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KODAI MATH. J.
37 (2014), 396–404

ALMOST COMPLETE INTERSECTIONS AND STANLEY'S CONJECTURE

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Abstract

Let K be a field and I a monomial ideal of the polynomial ring $S = K[x_1, \dots, x_n]$. We show that if either: 1) I is almost complete intersection, 2) I can be generated by less than four monomials; or 3) I is the Stanley-Reisner ideal of a locally complete intersection simplicial complex on $[n]$, then Stanley's conjecture holds for S/I.

1. Introduction

Throughout this paper, let K be a field and I a monomial ideal of the polynomial ring $S = K[x_1, \dots, x_n]$.

A decomposition of S/I as direct sum of K-vector spaces of the form $\mathscr{D} : S/I = \bigoplus_{i=1}^{r} u_i K[Z_i]$, where u_i is a monomial in S and $Z_i \subseteq \{x_1, \dots, x_n\}$, is called a *Stanley decomposition* of S/I. The number $\operatorname{sdepth} \mathscr{D} := \min\{|Z_i| : i = 1, \dots, r\}$ is called *Stanley depth* of $\mathscr{D}$. The *Stanley depth* of S/I is defined to be

$$\operatorname{sdepth} S/I := \max\{\operatorname{sdepth} \mathscr{D} : \mathscr{D} \text{ is a Stanley decomposition of } S/I\}.$$

Stanley conjectured [St] that $\operatorname{depth} S/I \le \operatorname{sdepth} S/I$. This conjecture is known as Stanley's conjecture. Recently, this conjecture was extensively examined by several authors; see e.g. [A1], [A2], [HP], [HSY], [P], [R], [S2] and [S3]. On the other hand, the present third author [S2] conjectured that there always exists a Stanley decomposition $\mathscr{D}$ of S/I such that the degree of each u_i is at most $\operatorname{reg} S/I$. We refer to this conjecture as h-regularity conjecture. It is known that for square-free monomial ideals, these two conjectures are equivalent. Our main aim in this paper is to determine some classes of monomial ideals such that these conjectures are true for them.

2010 *Mathematics Subject Classification*. 13F20; 05E40; 13F55.

Key words and phrases. Almost complete intersection monomial ideals; clean; locally complete intersection monomial ideals; pretty clean.

The research of the second and third authors is supported by grants from IPM (no. 90130212 and no. 90130062, respectively).

Received July 17, 2013; revised November 5, 2013.

A basic fact in commutative algebra says that there exists a finite chain

$$\mathcal{F} : I = I_0 \subset I_1 \subset \cdots \subset I_r = S$$

of monomial ideals such that $I_i/I_{i-1} \cong S/\mathfrak{p}_i$ for monomial prime ideals $\mathfrak{p}_i$ of S. Dress [D] called the ring S/I *clean* if there exists a chain $\mathcal{F}$ such that all the $\mathfrak{p}_i$ are minimal prime ideals of I. By [HSY, Proposition 2.2] if I is complete intersection, then the ring S/I is clean. Lemmas 2.4 and 2.8 provide two other classes of clean rings.

Herzog and Popescu [HP] called the ring S/I *pretty clean* if there exists a chain $\mathcal{F}$ such that for all $i < j$ for which $\mathfrak{p}_i \subseteq \mathfrak{p}_j$, it follows that $\mathfrak{p}_i = \mathfrak{p}_j$. Obviously, cleanness implies pretty cleanness and when I is square-free, it is known that these two concepts coincide; see [HP, Corollary 3.5].

If S/I is pretty clean, then S/I is sequentially Cohen-Macaulay and depth of S/I is equal to the minimum of the dimension of $S/\mathfrak{p}$, where $\mathfrak{p} \in \mathrm{Ass}_S\, S/I$; see [S1] for an easy proof. If S/I is pretty clean, then [HP, Theorem 6.5] asserts that Stanley's conjecture holds for S/I. In fact, if S/I is pretty clean, then [HVZ, Proposition 1.3] yields that depth S/I = sdepth S/I. Also if S/I is pretty clean, then by [S2, Theorem 4.7] h-regularity conjecture holds for S/I.

