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UNIQUENESS OF NON-TOPOLOGICAL SOLUTIONS FOR THE CHERN-SIMONS SYSTEM WITH TWO HIGGS PARTICLES

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Abstract

We study the non-topological radial solutions of the Abelian Chern-Simons equation with two Higgs particles. We establish the non-degeneracy property of linearized equation and the uniqueness property for the corresponding non-topological radial solutions.

1. Introduction

The Chern-Simons theories were developed to explain certain particle physics, condensed physics, superconductivity, quantum mechanics and so on. The Chern-Simons equations of various models correspond to non-linear elliptic equations, which are both interesting and challenging.

In this paper, we consider the non-linear elliptic system

$$\begin{cases} \Delta u + e^v(1 - e^u) = \mu \\ \Delta v + e^u(1 - e^v) = \nu \end{cases} \quad \text{in } \mathbf{R}^2 \tag{1.1}$$

where μ and ν are finite measure of the form $4\pi \sum_s \delta_{p_s}$ on $\mathbf{R}^2$. This system arises in a relativistic Abelian Chern-Simons model involving two Higgs scalar fields and two gauge fields. We refer to [8] for the details on the derivation of this system and [6, 7] for the background physics. For the past twenty years, the Abelian Chern-Simons equation with one Higgs particle,

$$\Delta u + e^u(1 - e^u) = 4\pi \sum_s \delta_{p_s},$$

has been extensively studied. We refer reader to [13, 1, 14, 2, 12, 3, 5, 10] and references therein for the recent developments.

In the literature, there are two natural boundary conditions for the solutions of (1.1) at infinity, namely,

(1) $\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = 0$

(2) $\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = -\infty$

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If (u, v) satisfies the boundary condition (1), then (u, v) is called a *topological solution* of (1.1).

In [8], Lin-Ponce-Yang showed the existence of topological solutions for (1.1) for any given set of singularities.

THEOREM 1.1 ([8]). *For any given sets* $\{p_1, \dots, p_{N_1}\}$ *and* $\{q_1, \dots, q_{N_2}\}$ *on* $\mathbf{R}^2$, (1.1) *with* $\mu = 4\pi \sum_{i=1}^{N_1} \delta_{p_i}$ *and* $\nu = 4\pi \sum_{j=1}^{N_2} \delta_{q_j}$ *possesses a topological solution.*

Lin-Prajapat [9] proved the existence of maximal and mountain-pass solutions of (1.1) on a torus. Lin-Yan [11] studied the bubbling solutions of (1.1) on a torus. Chern-Chen-Lin [4] studied the radial solutions of (1.1) with all $\{p_i\}_{i=1}^{N_1}$ and $\{q_j\}_{j=1}^{N_2}$ to be the origin.

$$\begin{cases} \Delta u + e^v(1 - e^u) = 4\pi N_1 \delta_0 \\ \Delta v + e^u(1 - e^v) = 4\pi N_2 \delta_0 \end{cases} \text{in } \mathbf{R}^2. \tag{1.2}$$

They showed the uniqueness of topological solutions of (1.2) by studying the nondegeneracy property of linearized equations and classify all entire radial solutions of (1.2) according to their behaviours at ∞:

Type (I): $\lim_{r\to\infty}(u(r), v(r)) = (0, 0)$.

Type (II): $\lim_{r\to\infty}(u(r), v(r)) = (-\infty, -\infty)$ with $\beta_1 < \infty$ and $\beta_2 < \infty$. For this case, (u, v) is called a *non-topological solution.*

Type (III): $\lim_{r\to\infty}(u(r), v(r)) = (-\infty, -\infty)$, and either $2N_1 < \beta_1 \le 2N_1 + 2$, $\beta_2 = \infty$ or $\beta_1 = \infty$, $2N_2 < \beta_2 \le 2N_2 + 2$.

Type (IV): $\lim_{r\to\infty}(u(r), v(r)) = (-c_u, \infty)$ or $\lim_{r\to\infty}(u(r), v(r)) = (-\infty, -c_v)$ for some constants $c_u > 0$ and $c_v > 0$.

Type (V): $\lim_{r\to\infty}(u(r), v(r)) = (+\infty, -\infty)$ or $\lim_{r\to\infty}(u(r), v(r)) = (-\infty, +\infty)$.

