm{id}$.

Let $\{Y_\alpha\}$ be a totally ordered set in F_I. Then the intersection $\bigcap_\alpha Y_\alpha$ is nonempty. Let us show that it belongs to F_I, namely,

$$\Phi(\mathscr{G}_2)\left(\bigcap_\alpha Y_\alpha\right) \cap I = \bigcap_\alpha Y_\alpha. \tag{5.1}$$

For the inclusion $\subset$, we have

$$\Phi(\mathscr{G}_2)\left(\bigcap_\alpha Y_\alpha\right)\cap I \subset \left(\bigcap_\alpha \Phi(\mathscr{G}_2)(Y_\alpha)\right)\cap I = \bigcap_\alpha(\Phi(\mathscr{G}_2)(Y_\alpha)\cap I) = \bigcap_\alpha Y_\alpha.$$

For the other inclusion, notice that

$$\bigcap_\alpha Y_\alpha \subset \Phi(\mathscr{G}_2)\left(\bigcap_\alpha Y_\alpha\right) \quad \text{and} \quad \bigcap_\alpha Y_\alpha \subset I.$$

Therefore by Zorn's lemma, there is a minimal element Y_2 in F_I. The set Y_2 is not finite. In fact, if it is finite, the set $X_2 = \psi(Y_2)$ in F is discrete, and there would be a nontrivial homomorphism from $\Phi(\mathscr{G}_2)$ to $\mathbf{Z}$, contrary to the fact that $\mathscr{G}_2$, (and hence $\Phi(\mathscr{G}_2)$) is simple.

Now the correspondence ϕ and ψ preserve the inclusion. This shows that there is no nonempty $\Phi(\mathscr{G}_2)$ invariant closed proper subset of $X_2 = \psi(Y_2)$. In other words, any $\Phi(\mathscr{G}_2)$ orbit contained in X_2 is dense in X_2. Therefore X_2 is either $\mathbf{R}$ itself or a locally Cantor set. In the former case, any nonempty $\Phi(\mathscr{G}_2)$ invariant closed subset must be $\mathbf{R}$ itself.

Let us show that in the latter case, X_2 satisfies the desired property: X_2 is contained in any nonempty $\Phi(\mathscr{G}_2)$ invariant closed subset. For this, we only need to show that the $\Phi(\mathscr{G}_2)$ orbit of any point x in $\mathbf{R}\backslash X_2$ accumulates to a point in X_2. Let (a,b) be the connected component of $\mathbf{R}\backslash X_2$ that contains x. Then there is a sequence $g_k \in \mathscr{G}_2$ $(k \in \mathbf{N})$ such that $\Phi(g_k)(a)$ accumulates to a and that $\Phi(g_k)(a)$'s are mutually distinct. Then the intervals $\Phi(g_k)((a,b))$ are mutually disjoint, and consequently $\Phi(g_k)(x)$ converges to a. This concludes the proof that X_2 is contained in any nonempty $\Phi(\mathscr{G}_2)$ invariant closed subset.

Let $\mathscr{F}_1$ be the subset of $\mathscr{G}_1$ formed by the elements g such that $\mathrm{Fix}(\Phi(g)) \neq \emptyset$. Again by a theorem of Hölder, $\mathscr{F}_1$ contains a nontrivial element. Now we have

$$X_2 \subset \mathrm{Fix}(\Phi(g)) \quad \text{for any } g \in \mathscr{F}_1. \tag{5.2}$$

This shows that $\mathscr{F}_1$ is a subgroup, normal by the definition. Since $\mathscr{G}_1$ is simple, $\mathscr{F}_1 = \mathscr{G}_1$. Finally again by (5.2), we have $\mathrm{Fix}(\Phi(\mathscr{G}_1)) \neq \emptyset$. This contradicts the fact that $\Phi(\mathscr{G}_2)$, being conjugate to $\Phi(\mathscr{G}_1)$, must also have a free element. The contradiction shows that any element of the image Φ has nonempty fixed point set.

6. Proof of Theorem 3

We first prove Theorem 3 for $p = 0$. Assume $r \neq 2$ and let Φ be a nontrivial homomorphism from $F = \mathrm{Diff}^r_+(S^1)$ to $\mathrm{Diff}^0(S^1)$. Since F is simple, Φ is injective and the image of Φ is contained in $\mathrm{Diff}^0_+(S^1)$. For any $x \in S^1$, denote by F_x the isotropy subgroup at x.

Proposition 6.1. *For any $x \in S^1$, the fixed point set* $\mathrm{Fix}(\Phi(F_x))$ *is a singleton.*

The proof uses Theorem 5.2 in [**M**′], which states as follows.