We prove that if the monomial ideal I is either almost complete intersection or it can be generated by less than four monomials, then S/I is pretty clean. Thus, for such monomial ideals both Stanley's and h-regularity conjectures hold. Also, we show that if I is the Stanley-Reisner ideal of a locally complete intersection simplicial complex on $[n]$, then S/I satisfies Stanley's conjecture.

2. Main results

A *simplicial complex* Δ on $[n] := \{1, \dots, n\}$ is a collection of subsets of $[n]$ with the property that if $F \in \Delta$, then all subsets of F are also in Δ. Any singleton element of Δ is called a *vertex*. An element of Δ is called a *face* of Δ and the maximal faces of Δ, under inclusion, are called *facets*. We denote by $\mathcal{F}(\Delta)$ the set of all facets of Δ. The *dimension* of a face F is defined as $\dim F = |F| - 1$, where $|F|$ is the number of elements of F. The dimension of the simplicial complex Δ is the maximal dimension of its facets. A simplicial complex Δ is called *pure* if all facets of Δ have the same dimension. We denote the simplicial complex Δ with facets $F_1, \dots, F_t$ by $\Delta = \langle F_1, \dots, F_t \rangle$. According to Björner and Wachs [BW], a simplicial complex Δ is said to be (*non-pure*) *shellable* if there exists an order $F_1, \dots, F_t$ of the facets of Δ such that for each $2 \le i \le t$, $\langle F_1, \dots, F_{i-1} \rangle \cap \langle F_i \rangle$ is a pure $(\dim F_i - 1)$-dimensional simplicial complex. If Δ is a simplicial complex on $[n]$, then the *Stanley-Reisner ideal* of Δ, I_Δ, is the square-free monomial ideal generated by all monomials $x_{i_1} x_{i_2} \cdots x_{i_t}$ such that $\{i_1, i_2, \dots, i_t\} \notin \Delta$. The *Stanley-Reisner ring* of Δ over the field K is the K-algebra $K[\Delta] := S/I_\Delta$. Any square-free monomial ideal I is the Stanley-Reisner ideal of some simplicial complex Δ on $[n]$. If $\mathcal{F}(\Delta) = \{F_1, \dots, F_t\}$, then $I_\Delta = \bigcap_{i=1}^{t} \mathfrak{p}_{F_i}$, where $\mathfrak{p}_{F_i} := (x_j : j \notin F_i)$; see [BH, Theorem 5.1.4].

Recall that the *Alexander dual* $\Delta^\vee$ of a simplicial complex Δ is the simplicial complex whose faces are $\{[n]\backslash F \mid F \notin \Delta\}$. Let I be a square-free monomial ideal of S. We denote by $I^\vee$, the square-free monomial ideal which is generated by all monomials $x_{i_1} \cdots x_{i_k}$, where $(x_{i_1}, \dots, x_{i_k})$ is a minimal prime ideal of I. It is easy to see that for any simplicial complex Δ, one has $I_{\Delta^\vee} = (I_\Delta)^\vee$. A monomial ideal I of S is said to have *linear quotients* if there exists an order $u_1, \dots, u_m$ of $G(I)$ such that for any $2 \le i \le m$, the ideal $(u_1, \dots, u_{i-1}) :_S u_i$ is generated by a subset of the variables.

LEMMA 2.1. *Let I be a square-free monomial ideal of S. Then S/I is clean if and only if $I^\vee$ has linear quotients.*

Proof. Dress [D, Theorem on page 53] proved that a simplicial complex Δ is (non-pure) shellable if and only if $K[\Delta]$ is a clean ring. On the other hand, by [HHZ, Theorem 1.4], a simplicial complex Δ is (non-pure) shellable if and only if $I_{\Delta^\vee}$ has linear quotients. Combining these facts, yields our claim. □

LEMMA 2.2. *Let I and J be two monomial ideals of S. Assume that $I = uJ$ for some monomial u in S and* $\operatorname{ht} J \ge 2$. *If S/J is pretty clean, then S/I is pretty clean too.*

Proof. With the proof of [S3, Lemma 1.9], the claim is immediate. □

In what follows for a monomial ideal I of S, we denote the number of elements of $G(I)$ by $\mu(I)$.