Here,

$$\beta_1 = \int_0^\infty e^v(1 - e^u) r \, dr, \quad \beta_2 = \int_0^\infty e^u(1 - e^v) r \, dr.$$

However, the issues of the uniqueness of the non-topological solutions have not been understood yet.

In this paper, we consider the radial solution of (1.1) with $\mu = \nu = 0$:

$$\begin{cases} \Delta u + e^v(1 - e^u) = 0 \\ \Delta v + e^u(1 - e^v) = 0 \end{cases} \text{in } \mathbf{R}^2. \tag{1.3}$$

The following theorem is the main theorem of this paper. We prove the uniqueness result for the non-topological radial solutions of (1.3).

THEOREM 1.2. *For any given pair* (β_1, β_2) *with* $2 < \beta_i < \infty$, $i = 1, 2$, *and* $(\beta_1 - 2)(\beta_2 - 2) > 4$, *there exists a unique non-topological radial solution of* (1.3) *satisfies*

$$\int_0^\infty e^v(1 - e^u) r \, dr = \beta_1 \quad \text{and} \quad \int_0^\infty e^u(1 - e^v) r \, dr = \beta_2.$$

The proof is based on the non-degeneracy of linearized equations. The linearized equations at (u, v) of (1.2) is called *degenerate* if there exists a nonzero bounded solution (A, B) of

$$\begin{cases} \Delta A + e^{v}(1 - e^{u})B - e^{u+v}A = 0 \\ \Delta B + e^{u}(1 - e^{v})A - e^{u+v}B = 0 \end{cases} \text{in } \mathbf{R}^2. \tag{1.4}$$

In Sec. 2, we will show the non-degeneracy of linearized equations at a non-topological solution. We hope that the method developed here could be helpful for a similar non-linear elliptic systems, like A_2, B_2 and G_2 Chern-Simons system. For the single Chern-Simons-Higgs model, the uniqueness result for the non-topological solution with one singularity at the origin was proved in [3]. Is there any uniqueness result for the non-topological solutions of (1.2) with (β_1, β_2) satisfying

$$(\beta_1 - 2(N_1 + 1))(\beta_2 - 2(N_2 + 1)) > 4(N_1 + 1)(N_2 + 1)?$$

We will come back to this issue in a coming paper.

The paper is organized as follows. We investigate the non-degeneracy property of the linearized equations on the non-topological solutions of (1.3) in Sec. 2. Theorem 1.2 is proved in Sec. 3.

2. Non-degeneracy of linearized equations

In this section, we consider the radial solutions of

$$\begin{cases} \Delta u + e^{v}(1 - e^{u}) = 0 \\ \Delta v + e^{u}(1 - e^{v}) = 0 \end{cases} \text{in } \mathbf{R}^2. \tag{2.1}$$

Denote $(u(r; \alpha_1, \alpha_2), v(r; \alpha_1, \alpha_2))$ be the solution of (2.1) with $(u(0), v(0)) = (\alpha_1, \alpha_2)$ and

$$\beta_1(\alpha_1, \alpha_2) = \int_0^\infty e^{v(r;\alpha_1,\alpha_2)}(1 - e^{u(r;\alpha_1,\alpha_2)})r\,dr,$$

$$\beta_2(\alpha_1, \alpha_2) = \int_0^\infty e^{u(r;\alpha_1,\alpha_2)}(1 - e^{v(r;\alpha_1,\alpha_2)})r\,dr.$$

Remark 2.1. For any solution of (2.1) with $\beta_1 < +\infty$ and $\beta_2 < +\infty$, we can prove that (u, v) is symmetric with respect to some point $p \in \mathbf{R}^2$. The proof can be obtained via the method of moving planes.

Denote the set of the initial conditions of the non-topological solutions of (2.1).

$$\Omega = \{(\alpha_1, \alpha_2) \in \mathbf{R}^2 \mid u(0) = \alpha_1, v(0) = \alpha_2, \text{ such that } u(r) < 0 \text{ and } v(r) < 0 \text{ and } \beta_1 < +\infty, \beta_2 < +\infty\}$$

In [4], Chern-Chen-Lin prove

Proposition 2.2. *Ω is simply connected.*

Let

$$\phi_i(r) = \frac{\partial u}{\partial \alpha_i}, \quad i = 1, 2.$$

$$\psi_i(r) = \frac{\partial v}{\partial \alpha_i}, \quad i = 1, 2.$$

Then (ϕ_i, ψ_i), $i = 1, 2$ satisfied the linearized equation

$$\text{(2.2)} \qquad \begin{cases} \Delta\phi_i - e^{u+v}\phi_i + e^v(1 - e^u)\psi_i = 0 \\ \Delta\psi_i - e^{u+v}\psi_i + e^u(1 - e^v)\phi_i = 0 \\ \phi_1(0) = \psi_2(0) = 1, \ \phi_2(0) = \psi_1(0) = 0, \ \phi_i'(0) = \psi_i'(0) = 0. \end{cases}$$

By [4], we have the monotone property of (ϕ_i, ψ_i), $i = 1, 2$.

