

## CONVERGENCE OF LOOP ERASED RANDOM WALKS ON A PLANAR GRAPH TO A CHORDAL SLE(2) CURVE

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### Abstract

In this paper we consider the ‘natural’ random walk on a planar graph and scale it by a small positive number  $\delta$ . Given a simply connected domain  $D$  and its two boundary points  $a$  and  $b$ , we start the scaled walk at a vertex of the graph nearby  $a$  and condition it on its exiting  $D$  through a vertex nearby  $b$ , and prove that the loop erasure of the conditioned walk converges, as  $\delta \downarrow 0$ , to the chordal SLE<sub>2</sub> that connects  $a$  and  $b$  in  $D$ , provided that an invariance principle is valid for both the random walk and the dual walk of it. Our result is an extension of one due to Dapeng Zhan [12] where the problem is considered on the square lattice. A convergence to the radial SLE<sub>2</sub> has been obtained by Lawler, Schramm and Werner [3] for the square and triangular lattices and by Yadin and Yehudayoff [10] for a wide class of planar graphs. Our proof, though an adaptation of that of [3] and [10], involves some new ingredients that arise from two sources: one for dealing with a martingale observable that is different from that used in [3] and [10] and the other for estimating the harmonic measures of the random walk started at a boundary point of a domain.

### 1. Introduction

The Schramm-Loewner evolutions driven by Brownian motion  $\sqrt{\kappa}B(t)$  of variance  $\kappa$ , abbreviated as SLE <sub>$\kappa$</sub> , introduced by Oded Schramm [6], have been studied from various points of view and are now recognized to well describe the scaling limits of certain lattice models of both physical and mathematical interest. Lawler, Schramm and Werner [3] have proved that the scaling limit of a loop erased random walk (or loop erasure (for the definition, see p. 9) of random walk, abbreviated as LERW) on either of the square and triangular lattices is the radial SLE<sub>2</sub>. Dapeng Zhan [12] have studied LERW’s on the square lattice but in a multiply connected domain and derived the convergence of them. In the case of a simply connected domain in particular, he has proved the convergence to the chordal SLE<sub>2</sub>. Yadin and Yehudayoff [10] extend the result of [3], the convergence of LERW to a radial SLE to that for the natural random walks on

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planar graphs under a natural setting of the problem. In this paper we consider the LERW in a similar setting to [10] and show that LERW conditioned to connecting two boundary points in a simply connected domain converges to a chordal  $\text{SLE}_2$  curve.

Here we state our result in an informal way by using the terminology familiar in the theory of SLE of which we shall give a brief exposition in the next section. Let  $V$  be the set of vertices of a planar graph on which a random walk (of discrete time) is defined and supposed to satisfy invariance principle in the sense that the linear interpolation of its space-scaled trajectory converges to that of Brownian motion (in a topology where two curves are identified if they agree by some change of time parameter). For each  $\delta > 0$  we make the scale change of the space by  $\delta : V_\delta = \{\delta v : v \in V\}$ , the set of scaled lattice points and accordingly we make the  $\delta$ -scaling of our random walk so that it moves on  $V_\delta$ . Given a simply connected bounded domain  $D$  and two distinct boundary points  $a$  and  $b$  of it, let  $\gamma_\delta$  denote the loop erasure of the random walk scaled by  $\delta$  that starts a vertex  $a_\delta$  of  $V_\delta$  nearby  $a$  and is conditioned to exit  $D \cap V_\delta$  through a vertex  $b_\delta$  nearby  $b$  so that  $\gamma_\delta$  is a random self-avoiding path on  $D \cap V_\delta$  connecting  $a_\delta$  and  $b_\delta$ , which may be regarded as a ‘path’ in the planar graph. We prove that the polygonal curve given by linearly interpolating  $\gamma_\delta$  converges to the chordal  $\text{SLE}_2$  curve connecting  $a$  and  $b$  in  $D$  under a certain natural assumption on  $D$ , the pair  $a, b$ , the planar graph and the random walk (Theorem 5.6).

For obtaining the result as stated above we first prove the convergence of the driving function of the loop erasure (Theorem 5.1). The proof is made in a way similar to [3], [10] and [7]. In [7] the harmonic explorer, an evolution of a self avoiding random curve, is introduced and proved to converge to a chordal  $\text{SLE}_4$  curve. For the proof a suitably chosen martingale associated with the evolving random curve, called martingale observable, plays a dominant role. Not as in [7] we take the martingale observable given by the ratio of harmonic measures of a (random) point relative to two points, the starting site of the walk and a suitably chosen site in a random domain defined by the loop erasure. This martingale is suggested in [3] as a suitable candidate of a martingale observable but we need to normalize it in an appropriate way; moreover we must change the normalization as the loop erasure grows. We apply the approximation result on the harmonic measure (Poisson kernel) proved in [10]. To this end we need a delicate probability estimate, since our random walk starts at a boundary point and we must deal with the conditional law given that it exits  $D \cap V_\delta$  through another boundary point.

We deduce the convergence of the loop erasure in a uniform topology from that of the driving function under the hypothesis that not only the random walk but also the dual walk of it satisfy the invariance principle (Theorem 5.6). For the deduction we prove Proposition 4.1 asserting that the law of the time reversal of loop erasure of a walk agrees with the law of loop erasure of the time reversal of the same walk.

By the way, Proposition 4.1 provides an improvement of the convergence to a radial  $\text{SLE}_2$ . In [10] the loop erasure is unti-chronological (loops are discarded

in the reverse order). The reason is that one wants to consider the loop erasure determined from the boundary. Because the radial  $SLE_2$  starts at a boundary point and stops at an inner point, and one wants to use a domain Markov property of the loop erasure. In [3], they used the reversibility property of the loop erasure of a simple random walk proved by Lawler [2]. Proposition 4.1 implies that the convergence to  $SLE_2$  in the result of Yadin and Yehudayoff is valid also for LERW with the loops discarded in the chronological order instead of anti-chronological order.

The rest of the paper is organized as follows. In Sections 2 and 3 we give brief expositions of the Loewner evolution and SLE, respectively, and the fundamental results relevant to the present issue or used in the proof of our results. In Section 4, consisting of three subsections, we first give the framework of our problem, the planar graph as well as the random walk on it, and bring in the LERW together with results associated with it (Subsection 4.1); we then present a martingale associated with the LERW (Subsection 4.2); we also present the result of [10] which asserts an approximation of the harmonic measure of our random walk by the classical Poisson kernel and a trivial lemma of the planar graph (Subsection 4.3). The statement and proof of the main result of the present paper are given in Section 5. The convergence of the loop erasure to  $SLE_2$  curve with respect to the driving function is given in Subsection 5.1, where a certain probability estimate proved in Section 6 is taken for granted. The convergence of the loop erasure to  $SLE_2$  curve in a uniform topology is given in Subsection 5.2, where we prove the invariance of law of LERW in (a double) time reversion. In Section 6 we verify the aforementioned probability estimate which plays an crucial role in the proof of our result, a probability estimate of the scaled random walk on  $D \cap V_\delta$  starting at a boundary vertex under the conditional law given that it exists the domain through another boundary vertex that is specified in advance.

## 2. Loewner chain

In this section, consisting of four subsections, we give a brief exposition of the Loewner evolution and some results relevant to the present issue. The standard results in the theory as given in Lawler’s book [1] are stated under the heading as **P 2.k** ( $k = 1, 2, \dots$ ).

### 2.1. Conformal map and half-plane capacity

Let  $\mathbf{H} := \{z \in \mathbf{C} : \text{Im } z > 0\}$  be the upper half plane. A bounded subset  $A \subset \mathbf{H}$  is called a compact **H**-hull if  $A = \bar{A} \cap \mathbf{H}$  and  $\mathbf{H} \setminus A$  is a simply connected domain. Let  $\mathcal{Q}$  denote the set of compact **H**-hulls. For any  $A \in \mathcal{Q}$ , there exists a unique conformal map  $g_A : \mathbf{H} \setminus A \rightarrow \mathbf{H}$  satisfying  $|g_A(z) - z| \rightarrow 0$  as  $z \rightarrow \infty$ . The half-plane capacity  $\text{hcap}(A)$  is defined by

$$\text{hcap}(A) := \lim_{z \rightarrow \infty} z(g_A(z) - z).$$

Then,  $g_A$  has the expansion

$$g_A(z) = z + \frac{\text{hcap}(A)}{z} + O\left(\frac{1}{|z|^2}\right), \quad z \rightarrow \infty.$$

The half-plane capacity has some nice properties, of which we need the following.

**P 2.1** (p69–71). *If  $r > 0$ ,  $x \in \mathbf{R}$ ,  $A \in \mathcal{Q}$ , then*

$$\text{hcap}(rA) = r^2 \text{hcap}(A), \quad \text{hcap}(A + x) = \text{hcap}(A).$$

*If  $A, B \in \mathcal{Q}$ ,  $A \subset B$ , then*

$$\text{hcap}(B) = \text{hcap}(A) + \text{hcap}(g_A(B \setminus A)).$$

## 2.2. Chordal Loewner chain in $\mathbf{H}$

A chordal Loewner chain is the solution of a type of Loewner equation that describes the evolution of a curve growing from the boundary to the boundary of a domain in  $\mathbf{C}$ . In this section we consider the special case when the domain is  $\mathbf{H} := \{z \in \mathbf{C} : \text{Im } z > 0\}$ , the upper half plane and the curve grows from the origin to the infinity in  $\mathbf{H}$ . Suppose that  $\gamma : [0, \infty) \rightarrow \bar{\mathbf{H}}$  is a simple curve with  $\gamma(0) = 0$ ,  $\gamma(0, \infty) \subset \mathbf{H}$ . Then, for each  $t \geq 0$ , there exists a unique conformal map  $g_t : \mathbf{H} \setminus \gamma(0, t] \rightarrow \mathbf{H}$  satisfying  $|g_t(z) - z| \rightarrow 0$  as  $z \rightarrow \infty$ . It is noted that  $g_t$  can be continuously extended to the (two sided) boundary of  $\mathbf{H} \setminus \gamma(0, t]$  along  $\gamma(0, t]$ . If  $\gamma$  is parametrized by half plane capacity (i.e., if  $\lim_{z \rightarrow \infty} z(g_t(z) - z) = 2t$ ),  $g_t$  satisfies the following differential equation

$$(1) \quad \frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - U(t)}, \quad g_0(z) = z,$$

where  $U(t) = g_t(\gamma(t))$  and  $U(\cdot)$  is a  $\mathbf{R}$ -valued continuous function (see [1]). We call the equation (1) the chordal Loewner equation and  $U(\cdot)$  the driving function.

Conversely, suppose that  $U(\cdot) : [0, \infty) \rightarrow \mathbf{R}$ , a continuous function, is given in advance, for  $z \in \mathbf{H}$ , solve the ordinary differential equation (1) to obtain the solution  $g_t(z)$  up to the time  $T_z := \sup\{t > 0 : |g_t(z) - U(t)| > 0\}$  and put  $K_t := \{z \in \mathbf{H} : T_z \leq t\}$ . Then for  $t > 0$ ,  $g_t(z)$  is a conformal map from  $\mathbf{H} \setminus K_t$  to  $\mathbf{H}$ . The family  $(g_t)_{t \geq 0}$  describes the evolution of hulls  $(K_t)_{t \geq 0}$  corresponding to  $U(\cdot)$  and growing from the boundary to  $\infty$ . Therefore, we have a one-to-one correspondence between  $U(\cdot)$  and  $(K_t)_{t \geq 0}$ . If  $U(\cdot)$  is the driving function of a simple curve  $\gamma$ , we can recover  $\gamma$  from  $U(\cdot)$  by the formula  $\gamma(t) = g_t^{-1}(U(t))$  and we can write  $K_t = \gamma(0, t]$ . If  $U(\cdot)$  is sufficiently nice, then  $(K_t)_{t \geq 0}$  is generated by a curve  $\gamma$  with  $\gamma(0) \in \mathbf{R}$ ,  $\lim_{t \rightarrow \infty} \gamma(t) = \infty$  (i.e., for any  $t \geq 0$ ,  $\mathbf{H} \setminus K_t$  is the unbounded component of  $\mathbf{H} \setminus \gamma(0, t]$ ). However, there exists a continuous function  $U(\cdot)$  such that  $(K_t)_{t \geq 0}$  can not be generated by a curve. There is known a sufficient condition for  $U(\cdot)$  to drive a curve as given by

**P 2.2** (p108). *Suppose for some  $r < \sqrt{2}$  and all  $s < t$ ,*

$$|U(t) - U(s)| \leq r\sqrt{t - s}.$$

*Then  $(K_t)_{t \geq 0}$  is generated by a simple curve.*

The family  $g_t$ ,  $t \geq 0$  is called the (chordal) Loewner chain generated by a curve  $\gamma$  or driven by a function  $U(t)$ . In summary, a simple curve  $\gamma$  brings out a Loewner chain, whereby it determines the driving function  $U(t)$ , and conversely a continuous function  $U(t)$  with appropriate regularity generates a curve through the Loewner chain driven by  $U(t)$ .