DEFINITION 2.3. A monomial ideal I of S is said to be *almost complete intersection* if $\mu(I) = \operatorname{ht} I + 1$.

LEMMA 2.4. *Let I be an almost complete intersection square-free monomial ideal of S. Then S/I is clean.*

Proof. The claim is obvious when $\operatorname{ht} I = 0$. Let $\operatorname{ht} I = 1$. Then $I = (u_1, u_2)$ for some monomials u_1 and u_2. We can write I as $I = u(u_1', u_2')$, where $u = \gcd(u_1, u_2)$ and u_1', u_2' are monomials forming a regular sequence on S. So in this case, the claim is immediate by Lemma 2.2 and [HSY, Proposition 2.2]. Now, assume that $h := \operatorname{ht} I \ge 2$. By [KTY, Theorem 4.4] I can be written in one of the following forms, where $A_1, A_2, \dots, B_1, B_2, \dots$ are non-trivial square-free monomials which are pairwise relatively prime, and p, p' are integers with $2 \le p \le h$ and $1 \le p' \le h$.

1) $I_1 = (A_1B_1, A_2B_2, \dots, A_pB_p, A_{p+1}, \dots, A_h, B_1B_2 \cdots B_p)$.
2) $I_2 = (A_1B_1, A_2B_2, \dots, A_{p'}B_{p'}, A_{p'+1}, \dots, A_h, A_{h+1}B_1B_2 \cdots B_{p'})$.
3) $I_3 = (B_1B_2, B_1B_3, B_2B_3, A_4, \dots, A_{h+1})$.
4) $I_4 = (A_1B_1B_2, B_1B_3, B_2B_3, A_4, \dots, A_{h+1})$.
5) $I_5 = (A_1B_1B_2, A_2B_1B_3, B_2B_3, A_4, \dots, A_{h+1})$.
6) $I_6 = (A_1B_1B_2, A_2B_1B_3, A_3B_2B_3, A_4, \dots, A_{h+1})$.

Let $I = I_1$. Since $A_1, A_2, \ldots, A_p, A_{p+1}, \ldots, A_h, B_1, B_2, \ldots, B_p$ are pairwise relatively prime, it turns out that $A_{p+1}, \ldots, A_h$ is a regular sequence on $S/(A_1B_1, A_2B_2, \ldots, A_pB_p, B_1B_2 \cdots B_p)$. So, in view of [R, Theorem 2.1], we may and do assume that $I = (A_1B_1, A_2B_2, \ldots, A_pB_p, B_1B_2 \cdots B_p)$. Next, we are going to show that I is of forest type. Let G be a subset of $\{A_1B_1, A_2B_2, \ldots, A_pB_p, B_1B_2 \cdots B_p\}$ with at least two elements. If $B_1B_2 \cdots B_p \notin G$, then any $a \in G$ can be taken as a leaf and any $b \in G$ different from a can be taken as a branch for this leaf. If $B_1B_2 \cdots B_p \in G$, then any $a \in G$ different from $B_1B_2 \cdots B_p$ can be taken as a leaf and then $B_1B_2 \cdots B_p$ is a branch for this leaf. So, I is of forest type. Thus, since I is square-free, by [SZ, Theorem 1.5], we obtain that S/I is clean. By the similar argument, one can see that if $I = I_2$, then S/I is clean. Set