Proposition 2.3. *$\phi_1(r) > 0$ and $\psi_2(r) > 0$ for $r \geq 0$, and $\phi_2(r) < 0$ and $\psi_1(r) < 0$ for $r \geq 0$ provided that $(\alpha_1, \alpha_2) \in \Omega$.*

It is not difficult to show that

$$\phi_1(r) = A_1 \log r + O(1), \quad \phi_2(r) = -B_1 \log r + O(1),$$
$$\psi_1(r) = -B_2 \log r + O(1), \quad \psi_2(r) = A_2 \log r + O(1),$$

as $r \to +\infty$, for some $A_i > 0$ and $B_i > 0$, $i = 1, 2$. More precisely,

$$A_1 = \int_0^\infty (e^{u+v}\phi_1 - e^v(1 - e^u)\psi_1) r\, dr, \quad B_1 = \int_0^\infty (-e^{u+v}\phi_2 + e^v(1 - e^u)\psi_2) r\, dr,$$

$$B_2 = \int_0^\infty (-e^{u+v}\psi_1 + e^u(1 - e^v)\phi_1) r\, dr, \quad A_2 = \int_0^\infty (e^{u+v}\psi_2 - e^u(1 - e^v)\phi_2) r\, dr.$$

It is clear that both (ϕ_1, ψ_1) and (ϕ_2, ψ_2) satisfy the linearized equation (1.4). In fact, any solution (ϕ, ψ) of the linearized equation comes from a linear combination of (ϕ_1, ψ_1) and (ϕ_2, ψ_2).

We want to show that the linearized equation is non-degenerate. It is not difficult to see it is equivalent to saying that for any solution (ϕ, ψ) of the linearized equation, $|\phi(x)| + |\psi(x)|$ is not bounded in $[0, \infty)$. Thus, it is equivalent to

$$det \begin{pmatrix} A_1 & -B_1 \\ -B_2 & A_2 \end{pmatrix} \neq 0.$$

We will utilize the following the Pohozaev's identity for (1.3) to prove it.

$$\int_0^R (e^v + e^u - e^{u+v}) r\, dr = \frac{R^2 u'(R) v'(R) + R^2(e^{u(R)} + e^{v(R)} - e^{(u+v)(R)})}{2}.$$

By differentiating the above identity with respect to the initial value of (u, v), we have

$$\int_0^R (e^u\phi + e^v\psi - e^u e^v(\phi + \psi))r\,dr = \frac{R^2}{2}[u'(R)\psi'(R) + v'(R)\psi'(R)] + \frac{R^2}{2}[e^{u(R)}\phi(R) + e^{v(R)}\psi(R) - e^{(u+v)(R)}(\psi + \psi)(R)]. \tag{2.3}$$

THEOREM 2.4.

$$det\begin{pmatrix} A_1 & -B_1 \\ -B_2 & A_2 \end{pmatrix} \neq 0.$$

Proof. Without loss of generality, we assume that

$$(u, v)(0) = (\alpha_1^*, \alpha_2^*) \quad \text{with } \alpha_2^* > \alpha_1^*.$$

In the followings, we set

$$\phi_c(r; \alpha_1, \alpha_2) = \phi_1(r; \alpha_1, \alpha_2) + c\phi_2(r; \alpha_1, \alpha_2)$$

and

$$\psi_2(r; \alpha_1, \alpha_2) = \psi_1(r; \alpha_1, \alpha_2) + c\psi_2(r; \alpha_1, \alpha_2)$$

for some constant c. Then (ϕ_c, ψ_c) is a solution of the linearized equation. If $c \leq 0$, then it is clear that (ϕ_c, ψ_c) is unbounded in $[0, \infty)$. Hence, we may only consider the case $c > 0$. Let

$$\widehat{\alpha_2} = \sup\{\alpha < \alpha_2^* \mid \text{For all } \beta \in (\alpha_1^*, \alpha], \text{ there is } c(\beta) > 0 \text{ such that both } \phi_c(r; \alpha_1^*, \beta) \text{ and } \psi_c(r; \alpha_1^*, \beta) \text{ change sign once and only once.}\}$$

STEP 1. If both $\phi_c(r)$ and $\psi_c(r)$ change sign once and only once, then at least one of them is unbounded on $[0, \infty)$.