**PROPOSITION 2.3** (Lemma 2.1. in [3]). *There exists a constant  $C > 0$  such that the following holds. Let  $K_t$  be the corresponding hull for a Loewner chain driven by a continuous function  $U(t)$ . Set*

$$k(t) := \sqrt{t} + \sup\{|U(s) - U(0)| : 0 \leq s \leq t\}.$$

*Then, for any  $t > 0$ ,*

$$C^{-1}k(t) \leq \text{diam}(K_t) \leq Ck(t).$$

**2.3. Chordal Loewner chains in simply connected domains**

Let  $D \Subset \mathbf{C}$  be a simply connected domain and  $\partial D$  a set of prime ends. If  $D$  is a Jordan domain, then  $\partial D$  may be identified with the topological boundary of  $D$ . Let  $a, b$  be distinct points on  $\partial D$ . For  $p \in D$ , we define the inner radius of  $D$  with respect to  $p$ ,

$$\text{rad}_p(D) := \inf\{|z - p| : z \notin D\}.$$

Let  $\phi : D \rightarrow \mathbf{H}$  be a conformal map with  $\phi(a) = 0$ ,  $\phi(b) = \infty$ . Although  $\phi$  is not unique, any other such map can be written as  $r\phi$  for some  $r > 0$ . For a simple curve  $\gamma : (0, T) \rightarrow D$  connecting  $a$  and  $b$  so that  $\gamma(0+) = a$  and  $\gamma(T-) = b$ , let  $g_t$  be the Loewner chain generated by the curve  $\phi \circ \gamma : (0, T) \rightarrow \mathbf{H}$  and put

$$\phi_t = g_t \circ \phi, \quad t \in [0, \infty).$$

We reparametrize the curve  $\gamma$  so that the curve  $\phi \circ \gamma$  in  $\mathbf{H}$  is parametrized by half plane capacity. Denote by  $(\gamma(t))$  the function representing the curve in this parametrization, so that  $2t = \text{hcap}(\phi \circ \gamma[0, t])$ . The driving function  $U(t)$  of the chain  $g_t$  is then given by

$$U(t) = \phi_t(\gamma(t)).$$

The family of conformal maps  $\phi_t$ ,  $t \geq 0$  may also be called a chordal Loewner chain (in  $D$ ) with driving function  $U(t)$ . For each  $s > 0$ ,  $\phi_s$  conformally maps  $D(s) := D \setminus \gamma(0, s]$  onto  $\mathbf{H}$  with  $\phi_s(a_s) = U(s)$ ,  $\phi_s(b) = \infty$ , where  $a_s = \gamma(s)$  and the curve  $\gamma^{(s)}(t) := \gamma(s + t)$  connects  $a_s$  and  $b$  in  $D(s)$ . On putting

$$(2) \quad g_t^{(s)} = g_{s+t} \circ g_s^{-1} \quad \text{and} \quad \phi_t^{(s)} = \phi_{s+t},$$

substitution into  $U(s+t) = \phi_{s+t}(\gamma(s+t))$  yields

$$(3) \quad U(t+s) = \phi_t^{(s)}(\gamma^{(s)}(t)).$$

It follows from (2) that  $\phi_t^{(s)} = g_t^{(s)} \circ \phi_s$  and  $g_t^{(s)}$  (and  $\phi_t^{(s)}$ ) is the Loewner chain generated by the curve  $\gamma^{(s)}$ ; and also, from (3) that  $U^{(s)}(t) := U(s+t)$  is the driving function of the chain  $\phi_t^{(s)}$  in  $D(s)$ .

Define  $p(t) \in D$  by

$$\phi_t(p(t)) = U(t) + i.$$

$p(t)$  serves as a reference point for the study of the conformal map  $\phi_t$ . (See Proposition 4.5 and the remark advanced before Lemma 5.3.)

LEMMA 2.4. *Let  $T > 1$  and  $\epsilon > 0$ , and, given a pair  $(D, \gamma)$ , put  $\tilde{T} := \sup\{t \in [0, T] : |U(t)| < 1/\epsilon\}$ . Then there exists a constant  $c(T, \epsilon) > 0$ , which may also depend on  $(D, \gamma(0))$  but does not on  $(\gamma(t), t > 0)$ , such that*

$$\text{rad}_{p(t)}(D(t)) \geq c(T, \epsilon) \text{rad}_{p(0)}(D) \quad \text{for } t < \tilde{T}.$$

*Proof.* We claim that

$$(4) \quad |\phi(p(t)) - \phi(\gamma(t'))| \geq 2^{-1}e^{-4\tilde{T}} \quad \text{if } t' \leq t < \tilde{T}.$$

Let  $t' \leq t < \tilde{T}$  and  $z = \phi(\gamma(t'))$ , and put

$$y(s) = g_s(\phi(p(t))) - g_s(z), \quad 0 \leq s \leq t.$$

We prove  $|y(0)| = |\phi(p(t)) - z| \geq 2^{-1}e^{-4\tilde{T}}$ . Recalling that  $\text{Im } g_s(w)$  is decreasing in  $s$  for any  $w \in \mathbf{H}$ , we see that

$$(5) \quad \text{Im } g_s \circ \phi(p(t)) \geq \text{Im } g_t \circ \phi(p(t)) = 1 \quad \text{if } s \leq t.$$

Applying this with  $s = 0$  we have  $|y(0)| \geq 1/2$  if  $\text{Im } z \leq 1/2$ . Let  $\text{Im } z > 1/2$  and define  $\tau := \inf\{t \geq 0 : \text{Im } g_t(z) = 1/2\}$ . Then  $\tau < t' \leq t$  (since  $\text{Im } g_{t'}(z) = 0$ ) and the Loewner equation together with the inequality (5) shows

$$\left| \frac{d}{ds} y(s) \right| = \frac{2|y(s)|}{|g_s \circ \phi(p(t)) - U(s)| \cdot |g_s(z) - U(s)|} \leq 4|y(s)| \quad \text{for } 0 \leq s \leq \tau.$$

Hence  $|y(s)|$  is absolutely continuous and satisfies  $\frac{d}{ds}|y(s)| \leq 4|y(s)|$ , so that

$$|y(\tau)| \leq |y(0)|e^{4\tau}.$$

Using (5) again we have  $1/2 \leq \text{Im } y(\tau)$  so that  $1/2 \leq |y(0)|e^{4\tilde{T}}$ , which is the same as what we need to prove. Thus the claim (4) is verified.

It is proved in [8] (the proof of Corollary 4.3) that the set  $\{\phi(p(t)) : t < \tilde{T}\}$  is included in a compact set of  $\mathbf{H}$  depending only on  $T$  and  $\epsilon$ , whence according to the Koebe distortion theorem  $\text{rad}_{p(t)}(D) \geq c_0(T, \epsilon) \text{rad}_{p(0)}(D)$  for some constant

$c_0(T, \epsilon) > 0$ . For the proof of the lemma it therefore suffices to show that

$$|p(t) - \gamma(t')| \geq c_1(T, \epsilon) \operatorname{dist}(p(t), \partial D) \quad \text{for } t' \leq t < \tilde{T}.$$

To this end we may suppose  $|p(t) - \gamma(t')| < 2^{-1} \operatorname{dist}(p(t), \partial D)$ . Applying (4) and the distortion theorem in turn yields

$$2^{-1} e^{-4\tilde{T}} \leq |\phi(p(t)) - \phi(\gamma(t'))| \leq 16|p(t) - \gamma(t')| \cdot \frac{\operatorname{dist}(\phi(p(t)), \mathbf{R})}{\operatorname{dist}(p(t), \partial D)}.$$

We know that  $\operatorname{dist}(\phi(p(t)), \mathbf{R}) \leq M$  for some constant  $M = M(T, \epsilon) > 0$  from the result of [8] mentioned above. Hence  $|p(t) - \gamma(t')| \geq [e^{-4\tilde{T}}/32M] \operatorname{dist}(p(t), \partial D)$  as desired.  $\square$

**2.4. Metrics on curves**

Let  $\gamma, \gamma^j$  ( $j = 1, 2, \dots$ ) be curves which generate the Loewner chains. Let  $U(t)$  and  $U_j(t)$  be driving functions corresponding to  $\gamma$  and  $\gamma^j$ , respectively. If  $U_j(t)$  converges uniformly to  $U(t)$  on any bounded interval, then we will say that  $\gamma^j$  converges to  $\gamma$  with respect to the driving function.

Next, we consider the metric on the space of unparametrized curves in  $\mathbf{C}$ . Let  $f_1, f_2 : [0, 1] \rightarrow \mathbf{C}$  be a continuous, non-locally constant functions. If there exists a continuously increasing bijection  $\alpha : [0, 1] \rightarrow [0, 1]$  such that  $f_2 = f_1 \circ \alpha$ , then we will say  $f_1$  and  $f_2$  are the same up to reparametrization, denoted by  $f_1 \sim f_2$ . A unparametrized curve  $\gamma$  is defined to be an equivalence class modulo  $\sim$ . Let  $d_*$  be the spherical metric on  $\hat{\mathbf{C}}$ . We define the metric on the space of unparametrized curves by

$$(6) \quad d_{\mathcal{U}}(\gamma_1, \gamma_2) := \inf_{\alpha} \left[ \sup_{0 \leq t \leq 1} d_*(f_1(t), f_2 \circ \alpha(t)) \right],$$

where  $f_i$  any function in the equivalence class  $\gamma_i$ , and the infimum is taken over all reparametrizations  $\alpha$  which are continuously increasing bijections of  $[0, 1]$ . We often denote by the same notation  $\gamma$  a parametrized curve as well as an unparametrized curve. Let us denote by  $\gamma^-$  the time reversal of  $\gamma$ .

The convergence with respect to the driving function is weaker than the convergence with respect to the metric  $d_{\mathcal{U}}$ . We will consider a sufficient condition for the convergence with respect to the metric  $d_{\mathcal{U}}$  when we have the convergence with respect to the driving function. Let  $D \subsetneq \mathbf{C}$  be a simply connected domain and  $\partial D$  be the set of prime ends of  $D$ . Let  $a, b \in \partial D$  be distinct points. Let  $\phi : D \rightarrow \mathbf{H}$  be a conformal map with  $\phi(a) = 0, \phi(b) = \infty$ . Let  $\phi^- : D \rightarrow \mathbf{H}$  be a conformal map with  $\phi^-(b) = 0, \phi^-(a) = \infty$ .

**THEOREM 2.5** (Theorem 1.2 in [8]). *Let  $\{\gamma^j\}$  be a sequence of simple curves travelling from  $a$  to  $b$  in  $D$ . Suppose that there exists simple curves  $\gamma$  and  $\eta$  such that  $\phi \circ \gamma^j$  converges to  $\phi \circ \gamma$  with respect to the driving function and  $\phi^- \circ \gamma^{j-}$  converges to  $\phi^- \circ \eta$  with respect to the driving function. Then  $\gamma^- = \eta$  and  $\gamma^j$  converges to  $\gamma$  with respect to the metric  $d_{\mathcal{U}}$ .*

### 3. Schramm-Loewner evolutions

#### 3.1. SLE in the upper half plane

Let  $B_t$  be a one-dimensional standard Brownian motion with  $B_0 = 0$ . A chordal Schramm-Loewner evolution with parameter  $\kappa > 0$  (abbreviated as chordal  $SLE_\kappa$ ) is the random family of conformal map  $g_t$  obtained from the chordal Loewner equation

$$(7) \quad \frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z \quad (z \in \mathbf{H}).$$

Let  $K_t$  be an evolving (random) hull corresponding to  $SLE_\kappa$ . Because  $B_t$  is not  $(1/2)$ -Hölder continuous, we can not use **P2.2** and it is not easy to see whether  $K_t$  is generated by a curve. However, according to the following results  $K_t$  is actually generated by a curve with full probability.