$$J := (C_1B_1B_2, C_2B_1B_3, C_3B_2B_3, A_4, \ldots, A_{h+1}),$$

where C_i is either A_i or 1 for each $i = 1, 2, 3$. Since each of I_3, I_4, I_5 and I_6 are the particular cases of the ideal J, we can finish the proof by showing that S/J is clean. Since, by the assumption $A_4, \ldots, A_{h+1}$, B_1, B_2, B_3, C_1, C_2, C_3 are pairwise relatively prime, it follows that $A_4, \ldots, A_{h+1}$ is a regular sequence on $S/(C_1B_1B_2, C_2B_1B_3, C_3B_2B_3)$. So by [R, Theorem 2.1], we can assume that $J = (C_1B_1B_2, C_2B_1B_3, C_3B_2B_3)$. Set $T := k[u, v, w, x, y, z]$ and $L := (uxy, vxz, wyz)$. Since B_1, B_2, B_3, C_1, C_2, C_3 is a regular sequence on S, by [HSY, Proposition 3.3], the cleanness of T/L implies the cleanness of S/J. So, by Lemma 2.1, it is enough to prove that $L^\vee$ has linear quotients. As

$$L = (x, y) \cap (x, z) \cap (x, w) \cap (y, z) \cap (y, v) \cap (z, u) \cap (u, v, w),$$

one has $L^\vee = (xy, xz, xw, yz, yv, zu, uvw)$, which clearly has linear quotients by the given order. $\square$

Let $u = \prod_{i=1}^n x_i^{a_i}$ be a monomial in $S = K[x_1, \ldots, x_n]$. Then

$$u^p := \prod_{i=1}^{n} \prod_{j=1}^{a_i} x_{i,j} \in K[x_{1,1}, \ldots, x_{1,a_1}, \ldots, x_{n,1}, \ldots, x_{n,a_n}]$$

is called the *polarization* of u. Let I be a monomial ideal of S with $\mathrm{G}(I) = \{u_1, \ldots, u_m\}$. Then the ideal $I^p := (u_1^p, \ldots, u_m^p)$ of $T := K[x_{i,j} : i = 1, \ldots, n, j = 1, \ldots, a_i]$ is called the *polarization* of I. [S3, Theorem 3.10] implies that S/I is pretty clean if and only if T/I^p is clean.

Recently, Cimpoeaş [C1] proved that if I is an almost complete intersection monomial ideal of S, then Stanley's conjecture holds for S/I. The next result shows that in this case S/I is even pretty clean.

THEOREM 2.5. *Let I be an almost complete intersection monomial ideal of S. Then S/I is pretty clean.*

Proof. From [F, Proposition 2.3], one has $\operatorname{ht} I = \operatorname{ht} I^p$. On the other hand $\mu(I) = \mu(I^p)$, and so I^p is an almost complete intersection square-free monomial ideal of T. Hence, by Lemma 2.4, the ring T/I^p is clean. Now, [S3, Theorem 3.10] implies that S/I is pretty clean, as desired. □

In the situation of Theorem 2.5, there is no need that S/I is clean. For instance, although (x^2, xy) is an almost complete intersection monomial ideal, the ring $k[x, y]/(x^2, xy)$ is not clean.

In [C2, Theorem 2.3], it is shown that if I is a monomial ideal of S with $\mu(I) \leq 3$, then Stanley's conjecture holds for S/I. The next result extends this fact.

Corollary 2.6. *Let I be a monomial ideal of S. If $\mu(I) \leq 3$, then S/I is pretty clean.*

Proof. Clearly, we may assume that I is non zero. Assume that $\mu(I) = 3$ and $\operatorname{ht} I = 1$. Then $I = uJ$, where u is a monomial in S and J is a monomial ideal of S with $\mu(J) = 3$ and $\operatorname{ht} J \geq 2$. By Lemma 2.2, it is enough to prove that S/J is pretty clean. If $\operatorname{ht} J = 2$, then $\mu(J) = \operatorname{ht} J + 1$, and so by Theorem 2.5, S/J is pretty clean. If $\operatorname{ht} J = 3$, then J is complete intersection, and hence by [HSY, Proposition 2.2], S/J is pretty clean.