Note that $\phi_c(0) = \phi_1(0) > 0$ and $\psi_c(0) = c\psi_2(0) > 0$. We know that (ϕ_c, ψ_c) satisfies

$$\begin{cases} \Delta\phi_c + e^v(1 - e^u)\psi_c - e^{u+v}\phi_c = 0 \\ \Delta\psi_c + e^u(1 - e^v)\phi_c - e^{u+v}\psi_c = 0 \end{cases} \tag{2.4}$$

Now, suppose both ϕ_c and ψ_c are bounded. Then

$$\int_0^\infty e^v(1 - e^u)\psi_c r\,dr = \int_0^\infty e^{u+v}\phi_c r\,dr \tag{2.5}$$

and

$$\int_0^\infty e^u(1-e^v)\phi_c r\,dr = \int_0^\infty e^{u+v}\psi_c r\,dr. \tag{2.6}$$

By applying the Pohozaev's identity, we have

$$\int_0^\infty (e^u\phi_c + e^v\psi_c)r\,dr = \int_0^\infty e^{u+v}(\phi_c+\psi_c)r\,dr. \tag{2.7}$$

On the other hand, by (2.5) and (2.6),

$$\int_0^\infty (e^u\phi_c + e^v\psi_c)r\,dr = 2\int_0^\infty e^{u+v}(\phi_c+\psi_c)r\,dr$$

together with (2.7), it implies

$$\int_0^\infty e^{u+v}(\phi_c+\psi_c)r\,dr = 0.$$

Again, by (2.5) and (2.6), one obtains

$$\int_0^\infty e^u\phi_c r\,dr = \int_0^\infty e^v\psi_c r\,dr = 0.$$

Suppose that $\phi_c(r_1)=0$ and $\psi_c(r_2)=0$ for some $r_1>0$ and $r_2>0$. Since $u'(r)<0$ and $v'(r)<0$ for $r>0$, we have

$$(e^{v(r_1)} - e^{v(r)})\phi_c(r) < 0 \quad \text{if } r \neq r_1.$$

Hence,

$$\int_0^\infty e^{u(r)}(e^{v(r_1)} - e^{v(r)})\phi_c(r)r\,dr < 0,$$

and it implies

$$0 = e^{v(r_1)}\int_0^\infty e^{u(r)}\phi_c(r)r\,dr < \int_0^\infty e^{(u+v)(r)}\phi_c(r)r\,dr.$$

Similarly, we have

$$0 = e^{u(r_2)}\int_0^\infty e^{v(r)}\psi_c(r)r\,dr < \int_0^\infty e^{(u+v)(r)}\psi_c(r)r\,dr.$$

But it yields

$$0 < \int_0^\infty e^{(u+v)(r)}(\phi_c(r)+\psi_c(r))r\,dr = 0,$$

a contradiction. This finishes the step 1.

STEP 2. Suppose $\phi_c(r)$ and $\psi_c(r)$ change sign only once. Then $\phi_c(r) = 0$ and $\psi_c(r) = 0$ has only one solution.

We want to prove if $\phi_c(r_0) = 0$, then $\phi_c'(r_0) < 0$. Suppose not, i.e. $\phi_c'(r_0) = 0$. Then there is $r_1 > r_0$ such that $\phi_c(r_1) > 0$ and r_1 is a local maximum point of ϕ_c.

Since

$$0 \le \Delta\phi_c(r_0) = e^{v(r_0)}(e^{u(r_0)} - 1)\psi_c(r_0),$$

one has

$$\psi_c(r_0) \le 0.$$

If $\psi_c(r_0) = 0$, then we have $\psi_c'(r_0) \neq 0$. Otherwise, (ϕ_c, ψ_c) satisfies a second order system of equations with $\phi_c(r_0) = \phi_c'(r_0) = 0$ and $\psi_c(r_0) = \psi_c'(r_0) = 0$. By the uniqueness of ODE, we have $\phi_c \equiv \psi_c \equiv 0$, a contradiction. Thus, $\psi_c'(r_0) < 0$.