**P 3.1** (p148). *With probability 1, the limit  $\gamma(t) := \lim_{z \rightarrow 0} g_t^{-1}(z + \sqrt{\kappa} B_t)$  exists for any  $t \geq 0$  and  $K_t$  is generated by the curve  $\gamma$ .*

This curve  $\gamma$  is called a chordal  $SLE_\kappa$  curve in  $\mathbf{H}$  from 0 to  $\infty$ . The following properties of  $SLE_\kappa$  curves are easily verified.

**P 3.2** (p148). *Suppose that  $\gamma$  is a chordal  $SLE_\kappa$  curve in  $\mathbf{H}$  and  $r > 0$ . Let  $\hat{\gamma}(t) := r^{-1}\gamma(r^2t)$ . Then,  $\hat{\gamma}$  has the same distribution as  $\gamma$ .*

**P 3.3** (p147). *Suppose that  $\gamma$  is a chordal  $SLE_\kappa$  curve in  $\mathbf{H}$ . Let  $\tau$  be a stopping time. Let  $\hat{\gamma}(t) := g_\tau(\gamma(t + \tau)) - \sqrt{\kappa} B_\tau$ . Then,  $\hat{\gamma}$  has the same distribution as  $\gamma$ .*

The behaviour of a chordal  $SLE_\kappa$  curve depends on the value of the parameter  $\kappa$ . There is three phases in the behaviour of a chordal  $SLE_\kappa$  curve. The two phases transitions take place at the values  $\kappa = 4$  and  $\kappa = 8$ .

**P 3.4** (p150–151). *Suppose that  $\gamma$  be a chordal  $SLE_\kappa$  curve in  $\mathbf{H}$ .*

- *If  $0 < \kappa \leq 4$ , then w.p.1,  $\gamma$  is a simple curve with  $\gamma(0, \infty) \subset \mathbf{H}$ .*
- *If  $4 < \kappa < 8$ , then w.p.1,  $\gamma(0, \infty) \cap \mathbf{H} \neq \mathbf{H}$  and  $\bigcup_{t>0} \bar{K}_t = \bar{\mathbf{H}}$ .*
- *If  $\kappa \geq 8$ , then w.p.1,  $\gamma$  is a space-filling curve, i.e.,  $\gamma[0, \infty) = \bar{\mathbf{H}}$ .*

#### 3.2. SLE in simply connected domains

Let  $\gamma$  be a chordal  $SLE_\kappa$  curve in  $\mathbf{H}$  from 0 to  $\infty$ . As in the subsection 2.3 let  $D \subsetneq \mathbf{C}$  be a simply connected domain,  $\partial D$  a set of prime ends,  $a, b$  two distinct points on  $\partial D$  and  $\phi: D \rightarrow \mathbf{H}$  a conformal map with  $\phi(a) = 0$ ,  $\phi(b) = \infty$ . Although  $\phi$  is not unique, any other such map  $\tilde{\phi}$  can be written as  $r\phi$  for some  $r > 0$ . By **P 3.2**,  $\phi^{-1}(\gamma)$  is independent of the choice of the map up to a time change and we consider  $SLE_\kappa$  curves in  $D$  as unparametrized curves. A chordal  $SLE_\kappa$  curve in  $D$  from  $a$  to  $b$  is defined by  $\phi^{-1}(\gamma)$ .

The two properties stated in the next proposition, called the domain Markov property and conformal invariance, respectively, immediately follow from the definition of SLE.

**P 3.5.** *Let  $\gamma$  be a chordal  $SLE_\kappa$  curve in  $D$  from  $a$  to  $b$  and  $\mu_{a,b;D}$  be a law of  $\gamma$ . Let  $f : D \rightarrow D'$  be a conformal map. Then,*

$$\mu_{a,b;D}(\cdot|\gamma(0, t]) = \mu_{\gamma(t),b;D \setminus \gamma(0, t]}(\cdot),$$

and

$$f \circ \mu_{a,b;D}(\cdot) = \mu_{f(a),f(b);D'}(\cdot).$$

In the theory of SLE, it is easier to prove the convergence with respect to the driving function than in the metric  $d_{\mathcal{H}}$ . Theorem 2.5 implies the following result, which we shall apply the following result to derive the convergence with respect to  $d_{\mathcal{H}}$  of LERW from that of the driving function. Let  $\phi^- : D \rightarrow \mathbf{H}$  be a conformal map with  $\phi^-(b) = 0$ ,  $\phi^-(a) = \infty$ .

**THEOREM 3.6** ([8]). *Let  $\{\gamma^j\}$  be a sequence of simple random curves travelling from  $a$  to  $b$  in  $D$ . Let  $\kappa \leq 4$ , and  $\gamma(a, b)$  be the chordal  $SLE_\kappa$  curve in  $D$  from  $a$  to  $b$ .  $\phi \circ \gamma^j$  and  $\phi^- \circ \gamma^{j-}$  converge weakly to a chordal  $SLE_\kappa$  curve in  $\mathbf{H}$  with respect to the driving function. Then  $\gamma_j$  converges weakly to  $\gamma(a, b)$  with respect to  $d_{\mathcal{H}}$ .*

The reversibility of SLE holds at least for  $\kappa \leq 4$ .

**THEOREM 3.7** (Theorem 2.1 in [11]). *Let  $\kappa \leq 4$ . The time-reversal of a chordal  $SLE_\kappa$  curve in  $D$  from  $a$  to  $b$  has the same distribution as chordal  $SLE_\kappa$  curve in  $D$  from  $b$  to  $a$ .*

If  $\kappa > 8$ , then SLE curve is not reversible.

#### 4. Loop erased random walks

##### 4.1. Some property of LERW

For any  $u, v \in \mathbf{C}$ , we write  $[u, v] = \{(1-t)u + tv : 0 \leq t \leq 1\}$  for the line segment whose end points are  $u$  and  $v$ . Let  $V \subset \mathbf{C}$  be a countable subset with  $0 \in V$ . Let  $E : V \times V \rightarrow [0, \infty)$  and  $E = \{(u, v) : E(u, v) > 0\}$ . We call  $G = (V, E)$  a directed weighted graph. We assume that  $\sum_{v \in V} E(u, v) < \infty$  for every  $u \in V$ , and put

$$p(u, v) := \frac{E(u, v)}{\sum_{w \in V} E(u, w)}.$$

We call  $G$  that satisfies the following conditions a planar irreducible graph.

1.  $G$  is a planar graph.  
(i.e. for every distinct edges  $(u, v), (u', v') \in E$ ,  $[u, v] \cap [u', v'] \in \{\emptyset, \{u\}, \{v\}\}$ .)
2. For any compact set  $K \subset \mathbf{C}$ , the number of vertices  $v \in K$  is finite.
3. The Markov chain  $S(\cdot)$  on  $V$  with transition probability  $p(u, v)$  is irreducible.  
(i.e. for every  $u, v \in V$ , there exists  $n \in \mathbf{N}$  such that  $\mathbf{P}(S(n) = v \mid S(0) = u) > 0$ .)

We call  $S(\cdot)$  the natural random walk on  $G$ . For the remainder of this paper we think that  $G$  is a planar irreducible graph.

For any simply connected domain  $D \subsetneq \mathbf{C}$ , let  $V(D) := V \cap D$ . Define

$$\partial_{out}V(D) := \{(u, v) \in E : [u, v] \cap \partial D \neq \emptyset, u \in V(D)\}$$

and

$$\partial_{in}V(D) := \{(u, v) \in E : [u, v] \cap \partial D \neq \emptyset, v \in V(D)\}.$$

The first exit time from  $D$  is defined by

$$\tau_D := \begin{cases} \inf\{n \geq 1 : (S(n-1), S(n)) \in \partial V_{out}(D)\} & \text{if } S(0) \in V(D) \\ \inf\{n \geq 2 : (S(n-1), S(n)) \in \partial V_{out}(D)\} & \text{if } (S(0), S(1)) \in \partial_{in}V(D). \\ 0 & \text{otherwise} \end{cases}$$

We sometimes consider the edge  $(u, v) \in \partial_{out}V(D)$  as the vertex  $v$ , and the edge  $(u, v) \in \partial_{in}V(D)$  as the vertex  $u$ ; e.g., we write  $S(\tau_D) \in \partial_{out}V(D)$  and  $S(0) \in \partial_{in}V(D)$  and for a set  $J \subset \partial D$ , we write  $S(\tau_D) \in J$  instead of writing  $[S(\tau_D - 1), S(\tau_D)] \cap J \neq \emptyset$ .

**LOOP ERASURE.** Let  $\omega = (\omega_0, \omega_1, \dots, \omega_n)$  be a finite sequence of points. Let  $s_0 = \max\{k \geq 0 : \omega_0 = \omega_k\}$ . Inductively, we define  $s_m = \max\{k \geq 0 : \omega_{s_{m-1}+1} = \omega_k\}$ . If  $l = \min\{m \geq 0 : \omega_{s_m} = \omega_n\}$ , then the loop erasure of  $\omega$  is defined by

$$L[\omega] = (\omega_{s_0}, \omega_{s_1}, \dots, \omega_{s_l}).$$

The time-reversal of  $\omega$  is defined by

$$\omega^- = (\omega_n, \omega_{n-1}, \dots, \omega_0).$$

It is readily recognized that the operations  $L$  and  $-$  are not commutable, namely,  $L[\omega^-] \neq L[\omega]^-$  in general. If the transition probability  $p(u, v)$  is symmetric, then the following result has been proved by Lawler in [2]. For our purpose, we prove the following result without assuming that  $p(u, v)$  is symmetric.

**PROPOSITION 4.1.** *Let  $S(\cdot)$  be a natural random walk on  $G$ .*

$$\mathbf{P}(L[(S(0), S(1), \dots, S(\tau_D))^-] = \omega) = \mathbf{P}(L[(S(0), S(1), \dots, S(\tau_D))]^- = \omega).$$

*Remark.* Theorem 4.1 implies that the convergence to the radial SLE<sub>2</sub> in the result of Yadin and Yehudayoff (Theorem 1.1 in [10]) is valid also for LERW

with the loops discarded in the chronological order instead of anti-chronological order.

*Proof.* Let  $\omega = (\omega_0, \dots, \omega_n)$  and  $\omega_1, \dots, \omega_{n-1} \in V(D)$  be distinct and  $(\omega_{n-1}, \omega_n) \in \partial_{out} V(D)$ . Our task is to show the identity

$$(8) \quad \mathbf{P}(L[(S(0), \dots, S(\tau_D))] = \omega) = \mathbf{P}(L[(S(0), \dots, S(\tau_D))^-] = \omega^-).$$

Let  $q : V \times V \rightarrow [0, 1]$ . Set

$$G_q(x; D) = 1 + \sum_{k=0}^{\infty} \sum_{\omega' \subset D: \omega'_0=x, \omega'_k=x} q(\omega'_0, \omega'_1) \cdots q(\omega'_{k-1}, \omega'_k),$$

where the inner summation is taken over all paths  $\omega' = (\omega'_0, \dots, \omega'_k)$  in  $D$  such that  $\omega'_0 = x, \omega'_k = x$ .