THEOREM 6.2. *Any nontrivial homomorphism from* $\mathrm{PSL}(2,\mathbf{R})$ *to* $\mathrm{Diff}^0_+(S^1)$ *is the conjugation of the standard inclusion by a homeomorphism h.*

Proof of Proposition 6.1. Let $G_x = \mathrm{Diff}^r_c(S^1\backslash\{x\})$. First we shall show $\mathrm{Fix}(\Phi(G_x)) \neq \emptyset$. Let $\{U_n\}_{n\in\mathbf{N}}$ be an increasing sequence of precompact open intervals in $S^1\backslash\{x\}$ such that $\bigcup_n U_n = S^1\backslash\{x\}$ and let V_n be an open interval in $S^1\backslash\{x\}$ disjoint from U_n. Then $\Phi(\mathrm{Diff}^r_c(U_n))$ and $\Phi(\mathrm{Diff}^r_c(V_n))$ satisfy the conditions of Proposition 3.1. Therefore $\mathrm{Fix}(\Phi(\mathrm{Diff}^r_c(U_n)))$ is nonempty. Since $\Phi(G_x) = \bigcup_n \Phi(\mathrm{Diff}^r_c(U_n))$, we have $\mathrm{Fix}(\Phi(G_x)) = \bigcap_n \mathrm{Fix}(\Phi(\mathrm{Diff}^r_c(U_n)))$. But the RHS is a decreasing intersection of nonempty compact subsets. Therefore we get $\mathrm{Fix}(\Phi(G_x)) \neq \emptyset$.

Next let us show that $\mathrm{Fix}(\Phi(G_x))$ is a singleton. Assume there are two distinct points ξ_0 and ξ_1 in $\mathrm{Fix}(\Phi(G_x))$. By Theorem 6.2, there is a rotation $R \in \mathrm{PSL}(2,\mathbf{R})$ such that $\Phi(R)(\xi_0) = \xi_1$. Then ξ_1 is left fixed both by $\Phi(G_x)$ and $\Phi(RG_xR^{-1}) = \Phi(G_{R(x)})$, and hence by $\Phi(F)$, since G_x and $G_{R(x)}$ generate F. Especially $\Phi(\mathrm{PSL}(2,\mathbf{R}))$ admits a global fixed point, contradicting Theorem 6.2.

Finally since $\Phi(G_x)$ is a normal subgroup of $\Phi(F_x)$ and since $\mathrm{Fix}(\Phi(G_x))$ is a singleton, we have $\mathrm{Fix}(\Phi(F_x)) = \mathrm{Fix}(\Phi(G_x))$. $\square$

Define a map $h' : S^1 \to S^1$ by $\{h'(x)\} = \mathrm{Fix}(\Phi(F_x))$. Let us show that h' coincides with a homeomorphism h in Theorem 6.2. In fact, for any $x \in S^1$, consider a parabolic element $g \in \mathrm{PSL}(2,\mathbf{R})$ such that $g(x) = x$. Then, by Theorem 6.2, $h(x)$ is the unique fixed point of $\Phi(g)$. Therefore $h(x) = h'(x)$.

Now for any $f \in F$, we have

$$\begin{aligned}\{h(f(x))\} &= \mathrm{Fix}(\Phi(F_{f(x)})) = \mathrm{Fix}(\Phi(fF_xf^{-1})) = \mathrm{Fix}(\Phi(f)\Phi(F_x)\Phi(f)^{-1}) \\ &= \Phi(f)(\mathrm{Fix}(\Phi(F_x)) = \Phi(f)\{h(x)\}.\end{aligned}$$

This shows that $h \circ f = \Phi(f) \circ h$ for any $f \in F$. Namely Φ is the conjugation by a homeomorphism h. This completes the proof for $p = 0$.

Let us consider the case $p \geq 2$. We follow an argument in [**T**], and show that the homeomorphism h established above is in fact a C^p diffeomorphism. Denote $H = \mathrm{Diff}^p_+(S^1)$. First of all, since h is locally monotone and thus of bounded variation, it is differentiable Lebesgue almost everywhere. But we have $h \circ R_t = f_t \circ h$ $(\forall t \in S^1)$, where R_t is the rotation by t and $f_t \in H$. This shows that h is differentiable everywhere, with nonvanishing derivative.

Let us show that for any point $x \in S^1$, h is a C^p diffeomorphism on some neighbourhood of x. Choose $f \in F$ such that $f(x) = x$ and $f'(x) = \lambda \in (0,1)$. Then $h \circ f \circ h^{-1} \in H$ leaves $h(x)$ fixed and the derivative there is also λ. Notice that f also belongs to H, since $r \geq p$. By the Sternberg linearization theorem (Theorem 2 of [**S**]), there is a C^p diffeomorphism k_1 (resp. k_2) from a neighbourhood of x (resp. $h(x)$) to a neighbourhood of 0 in $\mathbf{R}$ such that $k_1 \circ f = L \circ k_1$ (resp. $k_2 \circ (h \circ f \circ h^{-1}) = L \circ k_2$), where L is a linear map of $\mathbf{R}$

with slope λ. (The assumption $p \neq 1$ is necessary for Theorem 2 of [**S**].) Then the composite $k_2 \circ h \circ k_1^{-1}$ is a local homeomorphism of $(\mathbf{R}, 0)$ commuting with L and differentiable at 0. It is easy to prove that $k_2 \circ h \circ k_1^{-1}$ is a linear map. (The linear contraction $L \times L$ of $\mathbf{R}^2$ leaves the graph of $k_2 \circ h \circ k_1^{-1}$ invariant.) Then h is a C^p diffeomorphism in a neighbourhood of x, as is required.

The proof of Theorem 4 is given by the same argument as above starting from the homeomorphism h in Theorem 6.2.

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Shigenori Matsumoto
Department of Mathematics
College of Science and Technology
Nihon University
1-8-14 Kanda, Surugadai, Chiyoda-ku
Tokyo, 101-8308
Japan
E-mail: matsumo@math.cst.nihon-u.ac.jp