Since $0 < \operatorname{ht} I \leq \mu(I)$, in all other cases, it follows that I is either complete intersection or almost complete intersection. Thus, the proof is completed by [HSY, Proposition 2.2] and Theorem 2.5. □

Definition 2.7 ([TY, Definition 1.1 and Lemma 1.2]). A simplicial complex Δ on $[n]$ is said to be *locally complete intersection* if $\{\{1\}, \{2\}, \dots, \{n\}\} \subseteq \Delta$ and $(I_\Delta)_{\mathfrak{p}}$ is a complete intersection ideal of $S_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Proj} S/I$.

A simplicial complex Δ is said to be *connected* if for any two facets F and G of Δ, there exists a sequence of facets $F = F_0, F_1, \dots, F_{q-1}, F_q = G$ such that $F_i \cap F_{i+1} \neq \emptyset$ for all $0 \leq i < q$. Also, a simplicial complex Δ on $[n]$ is said to be *n-pointed path* (resp. *n-gon*) if $n \geq 2$ (resp. $n \geq 3$) and, after a suitable change of variables,

$$\mathscr{F}(\Delta) = \{\{i, i+1\} \mid 1 \leq i < n\}$$

(resp.

$$\mathscr{F}(\Delta) = \{\{i, i+1\} \mid 1 \leq i < n\} \cup \{\{n, 1\}\}).$$

Clearly, any n-pointed path (resp. n-gon) is one-dimensional and pure.

Let Δ be a connected simplicial complex on $[n]$ which is locally complete intersection. Then, it is known that Δ is shellable; see e.g. [TY, Proposition 1.11 and Theorem 1.5]. Hence, by [D, Theorem on page 53] it turns out that S/I_Δ is clean. So, we record the following:

Lemma 2.8. *Let Δ be a connected simplicial complex on $[n]$ which is locally complete intersection. Then S/I_Δ is clean.*

Let Δ be as in Lemma 2.8. Then S/I_Δ is clean, and so [HP, Theorem 6.5] implies that S/I_Δ satisfies Stanley's conjecture. In Theorem 2.11, we prove that the later assertion holds without assuming that Δ is connected.

Proposition 2.9. *Let* $I \subset S_1 = K[x_1, \dots, x_m]$, $J \subset S_2 = K[x_{m+1}, \dots, x_n]$ *be two monomial ideals and* $S = K[x_1, \dots, x_m, x_{m+1}, \dots, x_n]$. *Assume that* $\operatorname{depth} S_1/I > 0$ *and* $\operatorname{depth} S_2/J > 0$. *Then Stanley's conjecture holds for* $S/(I, J, \{x_i x_j\}_{1 \le i \le m, m+1 \le j \le n})$.

Proof. For convenience, we set $Q_1 := (x_1, \dots, x_m)$, $Q_2 := (x_{m+1}, \dots, x_n)$ and $Q := (x_i x_j)_{1 \le i \le m, m+1 \le j \le n}$. So, $Q = Q_1 \cap Q_2$. Since $I \subseteq Q_1$ and $J \subseteq Q_2$, it follows that

$$(I, J, Q) = (I, J, Q_1) \cap (I, J, Q_2) = (J, Q_1) \cap (I, Q_2).$$

By the assumption, we have $Q_1 \notin \operatorname{Ass}_{S_1} S_1/I$ and $Q_2 \notin \operatorname{Ass}_{S_2} S_2/J$. Hence

$$(x_1, \dots, x_m, x_{m+1}, \dots, x_n) \notin \operatorname{Ass}_S S/(I, Q_2)$$

and

$$(x_1, \dots, x_m, x_{m+1}, \dots, x_n) \notin \operatorname{Ass}_S S/(J, Q_1),$$

and so

$$\operatorname{depth}\left(\frac{S}{(J, Q_1)} \oplus \frac{S}{(I, Q_2)}\right) > 0 = \operatorname{depth}\left(\frac{S}{Q_1 + Q_2}\right).$$

Now, in view of the exact sequence

$$0 \to \frac{S}{(J, Q_1) \cap (I, Q_2)} \to \frac{S}{(J, Q_1)} \oplus \frac{S}{(I, Q_2)} \to \frac{S}{Q_1 + Q_2} \to 0,$$