The probability of LERW is described by the following (See [2]).

$$\mathbf{P}(L[(S(0), \dots, S(\tau_D))] = \omega) = \prod_{j=0}^{n-1} p(\omega_j, \omega_{j+1}) G_p(\omega_j; D \setminus \{\omega_0, \dots, \omega_{j-1}\})$$

By the exchange lemma (the equation (12.2.3) in [2]), we get

$$(9) \quad \mathbf{P}(L[(S(0), \dots, S(\tau_D))] = \omega) = \prod_{j=0}^{n-1} p(\omega_j, \omega_{j+1}) G_p(\omega_j; D \setminus \{\omega_{j+1}, \dots, \omega_{n-1}\})$$

On the other hand,

$$\begin{aligned} \mathbf{P}(L[(S(0), \dots, S(\tau_D))^-] = \omega^-) &= \sum_{\omega' \subset D: L[(\omega')^-] = \omega^-} \prod_{i=0}^{|\omega'|-1} p(\omega'_i, \omega'_{i+1}) \\ &= \sum_{\omega' \subset D: L[\omega'] = \omega^-} \prod_{i=0}^{|\omega'|-1} p^*(\omega'_i, \omega'_{i+1}), \end{aligned}$$

where  $|\omega'|$  is the length of  $\omega'$  and  $p^*(x, y) := p(y, x)$ . This equation and decomposing  $\omega'$  between its last visit to  $\omega_{n-1}, \dots, \omega_0$  imply that

$$(10) \quad \begin{aligned} \mathbf{P}(L[(S(0), \dots, S(\tau_D))^-] = \omega^-) &= \prod_{j=0}^{n-1} p^*(\omega_{j+1}, \omega_j) G_{p^*}(\omega_j; D \setminus \{\omega_{n-1}, \dots, \omega_{j+1}\}) \\ &= \prod_{j=0}^{n-1} p(\omega_j, \omega_{j+1}) G_p(\omega_j; D \setminus \{\omega_{j+1}, \dots, \omega_{n-1}\}). \end{aligned}$$

Finally observe that  $G_p(x; D') = G_{p^*}(x; D')$ . Thus, (9) and (10) imply (8).  $\square$

Let  $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_l)$  be the loop erasure of the time-reversal of the natural random walk stopped on exiting  $D$ . By Proposition 4.1, we may think that  $\gamma$  is the time-reversal of the loop erasure. (In Section 5, we treat  $\gamma$  as the time-reversal of the loop erasure. But in this section, we treat  $\gamma$  as the loop erasure of the time-reversal because it is more suitable to consider the following properties of  $\gamma$ .)

Let  $D_j := D \setminus \bigcup_{i=0}^{j-1} [\gamma_i, \gamma_{i+1}]$ . For any  $j \in \mathbf{N}$ ,

$$n_j := \min\{n \geq 0 : S(n) = \gamma_j\}.$$

Because the loop erasure  $\gamma$  is determined from the boundary,  $\gamma$  has the following Markov property.

**PROPOSITION 4.2** (Lemma 3.2. in [3]). *Conditioned on  $\gamma[0, j]$ , the following holds.*

1.  $S[0, n_j]$  and  $S[n_j, \tau_D]$  are independent.
2.  $\gamma[j, l]$  has the same distribution as the loop erasure of time-reversal of the natural random walk  $S[0, \tau_{D_j}]$  conditioned to exit at  $\gamma_j$ .

#### 4.2. Martingale observable for LERW

Let  $D \Subset \mathbf{C}$  be a simply connected domain. Let  $S^x(\cdot)$  be a natural random walk on  $G$  started at  $x \in V$ . Let  $v_0 \in V(D) \cup \partial_{in} V(D)$  and  $\gamma$  be the loop erasure of time-reversal of the natural random walk  $S^{v_0}[0, \tau_D]$ . Let  $D_j := D \setminus \bigcup_{i=0}^{j-1} [\gamma_i, \gamma_{i+1}]$ . The hitting probability  $H_j(u, v)$  is defined by

$$H_j(u, v) := \mathbf{P}(S^u(\tau_{D_j}) = v).$$

Let  $\mathcal{F}_j$  be a filtration generated by  $\gamma[0, j]$ .

**PROPOSITION 4.3.** *For any  $w \in V(D)$ , let*

$$M_j := \frac{H_j(w, \gamma_j)}{H_j(v_0, \gamma_j)}.$$

*Then,  $M_j$  is a martingale with respect to  $\mathcal{F}_j$ .*

Lawler, Schramm and Werner [3] point out that the martingale  $M_j$  given above should be a possible martingale observable, although they don't adopt it but a martingale formed by the Green functions of evolving domains. They provide a curtailed proof that  $M_j$  is a martingale. Since  $M_j$  plays the central role in this paper we give a detailed proof of this fact.

*Proof.* First, we consider another representation of  $M_j$ . Let  $\hat{S}^x(\cdot)$  be a independent copy of  $S^x(\cdot)$  and  $L_x$  be the loop erasure of the time-reversal of  $\hat{S}^x[0, \tau_D]$ . We will denote by  $\mathbf{Q}$  the law of  $\hat{S}$ . Fix  $\gamma[0, j]$ . By Proposition 4.2,

$$\begin{aligned} \frac{\mathbf{Q}(L_w[0, j] = \gamma[0, j])}{\mathbf{Q}(L_{v_0}[0, j] = \gamma[0, j])} &= \frac{\mathbf{Q}(\hat{S}^w(\tau_{D_j}) = \gamma_j) \mathbf{Q}(L_{\gamma_j}[0, j] = \gamma[0, j])}{\mathbf{Q}(\hat{S}^{v_0}(\tau_{D_j}) = \gamma_j) \mathbf{Q}(L_{\gamma_j}[0, j] = \gamma[0, j])} \\ &= \frac{H_j(w, \gamma_j)}{H_j(v_0, \gamma_j)}. \end{aligned}$$

Therefore, we can write

$$M_j = \frac{\mathbf{Q}(L_w[0, j] = \gamma[0, j])}{\mathbf{Q}(L_{v_0}[0, j] = \gamma[0, j])}.$$

Hence,

$$\mathbf{E}[M_{j+1} | \gamma[0, j]] = \sum_{v \in V(D_j)} \mathbf{P}(\gamma_{j+1} = v | \gamma[0, j]) \cdot \frac{\mathbf{Q}(L_w[0, j] = \gamma[0, j], L_w(j+1) = v)}{\mathbf{Q}(L_{v_0}[0, j] = \gamma[0, j], L_{v_0}(j+1) = v)},$$

and, since  $\mathbf{P}(\gamma_{j+1} = v | \gamma[0, j]) = \mathbf{Q}(L_{v_0}(j+1) = v | L_{v_0}[0, j] = \gamma[0, j])$ , the right-hand side reduces to

$$\sum_{v \in V(D_j)} \frac{\mathbf{Q}(L_w[0, j] = \gamma[0, j], L_w(j+1) = v)}{\mathbf{Q}(L_{v_0}[0, j] = \gamma[0, j])} = \frac{\mathbf{Q}(L_w[0, j] = \gamma[0, j])}{\mathbf{Q}(L_{v_0}[0, j] = \gamma[0, j])} = M_j.$$

Thus,  $M_j$  is a martingale. □

**4.3. Estimates of discrete harmonic measures**

For  $\delta > 0$ , the graph  $G_\delta = (V_\delta, E_\delta)$  defined by

$$V_\delta = \{\delta u : u \in V\}, \quad E_\delta = \{(\delta u, \delta v) : E(u, v) > 0\}.$$

Let the Markov chain  $S_\delta(\cdot)$  on  $V_\delta$  be the scaling of  $S(\cdot)$  by a factor of  $\delta$ . We call  $S_\delta(\cdot)$  the natural random walk on  $G_\delta$ . Let  $S_\delta^x(\cdot)$  be a natural random walk on  $G_\delta$  started at  $x \in V_\delta$ . Similarly, we can define  $H_j^{(\delta)}(u, v)$ ,  $V_\delta(D)$ ,  $\partial_{out} V_\delta(D)$ ,  $\partial_{in} V_\delta(D)$ .

Let  $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$  be the unit disc.

**DEFINITION 4.4.** If the family of the random walks  $S_\delta^x$  satisfies the following condition, then we say that  $S_\delta^x$  satisfies invariance principle:

For any compact set  $K \subset \mathbf{D}$  and  $\epsilon > 0$ , there is some  $\delta_0 > 0$  such that the following holds. Let  $Z^x$  be a two-dimensional Brownian motion started at  $x$  stopped on exiting  $\mathbf{D}$ . For any  $0 < \delta < \delta_0$  and  $x \in K \cap V_\delta$ , there exists a coupling of  $S_\delta^x$  and  $Z^x$  satisfying

$$\mathbf{P}(d_w(S_\delta^x[0, \tau_{\mathbf{D}}], Z^x) > \epsilon) < \epsilon.$$

In view of the Skorokhod representation theorem the above condition is equivalent to holding that  $S_\delta^x$  weakly converges to  $Z^x$  uniformly for all  $x \in K$ .

In [10] (Lemma 1.2) the following result is proved.

PROPOSITION 4.5. *Suppose that  $S_\delta^x$  satisfies invariance principle. For any positive constants  $r, \epsilon$  and  $\eta < 1$ , there exists some  $\delta_0 > 0$  such that for all  $0 < \delta < \delta_0$  the following holds. Let  $D \subset \mathbf{D}$ , let  $p \in V_\delta(D)$  be such that  $\text{rad}_p(D) \geq r$ , and let  $\psi : D \rightarrow \mathbf{D}$  be a conformal map with  $\psi(p) = 0$ . Let  $y \in V_\delta(D)$  be such that  $|\psi(y)| < 1 - \eta$  and let  $a \in \partial_{\text{out}} V_\delta(D)$ . Then,*

$$\left| \frac{H_0^{(\delta)}(y, a)}{H_0^{(\delta)}(p, a)} - \frac{K_{\mathbf{D}}(\psi(y), \psi(a))}{K_{\mathbf{D}}(\psi(p), \psi(a))} \right| < \epsilon,$$

where  $K_{\mathbf{D}}$  stands for the Poisson kernel of  $\mathbf{D}$ .

The Poisson kernel of  $\mathbf{H}$  is given by

$$K_{\mathbf{H}}(u, v) := -\frac{1}{\pi} \text{Im} \left( \frac{1}{u - v} \right) = \frac{1}{\pi} \frac{\text{Im } u}{|u - v|^2}.$$

The result above may be translated in terms of  $K_{\mathbf{H}}$ . For our purpose we apply it in a rather trivial fashion. Let

COROLLARY 4.6. *Suppose that  $S_\delta^x$  satisfies invariance principle. For any constants  $r > 0, \epsilon > 0, \eta > 0$  and  $\lambda > 1$ , there exists some  $\delta_0 > 0$  such that for all  $0 < \delta < \delta_0$  the following holds. Let  $D \subset \mathbf{D}$ , let  $p \in D$  be such that  $\text{rad}_p(D) \geq r$ , and let  $\phi : D \rightarrow \mathbf{H}$  be a conformal map with  $\phi(p) = i$ . Let  $y, w \in V_\delta(D)$  be such that  $\text{Im } \phi(y) > \eta, \text{Im } \phi(w) > \eta$  and  $|\phi(y)| < \lambda, |\phi(w)| < \lambda$ . Then, for all  $a \in \partial_{\text{out}} V_\delta(D)$*

$$\left| \frac{H_0^{(\delta)}(w, a)}{H_0^{(\delta)}(y, a)} - \frac{K_{\mathbf{H}}(\phi(w), \phi(a))}{K_{\mathbf{H}}(\phi(y), \phi(a))} \right| < \epsilon.$$

*Proof.* Let  $p_\delta \in V_\delta(D)$  be a nearest point of  $p$ . Applying Proposition 4.5 with  $p = p_\delta$ ,

$$\begin{aligned} \frac{H_0^{(\delta)}(w, a)}{H_0^{(\delta)}(y, a)} &= \frac{H_0^{(\delta)}(w, a)/H_0^{(\delta)}(p_\delta, a)}{H_0^{(\delta)}(y, a)/H_0^{(\delta)}(p_\delta, a)} \\ &= \frac{K_{\mathbf{D}}(\psi(w), \psi(a))}{K_{\mathbf{D}}(\psi(y), \psi(a))} + O(\epsilon). \end{aligned}$$

Because the ratio of the Poisson kernel is conformal invariance, we find

$$\frac{K_{\mathbf{D}}(\psi(w), \psi(a))}{K_{\mathbf{D}}(\psi(y), \psi(a))} = \frac{K_{\mathbf{H}}(\phi(w), \phi(a))}{K_{\mathbf{H}}(\phi(y), \phi(a))}.$$

This completes the proof. □

Here we present the following trivial lemma for convenience of a later citation.

LEMMA 4.7. *Suppose that  $S_\delta^x$  satisfy invariance principle. For any  $\epsilon > 0$ , there exists  $\delta_0$  such that for all  $0 < \delta < \delta_0$ , the length of edges of  $G_\delta$  in  $\mathbf{D}$  is bounded above by  $\epsilon$ .*

*Proof.* Suppose that this Lemma is not true. Then, there exists  $\epsilon > 0$  such that for some sufficiently small  $\delta$ , there exists an edge  $e$  of  $G_\delta$  such that the length of  $e$  is bounded below by  $\epsilon$ . Since  $G_\delta$  is planar graph,  $S_\delta^x$  can not cross the edge  $e$ , so that it cannot behave as a Brownian path and the invariance principle fails to hold.  $\square$

### 5. Scaling limit

#### 5.1. Convergence with respect to the driving function

Let  $D \subsetneq \mathbf{C}$  be a simply connected domain and  $a, b$  two distinct points on  $\partial D$ . We say that  $\partial D$  is locally analytic at  $z \in \partial D$  if there exists a one-to-one analytic function  $f : \mathbf{D} \rightarrow \mathbf{C}$  with  $f(0) = z$  and  $f(\mathbf{D}) \cap D = f(\{w \in \mathbf{D} : \text{Im } w > 0\})$ . Let  $G = (V, E)$  be a planar irreducible graph and  $S_\delta^x$  a natural random walk on  $G_\delta$  started at  $x$  (see Section 4 for detailed description). Let  $\Gamma_\delta^{a,b}$  be a natural random walk on  $G_\delta$  started at  $a_\delta$  and stopped on exiting  $D$  and conditioned to hit  $\partial D$  at  $b_\delta$ , where  $a_\delta$  is a point of  $\partial_{in} V_\delta(D)$  close to  $a$  and  $b_\delta$  is a point of  $\partial_{out} V_\delta(D)$  close to  $b$  such that there exists a path on  $G_\delta$  connecting  $a_\delta$  and  $b_\delta$  in  $D$ . If  $\partial D$  is locally analytic at  $a$  and  $b$ , we can choose such  $a_\delta$  and  $b_\delta$ . Let  $\gamma_\delta^{a,b}$  be the loop erasure of  $\Gamma_\delta^{a,b}$ .

THEOREM 5.1. *Suppose that  $S_\delta^x$  satisfy invariance principle. Let  $D$  be a bounded simply connected domain and  $a, b$  be distinct points on  $\partial D$ . Suppose that  $\partial D$  is locally analytic at  $a$  and  $b$ . Let  $\phi : D \rightarrow \mathbf{H}$  be a conformal map with  $\phi(a) = 0, \phi(b) = \infty$ . Then,  $\phi \circ (\gamma_\delta^{b,a})^-$  converges weakly to the chordal  $SLE_2$  curve in  $\mathbf{H}$  as  $\delta \rightarrow 0$  with respect to the driving function.*

*Remark.* In order to assure the uniformity of invariance principle so imposed in Definition 4.4 it suffices to suppose it only for the walk starting at a point, e.g., the origin as is shown in [9].

We abbreviate  $(\gamma_\delta^{b,a})^- = \gamma = (\gamma_0, \gamma_1, \dots, \gamma_l)$ . By Proposition 4.1,  $\gamma$  has the same distribution as the loop erasure of the time-reversal of  $\Gamma_\delta^{b,a}$ . Hence, it is possible for  $\gamma$  to use results in Section 4. Let  $\mathcal{F}_j$  be a filtration generated by  $\gamma[0, j]$ . We may also think that  $\gamma[0, j]$  is the simple curve that is a linear interpolation.

Let  $U(t)$  be a driving function of  $\phi(\gamma)$  and  $g_t$  be a Loewner chain driven by  $U(t)$ . Let  $t_j := \frac{1}{2} \text{hcap } \phi(\gamma[0, j])$  and

$$U_j := U(t_j), \quad \phi_j := g_{t_j} \circ \phi \quad \text{and} \quad D_j := D \setminus \gamma[0, j].$$

Let  $p_j := \phi_j^{-1}(i + U_j)$ .  $p_j$  plays the role of a reference point, an ‘origin’, of  $D_j$ . In radial case, such a point is fixed at the origin. But in chordal case,  $p_j$

must be moved with  $j$ , so that there remains sufficient space around  $p_j$  in  $D_j$ , a sequence of reducing domains formed by encroachment of  $\gamma$  into  $D$ . (Cf. [7]).

We use the martingale introduced in Proposition 4.3, as in [3] and [10]. But we need to normalize it appropriately. We denote by  $S_\delta^b$  a natural random walk on  $G_\delta$  started at  $b_\delta$ . Let  $A := \phi^{-1}([-1, 1])$  and the normalization is made by multiplying  $\mathbf{P}(S_\delta^b(\tau_D) \in A)$ , which we name  $M_j$ :

$$(11) \quad M_j := \frac{H_j^{(\delta)}(w, \gamma_j)}{H_j^{(\delta)}(b, \gamma_j)} H_0^{(\delta)}(b; A),$$

(for any  $\delta > 0$  and  $w \in V_\delta(D)$ ), where we write  $H_0^{(\delta)}(b; A) := \mathbf{P}(S_\delta^b(\tau_D) \in A)$ .

Let  $D \not\subseteq \mathbf{C}$  be a simply connected domain,  $a, b$  two distinct points on  $\partial D$  and  $\phi : D \rightarrow \mathbf{H}$  a conformal map with  $\phi(a) = 0, \phi(b) = \infty$  as before. Let  $p = \phi^{-1}(i)$ . Put  $\Psi(z) = (z - i)/(z + i)$ . Define  $\psi := \Psi \circ \phi : D \rightarrow \mathbf{D}$ , which is a conformal map with  $\psi(b) = 1, \psi(p) = 0, \psi(a) = -1$ . Let  $\mathcal{D} = \mathcal{D}(r, R, \eta)$  be the collection of all quadruplets  $(D, a, b, p)$  such that  $\text{rad}_p(D) \geq r$  and  $D \subset \mathbf{RD}$  and  $\psi^{-1}$  has analytic extension in  $\{z \in \mathbf{C} : |z - 1| < \eta\}$ .

In the rest of this section let  $r, R$  and  $\eta$  be arbitrarily fixed positive constants and suppose the same hypothesis of Theorem 5.1 to be valid. We write  $\mathcal{D}$  for  $\mathcal{D}(r, R, \eta)$  and consider  $(D, a, b, p) \in \mathcal{D}$ . For dealing with the martingale observable  $M_j$  defined above the following lemma plays a significant role and  $\mathcal{D}(r, R, \eta)$  is introduced as a class for which the estimates given there is valid uniformly.

LEMMA 5.2. *There exists a number  $\lambda_0 = \lambda_0(\eta) > 1/2$  such that for any  $\varepsilon > 0$  and  $\lambda > \lambda_0$ , there exists numbers  $\delta_0 > 0$  and  $\alpha \in (0, 1/2)$  such that if  $(D, a, b, p) \in \mathcal{D}(r, R, \eta), 0 < \delta < \delta_0$  and  $D' = D \setminus \phi^{-1}(\{z : |z| < 2\lambda\})$ , then*

$$(12) \quad \mathbf{P}(\text{Im } \phi(S_\delta^b(\tau_{D'})) < \alpha\lambda \mid S_\delta^b(\tau_D) \in A) < \varepsilon,$$

and, if  $\text{diam}(\phi(\gamma[0, j])) < 1$ , then

$$(13) \quad \mathbf{P}(\text{Im } \phi(S_\delta^b(\tau_{D'})) < \alpha\lambda \mid S_\delta^b(\tau_{D_j}) = \gamma_j) < \varepsilon.$$

The proof of Lemma 5.2 is involved and postponed to the end of Section 6. For any  $\varepsilon > 0$ , let

$$m := \min\{j \geq 1 : t_j \geq \varepsilon^2 \text{ or } |U_j - U_0| \geq \varepsilon\}.$$

LEMMA 5.3. *There exists a constant  $C > 0$  and a number  $\varepsilon_0 > 0$  such that for each positive  $\varepsilon < \varepsilon_0$ , there exists  $\delta_0 > 0$  such that if  $(D, a, b, p) \in \mathcal{D}(r, R, \eta)$  and  $0 < \delta < \delta_0$ , then*

$$|\mathbf{E}[U_m - U_0]| \leq C\varepsilon^3,$$

and

$$|\mathbf{E}[(U_m - U_0)^2 - 2t_m]| \leq C\varepsilon^3.$$

(Although  $U_0 = 0$ , we write  $U_0$  in the formulae above to indicate how they show be when the starting position  $U_0 = \gamma_0$  is not mapped to the origin by  $\phi$ .)

*Proof.* This proof is broken into four steps. It consists of certain estimations of the harmonic functions that constitutes the martingale observable defined by (11).

*Step 1.* In this step we derive an expression, given in (16) below, of the ratio

$$H_j^{(\delta)}(b, \gamma_j) / H_0^{(\delta)}(b; A).$$

We take sufficiently small  $\epsilon_0 > 0$ , which we need in this proof. Given  $0 < \epsilon < \epsilon_0$  we take a number  $\lambda = 1/\epsilon^3$  that will be specified shortly. Let  $D' := D \setminus \phi^{-1}(B(U_0, 2\lambda) \cap \mathbf{H})$  (note that  $B(U_0, 2\lambda) = \{z : |z| < 2\lambda\}$ ). In the following we consider for  $j = 0, 1, 2, \dots$ , although we apply the resulting relation only for  $j = 0, m$ ,

$$H_j^{(\delta)}(b, \gamma_j) = \sum_{y \in V_\delta(D)} \mathbf{P}(S_\delta^b(\tau_{D'}) = y, S_\delta^b(\tau_{D_j}) = \gamma_j)$$

We split the sum on the right-hand side into two parts according as  $y$  is close to the boundary of  $D$  or not. The part of those  $y$  which are close to the boundary must be negligible.

Proposition 2.3 and the definition of  $m$  imply that  $\text{diam}(\phi(\gamma[0, m-1])) = O(\epsilon)$ . By Lemma 4.7, the harmonic measure from  $p$  of  $\gamma[m-1, m]$  in  $D_m$  is  $O(\epsilon)$  for sufficiently small  $\delta > 0$ . By conformal invariance of harmonic measure, the harmonic measure from  $\phi_{m-1}(p)$  of  $\phi_{m-1}(\gamma[m-1, m])$  in  $\mathbf{H} \setminus \phi_{m-1}(\gamma[m-1, m])$  is  $O(\epsilon)$ . This implies that  $\text{diam}(\phi_{m-1}(\gamma[m-1, m])) = O(\epsilon)$ , and we have

$$(14) \quad \text{diam}(\phi(\gamma[0, m])) = O(\epsilon).$$

By (14) and Lemma 5.2, we can choose  $\alpha = \alpha(\epsilon) < 1/2$  so that for all sufficiently small  $\delta > 0$ , for  $j = 0, m$ ,

$$\mathbf{P}(\text{Im } \phi(S_\delta^b(\tau_{D'})) < \alpha\lambda \mid S_\delta^b(\tau_{D_j}) = \gamma_j) = O(\epsilon^3).$$

This implies

$$\begin{aligned} \frac{\mathbf{P}(\text{Im } \phi(S_\delta^b(\tau_{D'})) < \alpha\lambda, S_\delta^b(\tau_{D_j}) = \gamma_j)}{\mathbf{P}(\text{Im } \phi(S_\delta^b(\tau_{D'})) \geq \alpha\lambda, S_\delta^b(\tau_{D_j}) = \gamma_j)} &= \frac{\mathbf{P}(\text{Im } \phi(S_\delta^b(\tau_{D'})) < \alpha\lambda \mid S_\delta^b(\tau_{D_j}) = \gamma_j)}{\mathbf{P}(\text{Im } \phi(S_\delta^b(\tau_{D'})) \geq \alpha\lambda \mid S_\delta^b(\tau_{D_j}) = \gamma_j)} \\ &= O(\epsilon^3). \end{aligned}$$