[V, Lemma 1.3.9] implies that

$$\operatorname{depth}\left(\frac{S}{(I, J, Q)}\right) = \operatorname{depth}\left(\frac{S}{(J, Q_1) \cap (I, Q_2)}\right) = 1.$$

Now the proof is complete, because [C2, Theorem 2.1] yields that for any monomial ideals L of S if $\operatorname{depth} S/L \le 1$, then Stanley's conjecture holds for S/L. □

Corollary 2.10. *Let* Δ_1 *and* Δ_2 *be two non-empty disjoint simplicial complexes and* $\Delta := \Delta_1 \cup \Delta_2$. *Then Stanley's conjecture holds for* S/I_Δ.

Proof. For two natural integers $m < n$, we may assume that Δ_1 and Δ_2 are simplicial complexes on $[m]$ and $\{m+1, \dots, n\}$, respectively. Then $K[\Delta_1] = K[x_1, \dots, x_m]/I_{\Delta_1}$ and $K[\Delta_2] = K[x_{m+1}, \dots, x_n]/I_{\Delta_2}$, and so

$$K[\Delta] = K[x_1, \dots, x_m, x_{m+1}, \dots, x_n]/(I_{\Delta_1}, I_{\Delta_2}, \{x_i x_j\}_{1 \le i \le m, m+1 \le j \le n}).$$

We claim that $\operatorname{depth}(K[x_1,\dots,x_m]/I_{\Delta_1}) > 0$ and $\operatorname{depth}(K[x_{m+1},\dots,x_n]/I_{\Delta_2}) > 0$. Because if for example $\operatorname{depth}(K[x_1,\dots,x_m]/I_{\Delta_1}) = 0$, then $I_{\Delta_1} = (x_1,\dots,x_m)$. But, this implies that $\Delta_1 = \emptyset$ which contradicts our assumption on Δ_1. Now, the claim is immediate by Proposition 2.9. □

THEOREM 2.11. *Let Δ be a locally complete intersection simplicial complex on $[n]$. Then Stanley's conjecture holds for S/I_Δ.*

Proof. If Δ is connected, then Lemma 2.8 yields the claim. Otherwise, by [TY, Theorem 1.15], Δ is the disjoint union of finitely many non-empty simplicial complexes. So, in this case the assertion follows by Corollary 2.10. □

In [HP, Corollary 4.3] it is shown that if S/I is pretty clean, then it is sequentially Cohen-Macaulay. In [S1] this fact is reproved by a different argument and, in addition, it is shown that depth of S/I is equal to the minimum of the dimension of $S/\mathfrak{p}$, where $\mathfrak{p} \in \operatorname{Ass}_S S/I$. Also if S/I is pretty clean, then by [S2, Theorem 4.7] h-regularity conjecture holds for S/I. This implies part a) of the following remark.

Remark 2.12. Let I be a monomial ideal of S.

a) Assume that either:

i) I is almost complete intersection,

ii) $\mu(I) \le 3$; or

iii) I is the Stanley-Reisner ideal of a connected simplicial complex on $[n]$ which is locally complete intersection.

Then both Stanley's and h-regularity conjectures hold for S/I. Also, in each of these cases S/I is sequentially Cohen-Macaulay and $\operatorname{depth} S/I = \min\{\dim S/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass}_S S/I\}$.

b) We know that if S/I is pretty clean, then Stanley's conjecture holds for S/I. By using Corollary 2.10, we can provide an example of a monomial ideal I of S such that Stanley's conjecture holds for S/I, while it is not pretty clean. To this end, let Δ_1, Δ_2 and Δ be as in Corollary 2.10 and $\dim \Delta_i > 0$, $i = 1, 2$. Evidently, Δ is not shellable, and so [D, Theorem on page 53] implies that S/I_Δ is not pretty clean. On the other hand, Stanley's conjecture holds for S/I_Δ by Corollary 2.10.

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