Therefore,

$$(15) \quad H_j^{(\delta)}(b, \gamma_j) = (1 + O(\epsilon^3)) \sum_{\substack{y \in V_\delta(D) \\ \text{Im } \phi(y) \geq \alpha\lambda}} \mathbf{P}(S_\delta^b(\tau_{D'}) = y, S_\delta^b(\tau_{D_j}) = \gamma_j).$$

By strong Markov property,

$$\begin{aligned} \frac{\mathbf{P}(S_\delta^b(\tau_{D'}) = y, S_\delta^b(\tau_{D_j}) = \gamma_j)}{H_0^{(\delta)}(b; A)} &= \frac{\mathbf{P}(S_\delta^b(\tau_{D'}) = y)\mathbf{P}(S_\delta^y(\tau_{D_j}) = \gamma_j)}{\mathbf{P}(S_\delta^b(\tau_D) \in A)} \\ &= \frac{\mathbf{P}(S_\delta^b(\tau_{D'}) = y)\mathbf{P}(S_\delta^y(\tau_D) \in A)}{\mathbf{P}(S_\delta^b(\tau_D) \in A)} \cdot \frac{\mathbf{P}(S_\delta^y(\tau_{D_j}) = \gamma_j)}{\mathbf{P}(S_\delta^y(\tau_D) \in A)}. \end{aligned}$$

Therefore, (15) implies

$$(16) \quad \frac{H_j^{(\delta)}(b, \gamma_j)}{H_0^{(\delta)}(b; A)} = (1 + O(\epsilon^3)) \sum_{\substack{y \in V_\delta(D) \\ \text{Im } \phi(y) \geq \alpha\lambda}} \mathbf{P}(S_\delta^b(\tau_{D'}) = y | S_\delta^b(\tau_D) \in A) \cdot \frac{H_j^{(\delta)}(y, \gamma_j)}{H_0^{(\delta)}(y; A)}.$$

*Step 2.* Let  $w \in V_\delta$  and  $y \in V_\delta(D)$  satisfy

$$(17) \quad \text{Im } \phi(w) \geq \frac{1}{2}, \quad |\phi(w) - U_0| \leq 3; \quad \text{Im } \phi(y) \geq \alpha\lambda, \quad \lambda \leq |\phi(y) - U_0| \leq 2\lambda.$$

Applying Corollary 4.6 to the domain  $D$  with a reference point  $p$ ,

$$(18) \quad \frac{H_0^{(\delta)}(w, \gamma_0)}{H_0^{(\delta)}(y, \gamma_0)} = \frac{\text{Im } \phi(w)/|\phi(w) - U_0|^2}{\text{Im } \phi(y)/|\phi(y) - U_0|^2} + O(\epsilon^3),$$

and the assumed invariance principle implies

$$(19) \quad H_0^{(\delta)}(y; A) = \frac{1}{\pi} \int_{-1}^1 \frac{\text{Im } \phi(y)}{|\phi(y) - x|^2} dx + O(\epsilon^3 \alpha / \lambda)$$

since  $|\phi(y) - U_0|^2 / \text{Im } \phi(y) \leq 2\lambda / \alpha$  (recall  $\alpha / \lambda$  must get small together with  $\epsilon$ ). The relations (17), (18) and (19) together imply

$$(20) \quad \begin{aligned} \frac{H_0^{(\delta)}(w, \gamma_0)}{H_0^{(\delta)}(y, \gamma_0)} H_0^{(\delta)}(y; A) &= \frac{\text{Im } \phi(w)}{\pi |\phi(w) - U_0|^2} \int_{-1}^1 \frac{|\phi(y) - U_0|^2}{|\phi(y) - x|^2} dx + O(\epsilon^3) \\ &= \frac{2}{\pi} \frac{\text{Im } \phi(w)}{|\phi(w) - U_0|^2} + O(\epsilon^3). \end{aligned}$$

From (16) and (20) we infer that

$$\begin{aligned} \frac{1}{M_0} &= \frac{H_0^{(\delta)}(b, \gamma_0)}{H_0^{(\delta)}(b; A) H_0^{(\delta)}(w; \gamma_0)} \\ &= (1 + O(\epsilon^3)) \sum_{\substack{y \in V_\delta(D) \\ \text{Im } \phi(y) \geq \alpha\lambda}} p(y) \left/ \left[ \frac{2}{\pi} \frac{\text{Im } \phi(w)}{|\phi(w) - U_0|^2} + O(\epsilon^3) \right] \right., \end{aligned}$$

where  $p(y) = \mathbf{P}(S_\delta^b(\tau_{D'}) = y \mid S_\delta^b(\tau_D) \in A)$ . In view of Lemma 5.2, we can suppose

$$(21) \quad \sum_{\substack{y \in V_\delta(D) \\ \text{Im } \phi(y) \geq \alpha\lambda}} p(y) = 1 + O(\epsilon^3),$$

by replacing  $\alpha$  by smaller one if necessary. Since  $\text{Im } \phi(w)/|\phi(w) - U_0|^2$  is bounded by a universal constant, we now conclude

$$(22) \quad \begin{aligned} M_0 &= \frac{2}{\pi} \frac{\text{Im } \phi(w)}{|\phi(w) - U_0|^2} + O(\epsilon^3) \\ &= \frac{2}{\pi} \text{Im} \left( \frac{-1}{\phi(w) - U_0} \right) + O(\epsilon^3). \end{aligned}$$

*Step 3.* We derive an analogous formula for  $M_m$ . Lemma 2.3 and (14) imply

$$(23) \quad t_m = O(\epsilon^2), \quad |U(s) - U(0)| = O(\epsilon) \quad \text{for } \forall s \in [0, t_m].$$

The Loewner equation (1) shows that

$$(24) \quad |g_t(z) - z| \leq t \cdot \sup_{0 \leq s \leq t} \frac{2}{|g_s(z) - U(s)|},$$

and, observing the imaginary part of the Loewner equation,

$$(25) \quad 1 \geq \frac{\text{Im } g_t(z)}{\text{Im } z} \geq \exp \left( -t \cdot \sup_{0 \leq s \leq t} \frac{2}{|g_s(z) - U(s)|^2} \right).$$

We also find  $\frac{d}{dt} \text{Im } g_t(z) \geq -2/\text{Im } g_t(z)$ , and this implies  $\frac{d}{dt} (\text{Im } g_t(z))^2 \geq -4$ .

By integrating this relation over  $[0, t]$ , we get  $(\text{Im } g_t(z))^2 \geq (\text{Im } z)^2 - 4t$ . Since  $t_m = O(\epsilon^2)$ , we have  $\text{Im } g_s \circ \phi(w) \geq 1/4$  for  $0 \leq s \leq t_m$ . Therefore, (24) gives

$$(26) \quad |g_s \circ \phi(w) - \phi(w)| = O(\epsilon^2) \quad \text{for } \forall s \in [0, t_m].$$

Let  $\sigma := \inf\{t \geq 0 : |g_t(z) - U(t)| \leq \lambda/2\}$ . Using (24), we get  $|g_\sigma(z) - z| \leq 4\sigma/\lambda$  and

$$|z - U(0)| \leq \frac{4\sigma}{\lambda} + \frac{\lambda}{2} + |U(\sigma) - U(0)|.$$

Thus, if  $|z - U(0)| > \lambda$ , then  $\sigma > t_m$ . This implies  $|g_s \circ \phi(y) - U(s)| \geq \lambda/2$  for  $0 \leq s \leq t_m$ . Therefore, (24) and (25) lead to

$$(27) \quad |\phi_m(y) - \phi(y)| = O(\epsilon^3) \quad \text{and} \quad \frac{\text{Im } \phi_m(y)}{\text{Im } \phi(y)} = 1 + O(\epsilon^3).$$

(26) and (27) imply

$$\operatorname{Im} \phi_m(w) \geq \frac{1}{3}, \quad |\phi_m(w) - U_m| \leq 4; \quad \operatorname{Im} \phi_m(y) \geq \frac{\alpha\lambda}{2}, \quad \frac{\lambda}{2} \leq |\phi_m(y) - U_m| \leq 3\lambda.$$

and it follows from Lemma 2.4 that  $\operatorname{rad}_{p_m}(D_m) \geq r'$  for some  $r' > 0$ . Therefore, we can apply Corollary 4.6 to the domain  $D_m$  with the reference point  $p_m$ , and hence the relation (19) implies

$$\begin{aligned} \frac{H_m^{(\delta)}(w, \gamma_m)}{H_m^{(\delta)}(y, \gamma_m)} H_0^{(\delta)}(y; A) &= \frac{\operatorname{Im} \phi_m(w)}{\pi |\phi_m(w) - U_m|^2} \int_{-1}^1 \frac{\operatorname{Im} \phi(y)}{\operatorname{Im} \phi_m(y)} \cdot \frac{|\phi_m(y) - U_m|^2}{|\phi(y) - x|^2} dx \\ &\quad + O(\epsilon^3). \end{aligned}$$

Thus, from (16), (21) and (27) we get

$$(28) \quad M_m = \frac{2}{\pi} \operatorname{Im} \left( \frac{-1}{\phi_m(w) - U_m} \right) + O(\epsilon^3).$$

*Step 4.* Proposition 4.3 implies that  $M_j$  is a martingale. Because  $m$  is a bounded stopping time,

$$\mathbf{E}[M_m - M_0] = 0.$$

Thus, (22) and (28) lead to

$$(29) \quad \mathbf{E} \left[ \operatorname{Im} \left( \frac{1}{\phi_m(w) - U_m} \right) - \operatorname{Im} \left( \frac{1}{\phi(w) - U_0} \right) \right] = O(\epsilon^3).$$

(23) and (26) imply

$$\frac{1}{g_s \circ \phi(w) - U(s)} = \frac{1}{\phi(w) - U_0} + O(\epsilon) \quad \text{for } \forall s \in [0, t_m].$$

By integrating this relation over  $[0, t_m]$ , Loewner equation and (23) show that

$$(30) \quad \phi_m(w) = \phi(w) + \frac{2}{\phi(w) - U_0} \cdot t_m + O(\epsilon^3).$$

Let  $f(u, v) = 1/(u - v)$ . Using (23) and (30), we Taylor-expand  $f(\phi_m(w), U_m) - f(\phi(w), U_0)$  with respect to  $\phi_m(w) - \phi(w)$  and  $U_m - U_0$ , up to  $O(\epsilon^3)$ . Observing imaginary part of this Taylor expansion, from (29) and (30) we get

$$(31) \quad \begin{aligned} \operatorname{Im} \left( \frac{1}{(\phi(w) - U_0)^2} \right) \mathbf{E}[U_m - U_0] + \operatorname{Im} \left( \frac{1}{(\phi(w) - U_0)^3} \right) \mathbf{E}[(U_m - U_0)^2 - 2t_m] \\ = O(\epsilon^3). \end{aligned}$$

Now, we consider two different choices of  $w$  under the constraint  $w \in V_\delta$  such that  $\operatorname{Im} \phi(w) \geq \frac{1}{2}$ ,  $|\phi(w)| \leq 3$ . By the Koebe distortion theorem we can find  $w$

satisfying  $\phi(w) - U_0 = i + O(\epsilon^3)$ . Then, (31) implies

$$(32) \quad \mathbf{E}[(U_m - U_0)^2 - 2t_m] = O(\epsilon^3).$$

Similarly, we can find  $w$  satisfying  $\phi(w) - U_0 = e^{i(\pi/3)} + O(\epsilon^3)$  and we get

$$(33) \quad \mathbf{E}[U_m - U_0] = O(\epsilon^3). \quad \square$$

As in Subsection 2.3, let  $D(t) = D \setminus \gamma[0, t]$ ,  $\phi_t = g_t \circ \phi$  and  $p(t) = \phi_t^{-1}(i + U(t))$ .

LEMMA 5.4. *Let  $T > 1$  and  $\epsilon > 0$ , and, given a quadruplet  $(D, a, b, p) \in \mathcal{D}$ , put  $\tilde{T} = \sup\{t \in [0, T] : |U(t)| < 1/\epsilon\}$ . Then, there exists  $\eta_1 = \eta_1(T, \epsilon) > 0$  and  $r_1 = r_1(T, \epsilon) > 0$  such that  $(D(t), \gamma(t), b, p(t)) \in \mathcal{D}(r_1, R, \eta_1)$  for all  $t < \tilde{T}$ .*

*Proof.* Let  $g_t^*(z) := g_t(z) - U(t)$ . Put  $\Psi(z) = (z - i)/(z + i)$ . Define the conformal map  $h_t : \mathbf{D} \setminus \psi[\gamma[0, t]] \rightarrow \mathbf{D}$  by

$$h_t(z) := \Psi \circ g_t^* \circ \Psi^{-1}(z).$$

Put  $\psi_t(z) := h_t \circ \psi(z)$  so that  $\psi_t : D(t) \rightarrow \mathbf{D}$  is a conformal map with  $\psi_t(\gamma(t)) = -1$ ,  $\psi_t(b) = 1$ ,  $\psi_t(p(t)) = 0$ . Clearly  $\partial(\mathbf{D} \setminus \psi[\gamma[0, t]])$  is locally analytic at 1 and  $h_t(1) = 1$ . On using the Loewner equation we infer that  $g_t'(z) = 1$  as  $z \rightarrow \infty$ , which implies  $h_t'(1) = 1$ . Now we can choose a positive  $\eta_1 < \eta/4$  such that if  $t < \tilde{T}$ , then  $\psi(\gamma[0, t])$  does not intersect with  $B := \{z \in \mathbf{C} : |z - 1| < 4\eta_1\}$ . Thus,  $h_t$  is analytically extended to  $B$  for  $t < \tilde{T}$ , so that in view of Koebe's 1/4 theorem  $h_t^{-1}$  has an analytic extension in  $\{z \in \mathbf{C} : |z - 1| < \eta_1\}$  for  $t < \tilde{T}$ . Since  $\psi_t^{-1} = \psi^{-1} \circ h_t^{-1}$  and  $\psi^{-1}$  is analytic on  $B$ ,  $\psi_t^{-1}$  has an analytic extension in  $\{z \in \mathbf{C} : |z - 1| < \eta_1\}$  for  $t < \tilde{T}$ . The existence of  $r_1$  is deduced from Lemma 2.4. Thus the assertion of the lemma has been proved.  $\square$

*Proof of Theorem 5.1.* Having proved Lemma 5.3 it is easy to adapt the arguments given in [7]. Let  $D$  be as in the theorem and take  $R$  so that  $D \subset \mathbf{RD}$ . Let  $r := \text{rad}_p(D)$ . From our hypothesis of local analyticity of  $\partial D$  at  $b$ , the function  $\psi$  has an analytic extension in a neighborhood of  $b$ . Thus, we can choose  $\eta > 0$  such that  $\psi^{-1}$  is analytic in  $\{z \in \mathbf{C} : |z - 1| < \eta\}$ , hence  $(D, a, b, p) \in \mathcal{D}(r, R, \eta)$ .

Let  $T > 1$  and  $\epsilon_1 > 0$  and put  $\tilde{T} = \sup\{t \in [0, T] : |U(t)| < 1/\epsilon_1\}$ . Let  $\epsilon > 0$  be small enough. Let  $m_0 = 0$  and define  $m_n$  inductively by

$$m_n := \min\{j > m_{n-1} : t_j - t_{m_{n-1}} \geq \epsilon^2 \text{ or } |U_j - U_{m_{n-1}}| \geq \epsilon\}.$$

Let  $N := \max\{n \in \mathbf{N} : t_{m_n} < \tilde{T}\}$ . By Lemma 5.4, we can take some positive constants  $r_1$  and  $\eta_1$  such that  $(D_{m_n}, \gamma_{m_n}, b, p_{m_n}) \in \mathcal{D}(r_1, R, \eta_1)$  for any  $n \leq N$ .

By the Markov property stated in Proposition 4.2, we find that  $\gamma^{(t_{m_n})}(\cdot) = \gamma(t_{m_n} + \cdot)$  is the same distribution as the time-reversal of the loop erasure of a natural random walk on  $G_\delta$  started at  $b_\delta$  and stopped on exiting  $D_{m_n}$  and conditioned to hit  $\partial D_{m_n}$  at  $\gamma_{m_n}$ . We apply Lemma 5.3 with  $(D_{m_n}, \gamma_{m_n}, b, p_{m_n})$  for any  $n \leq N$ . Then, we deduce from the fact stated at (3) that there exists  $\delta_0 =$

$\delta_0(\epsilon, \epsilon_1, T) > 0$  such that if  $\delta < \delta_0$ , then for any  $n \leq N$

$$\mathbf{E}[U_{m_{n+1}} - U_{m_n} \mid \gamma[0, m_n]] = O(\epsilon^3),$$

and

$$\mathbf{E}[(U_{m_{n+1}} - U_{m_n})^2 \mid \gamma[0, m_n]] = \mathbf{E}[2(t_{m_{n+1}} - t_{m_n}) \mid \gamma[0, m_n]] + O(\epsilon^3).$$

The rest of proof of Theorem 5.1 is the proof that  $U(t)$  weakly converges to  $\sqrt{2}B(t)$  uniformly on  $[0, T]$  as  $\delta \rightarrow 0$ , where  $B(t)$  is a one-dimensional standard Brownian motion with  $B(0) = 0$ . This proof follows from the above estimate and the Skorokhod embedding theorem as in [3] and [7]. (See Subsection 3.3 in [3] and Corollary 4.3 in [7].)  $\square$

**5.2. Convergence with respect to the metric  $d_{\mathcal{U}}$**

Now, we assume that there exists an invariant measure  $\pi$  for a natural random walk  $S(\cdot)$  on  $G$  such that  $0 < \pi(v) < \infty$  for any  $v \in V$ . Let  $p(u, v)$  be the transition probability for  $S(\cdot)$ . We consider the dual walk  $S^*(\cdot)$ . The transition probability of  $S^*(\cdot)$ , denoted by  $p^*(u, v)$ , is given by

$$p^*(u, v) := \frac{\pi(v)}{\pi(u)} p(v, u).$$

Then, the dual walk  $S^*(\cdot)$  is a natural random walk on some other planar irreducible graph. As in the case of  $S(\cdot)$ , we define  $(S^*)_{\delta}^x$ ,  $(\Gamma^*)_{\delta}^{a,b}$ ,  $(\gamma^*)_{\delta}^{a,b}$  corresponding to  $S^*(\cdot)$ . The following lemma is a relation between the time-reversal and the dual walk.

**PROPOSITION 5.5.** *Suppose that there exists an invariant measure  $\pi$  for a natural random walk  $S(\cdot)$  on  $G$  such that  $0 < \pi(v) < \infty$  for any  $v \in V$ . Then, the time-reversal of  $\Gamma_{\delta}^{a,b}$  has the same distribution as  $(\Gamma^*)_{\delta}^{b,a}$ . Similarly, the time-reversal of  $\gamma_{\delta}^{a,b}$  has the same distribution as  $(\gamma^*)_{\delta}^{b,a}$ .*

*Proof.* The first assertion immediately follows from the definition of the dual walk and the conditional probability. In addition to the first assertion, applying Proposition 4.1,

$$(\gamma_{\delta}^{a,b})^{-} = L[\Gamma_{\delta}^{a,b}]^{-} \stackrel{d}{=} L[(\Gamma_{\delta}^{a,b})^{-}] \stackrel{d}{=} L[(\Gamma^*)_{\delta}^{b,a}] = (\gamma^*)_{\delta}^{b,a},$$

where  $\stackrel{d}{=}$  means the same distribution. Hence, we get the second assertion.  $\square$

Let  $\eta^{a,b}$  be a chordal SLE<sub>2</sub> curve in  $D$  from  $a$  to  $b$ . Recall the metric  $d_{\mathcal{U}}$  defined by (6) in Subsection 2.4.

**THEOREM 5.6.** *Suppose that there exists an invariant measure  $\pi$  for a natural random walk  $S(\cdot)$  on  $G$  such that  $0 < \pi(v) < \infty$  for any  $v \in V$  and  $S_{\delta}^x$  and  $(S^*)_{\delta}^x$  satisfy invariance principle. Let  $D$  be a bounded simply connected domain and*

$a, b \in \partial D$  be distinct points. Suppose that  $\partial D$  is locally analytic at  $a$  and  $b$ . Then,  $\gamma_\delta^{a,b}$  converges weakly to  $\eta^{a,b}$  as  $\delta \rightarrow 0$  with respect to the metric  $d_M$ .

*Proof.* Let  $\phi : D \rightarrow \mathbf{H}$  be a conformal map with  $\phi(a) = 0, \phi(b) = \infty$  and Let  $\phi^- : D \rightarrow \mathbf{H}$  be a conformal map with  $\phi(b) = 0, \phi(a) = \infty$ . Theorem 5.1 implies that  $\phi^- \circ (\gamma_\delta^{a,b})^-$  converges weakly to a chordal SLE<sub>2</sub> with respect to the driving function. Because we also assume that  $(S^*)_\delta^x$  satisfy invariance principle, Theorem 5.1 implies that  $\phi \circ ((\gamma^*)_\delta^{b,a})^-$  converges weakly to a chordal SLE<sub>2</sub> with respect to the driving function. By Proposition 5.5,  $\gamma_\delta^{a,b}$  is the same distribution as  $((\gamma^*)_\delta^{b,a})^-$ . Hence,  $\phi \circ \gamma_\delta^{a,b}$  converges weakly to a chordal SLE<sub>2</sub> with respect to the driving function. Therefore, Theorem 3.6 completes the proof.  $\square$

**6. Estimates of hitting probabilities of the random walk started at a boundary point**

In this section we prove Lemma 5.2. To this end it is convenient to work in the disc  $\mathbf{D}$  instead of  $\mathbf{H}$ . Let  $D \subsetneq \mathbf{C}$  be a simply connected domain and  $a, b$  be distinct points on  $\partial D$ . Let  $\phi : D \rightarrow \mathbf{H}$  be a conformal map with  $\phi(a) = 0, \phi(b) = \infty$ . Let  $p := \phi^{-1}(i)$ . Put  $\Psi(z) = (z - 1)/(z + 1)$  and  $\psi = \Psi \circ \phi$  so that  $\psi$  is a conformal map of  $D$  onto  $\mathbf{D}$  with  $\psi(a) = -1, \psi(b) = 1, \psi(p) = 0$ . Let  $S_\delta^b$  be a natural random walk on  $G_\delta$  started at  $b_\delta$ , where  $b_\delta$  is a point of  $\partial_{in} V_\delta(D)$  close to  $b$ .

Recall the class  $\mathcal{D}(r, R, \eta_0)$ , which is the collection of all quadruplets  $(D, a, b, p)$  such that  $\text{rad}_p(D) \geq r$  and  $D \subset \mathbf{RD}$  and  $\psi^{-1}$  has analytic extension in  $\{z \in \mathbf{C} : |z - 1| < \eta_0\}$ . Throughout this section we consider the constants  $r, R$  and  $\eta_0$  to be fixed and write  $\mathcal{D}$  for  $\mathcal{D}(r, R, \eta_0)$ ; also suppose that  $S_\delta^x$  satisfies invariance principle.

For  $(D, a, b, p) \in \mathcal{D}$  and  $\eta < \eta_0 \wedge \frac{1}{2}$  put

$$U = U_\eta = \{z \in D : |\psi(z) - 1| < \eta\}$$

and for any number  $\alpha$  from the open interval  $(0, 1/2)$ ,

$$J_\alpha = \{z \in \partial U : \text{dist}(\psi(z), \partial \mathbf{D}) < \alpha \eta, z \in D\}.$$

**PROPOSITION 6.1.** *Let  $U = U_\eta$  and  $J_\alpha$  be as described above. Then for any  $\epsilon > 0$  there exists  $\delta_0 = \delta(\epsilon, \eta) > 0$  such that for all positive  $\delta < \delta_0, \alpha < \delta_0$  and for all  $(D, a, b, p) \in \mathcal{D}$ ,*

$$\mathbf{P}(S_\delta^b(\tau_U) \in J_\alpha \mid S_\delta^b(\tau_U) \in D) < \epsilon,$$

Here  $\delta_0$  may depend on the graph  $(V, E)$ .

*Remark.* It is only for this proposition that we need the condition of the analyticity about  $b$ . Without that condition the estimate of the proposition is obtained by Uchiyama [9].

*Proof.* This proof is an adaptation of a part of the arguments given in [9]. Put

$$C = \{z \in \partial D : \operatorname{Im} \psi(z) > 0, |\psi(z) - 1| < \eta/3\},$$

and

$$B = \{z \in \mathbf{C} : |\psi(z) - 1| < \eta/3\} \setminus \bar{U}, \quad \Omega = B \cup C \cup U.$$

Let

$$C_\delta = \{v \in V_\delta(D) : [u, v] \cap C \neq \emptyset \text{ for some } u \in V_\delta(B)\},$$

and  $v^*$  be a vertex in  $C_\delta$  such that  $\operatorname{Im} \psi(v^*)$  is closest to  $\eta/6$  among vertexes of  $C_\delta$ .

Let  $L$  denote the last time when the walk  $S_\delta^{v^*}$  in  $\Omega$  killed when it crosses the boundary  $\partial\Omega$  exits  $B$ :

$$L = \begin{cases} 1 + \max\{0 \leq n < \tau_\Omega : S_\delta^{v^*}(n) \in B\} & \text{if } S_\delta^{v^*}(\tau_\Omega) \notin \partial B \\ \infty & \text{if } S_\delta^{v^*}(\tau_\Omega) \in \partial B \end{cases}$$

We write  $T = \tau_U$ . Putting  $J_\alpha^+ = J_\alpha \cap \mathbf{H}$  we compute  $q = \mathbf{P}(S_\delta^{v^*}(\tau_\Omega) \in J_\alpha^+)$ , the probability that the walk exits  $\Omega$  through  $J_\alpha^+$ , which we rewrite as

$$q = \mathbf{P}(S_\delta^{v^*}(T) \circ \theta_L \in J_\alpha^+, L < \tau_\Omega),$$

where the shift operator  $\theta_L$  acts on  $T$  as well as on  $S_\delta^{v^*}$ . By employing the strong Markov property

$$\begin{aligned} q &= \sum_{n=0}^{\infty} \sum_{y \in C_\delta} \mathbf{P}(S_\delta^{v^*}(T) \circ \theta_n \in J_\alpha^+, L = n, S_\delta^{v^*}(n) = y) \\ &= \sum_{n=0}^{\infty} \sum_{y \in C_\delta} \mathbf{P}(S_\delta^{v^*}(T) \circ \theta_n \in J_\alpha^+, S_\delta^{v^*}(n) = y) \\ &= \sum_{n=0}^{\infty} \sum_{y \in C_\delta} \mathbf{P}(S_\delta^{v^*}(n) = y) \mathbf{P}(S_\delta^y(T) \in J_\alpha^+) \end{aligned}$$

The occurrence of the event  $S_\delta^y(T) \in J_\alpha^+$  for  $y \in C_\delta$  entails  $S_\delta^y(T) \in D$ , so that  $\mathbf{P}(S_\delta^y(T) \in J_\alpha^+) = \mathbf{P}(S_\delta^y(T) \in J_\alpha^+, S_\delta^y(T) \in D)$ . Hence, bringing in the conditional probability

$$p(y) = \mathbf{P}(S_\delta^y(T) \in J_\alpha^+ | S_\delta^y(T) \in D),$$

we infer that

$$q = \sum_{y \in C_\delta} G_\Omega(v^*, y) \mathbf{P}(S_\delta^y(T) \in D) p(y),$$

where  $G_\Omega$  stands for the Green function of the walk killed on exiting  $\Omega$ . We have

$$p(y) \geq p(b), \quad y \in C_\delta,$$

for, if  $\gamma^b$  denote a path joining  $b_\delta$  with  $J_\alpha^+$  in  $V_\delta(U)$ , then the walk starting at  $y \in C_\delta$  and conditioned on the event  $S_\delta^y(T) \in D$  must hit  $\gamma^b \cup J_\alpha^+$  before existing  $U$ . Observing the identity

$$\sum_{y \in C_\delta} G_\Omega(v^*, y) \mathbf{P}(S_\delta^y(T) \in D) = \mathbf{P}(S_\delta^{v^*}(\tau_\Omega) \in D),$$

we finally obtain

$$q \geq p(b) \mathbf{P}(S_\delta^{v^*}(\tau_\Omega) \in D).$$

This concludes  $p(b) < \epsilon/2$  since  $\mathbf{P}(S_\delta^{v^*}(\tau_\Omega) \in D) > 1/3$  and  $q < \epsilon/6$  for all sufficiently small  $\delta$  and  $\alpha$ . Let  $J_\alpha^- = J_\alpha \setminus J_\alpha^+$ . On defining  $C$  with  $\text{Im } \psi(z) \leq 0$  in place of  $\text{Im } \psi(z) > 0$  we repeat the same argument to show that  $\mathbf{P}(S_\delta^b(T) \in J_\alpha^- | S_\delta^b(T) \in D) < \epsilon/2$ .  $\square$

LEMMA 6.2. *Let  $A := \phi^{-1}([-1, 1])$ . For any  $\epsilon > 0$ , there exists  $\delta_0 = \delta_0(\epsilon, \eta) > 0$  such that the following holds. Let  $(D, a, b, p) \in \mathcal{D}$ . Then, for all  $0 < \delta < \delta_0$  and  $0 < \alpha < \delta_0$ ,*

$$\mathbf{P}(S_\delta^b(\tau_U) \in J_\alpha | S_\delta^b(\tau_D) \in A) < \epsilon.$$

*Proof.* By the definition of the conditional probability and the strong Markov property,

$$\begin{aligned} \frac{\mathbf{P}(S_\delta^b(\tau_U) \in J_\alpha | S_\delta^b(\tau_D) \in A)}{\mathbf{P}(S_\delta^b(\tau_U) \notin J_\alpha | S_\delta^b(\tau_D) \in A)} &= \frac{\mathbf{P}(S_\delta^b(\tau_U) \in J_\alpha, S_\delta^b(\tau_D) \in A)}{\mathbf{P}(S_\delta^b(\tau_U) \notin J_\alpha, S_\delta^b(\tau_D) \in A)} \\ &= \frac{\sum_{y \in J_\alpha} \mathbf{P}(S_\delta^b(\tau_U) = y) \mathbf{P}(S_\delta^y(\tau_D) \in A)}{\sum_{y \notin J_\alpha} \mathbf{P}(S_\delta^b(\tau_U) = y) \mathbf{P}(S_\delta^y(\tau_D) \in A)}. \end{aligned}$$

Because we assume invariance principle, the hitting probability  $\mathbf{P}(S_\delta^y(\tau_D) \in A)$  can be approximated by the same probability for a Brownian motion. Because the hitting probability for a Brownian motion is conformal invariant, we can calculate the hitting probability on the upper half plane instead of  $D$ . Therefore, we find that there exists a universal constant  $C$  such that for sufficiently small  $\delta$ ,

$$\frac{\sup_{y \in J_\alpha} \mathbf{P}(S_\delta^y(\tau_D) \in A)}{\inf_{y \notin J_\alpha} \mathbf{P}(S_\delta^y(\tau_D) \in A)} \leq C.$$

Thus, we obtain

$$\frac{\mathbf{P}(S_\delta^b(\tau_U) \in J_\alpha | S_\delta^b(\tau_D) \in A)}{\mathbf{P}(S_\delta^b(\tau_U) \notin J_\alpha | S_\delta^b(\tau_D) \in A)} \leq C \frac{\sum_{y \in J_\alpha} \mathbf{P}(S^b(\tau_U) = y)}{\sum_{y \notin J_\alpha} \mathbf{P}(S^b(\tau_U) = y)}.$$

Because

$$\frac{\sum_{y \in J_\alpha} \mathbf{P}(S^b(\tau_U) = y)}{\sum_{y \notin J_\alpha} \mathbf{P}(S^b(\tau_U) = y)} = \frac{\mathbf{P}(S_\delta^b(\tau_U) \in J_\alpha | S_\delta^b(\tau_U) \in D)}{\mathbf{P}(S_\delta^b(\tau_U) \notin J_\alpha | S_\delta^b(\tau_U) \in D)},$$

Proposition 6.1 completes the proof.  $\square$

LEMMA 6.3. For any  $\epsilon > 0$ , there exists  $\delta_0 = \delta_0(\epsilon, \eta) > 0$  such that the following holds. Let  $(D, a, b, p) \in \mathcal{D}$ . Then, for all  $0 < \delta < \delta_0$  and  $0 < \alpha < \delta_0$ ,

$$\mathbf{P}(S_\delta^b(\tau_U) \in J_x \mid S_\delta^b(\tau_D) = a_\delta) < \epsilon,$$

where  $a_\delta$  is a point of  $\partial_{\text{out}}V_\delta(D)$  close to  $a$ .

*Proof.* By the definition of the conditional probability,

$$\begin{aligned} \frac{\mathbf{P}(S_\delta^b(\tau_U) \in J_x \mid S_\delta^b(\tau_D) = a_\delta)}{\mathbf{P}(S_\delta^b(\tau_U) \notin J_x \mid S_\delta^b(\tau_D) = a_\delta)} &= \frac{\mathbf{P}(S_\delta^b(\tau_U) \in J_x, S_\delta^b(\tau_D) = a_\delta)}{\mathbf{P}(S_\delta^b(\tau_U) \notin J_x, S_\delta^b(\tau_D) = a_\delta)} \\ &= \frac{\mathbf{P}(S_\delta^b(\tau_U) \in J_x, S_\delta^b(\tau_D) = a_\delta \mid S_\delta^b(\tau_D) \in A)}{\mathbf{P}(S_\delta^b(\tau_U) \notin J_x, S_\delta^b(\tau_D) = a_\delta \mid S_\delta^b(\tau_D) \in A)}. \end{aligned}$$

Since the random walk conditioned on exiting  $D$  through  $A$  is Markovian, the right-hand side above may be written as

$$\frac{\sum_{y \in J_x} \mathbf{P}(S_\delta^b(\tau_U) = y \mid S_\delta^b(\tau_D) \in A) \mathbf{P}(S_\delta^y(\tau_D) = a_\delta \mid S_\delta^y(\tau_D) \in A)}{\sum_{y \notin J_x} \mathbf{P}(S_\delta^b(\tau_U) = y \mid S_\delta^b(\tau_D) \in A) \mathbf{P}(S_\delta^y(\tau_D) = a_\delta \mid S_\delta^y(\tau_D) \in A)}.$$

By Lemma 5.8. in [10], there exists a universal constant  $C$  such that for sufficiently small  $\delta$ ,

$$\frac{\sup_{y \in J_x} \mathbf{P}(S_\delta^y(\tau_D) = a_\delta \mid S_\delta^y(\tau_D) \in A)}{\inf_{y \notin J_x} \mathbf{P}(S_\delta^y(\tau_D) = a_\delta \mid S_\delta^y(\tau_D) \in A)} \leq C.$$

Hence, we obtain

$$\frac{\mathbf{P}(S_\delta^b(\tau_U) \in J_x \mid S_\delta^b(\tau_D) = a_\delta)}{\mathbf{P}(S_\delta^b(\tau_U) \notin J_x \mid S_\delta^b(\tau_D) = a_\delta)} \leq C \frac{\mathbf{P}(S_\delta^b(\tau_U) \in J_x \mid S_\delta^b(\tau_D) \in A)}{\mathbf{P}(S_\delta^b(\tau_U) \notin J_x \mid S_\delta^b(\tau_D) \in A)}$$

Therefore, Lemma 6.2 completes the proof.  $\square$

*Proof of Lemma 5.2.* By the mapping  $\Psi(z) = (z - i)/(z + i)$ , the half disc  $B_+(2\lambda) := B(U_0, 2\lambda) \cap \mathbf{H}$  is mapped to a small disc of radius  $\sim 1/2\lambda$  and centered at 1. For  $1/2\lambda < \eta_0$ , (12) follows from applying Lemma 6.2 with this small disc in place of  $U_\eta$ , the little discrepancy between them making no harm. If  $\text{diam}(\phi(\gamma[0, j])) < 1$ , the difference between  $B_+(2\lambda)$  and  $g_{t_j}(B_+(2\lambda))$  is insignificant for sufficiently large  $\lambda$ . Hence, we also have (13) by applying Lemma 6.3 with  $(D_j, \gamma_j, b, p_j)$ , which is legitimate because of Lemma 5.4.  $\square$

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