Hence, we conclude that either $\psi_c(r_0) = 0$ and $\psi_c'(r_0) < 0$, or $\psi_c(r_0) < 0$. It is easy to see that either case implies $\psi_c(r_1) \le 0$, due to the fact that ψ_c changes sign only once.

By the maximum principle,

$$0 < -\Delta\phi_c(r_1) + e^{(u+v)(r_1)}\phi_c(r_1) = e^{v(r_1)}(1 - e^{u(r_1)})\psi_c(r_1) \le 0,$$

because $\psi_c(r_1) < 0$ and $u(r_1) < 0$. This yields a contradiction and then the step 2 is proved.

STEP 3. $\alpha_2^* = \widehat{\alpha_2}$.

We denote $S = \{\beta \mid \text{Both } \phi_c(r; \alpha_1^*, \beta) \text{ and } \psi_c(r; \alpha_1^*, \beta) \text{ change sign once and only once for some } c > 0\}$. Clearly, $\widehat{\alpha_2} \le \alpha_2^*$. If $\widehat{\alpha_2} < \alpha_2^*$, then by definition, there exists $c_i = c(\beta_i) > 0$ such that $(\phi_{c_i}, \psi_{c_i}) = (\phi_{c_i}(r; \alpha_1^*, \beta_i), \psi_{c_i}(r; \alpha_1^*, \beta_i))$ changes sign once and only once, where $(\phi_{c_i}(r), \psi_{c_i}(r))$ is a solution of the linearized equation, where $\beta_i \to \widehat{\alpha_2}$. It is easy to see that $\varepsilon \le c_i \le \varepsilon^{-1}$ for some constant $\varepsilon > 0$. By passing to the limit, we find that (ϕ_c, ψ_c) is a solution of the linearized equation at $(u, v) = (u(r, \alpha_1, \widehat{\alpha_2}), v(r, \alpha_1, \widehat{\alpha_2}))$, and ϕ_c and ψ_c change their sign at most once. We want to prove both ϕ_c and ψ_c cannot be bounded. If both ϕ_c and ψ_c are bounded, then we have

$$\int_0^\infty e^u\phi_c(r)r\,dr = \int_0^\infty e^v\psi_c(r)\,dr = 0,$$

which implies both ϕ_c and ψ_c must change their sign. Thus, ϕ_c and ψ_c must change their sign once and only once. And then by the step 1, we know that ϕ_c and ψ_c cannot be both bounded.

We thus consider the following possible cases:

(a) Both ϕ_c and ψ_c change sign once and only once.
(b) $\phi_c(r) \geq 0$ and $\psi_c(r) \geq 0$ for $r \in [0, \infty)$.
(c) $\phi_c(r) \geq 0$ and $\psi_c(r)$ changes sign only once.
We need to exclude the last two cases.

Case (b). Since

$$\phi_c(r) = (A_1 - B_1 c) \log r + O(1) \quad \text{at } \infty$$

and

$$\psi_c(r) = (-B_2 + A_2 c) \log r + O(1) \quad \text{at } \infty,$$

we have $A_1 - B_1 c \geq 0$ and $-B_2 + A_2 c \geq 0$ (but equality can hold only for one equation), which implies

$$\frac{B_2}{A_2} \leq c \leq \frac{A_1}{B_1},$$

i.e.,

$$A_1 A_2 - B_1 B_2 > 0.$$

But, we can prove that (see step 4 below)

$$A_1 A_2 - B_1 B_2 \leq 0,$$

and it yields a contradiction. Hence this case is excluded.

Case (c). Let r_i be the last local maximum of $\phi_{c_i}(r) > 0$. Hence,

$$0 \leq -\Delta \phi_{c_i}(r_i) + e^{(u_i+v_i)(r_i)} \phi_{c_i}(r_i) = e^{v_i(r_i)}(1 - e^{u_i(r_i)}) \psi_{c_i}(r_i),$$

which implies $\psi_{c_i}(r_i) > 0$. Thus,

$$\psi_{c_i}(r) > 0 \quad \text{for } r \in [0, r_i].$$

If $r_i \to +\infty$, then we have $\psi_c(r) \geq 0$, a contradiction to the assumption of this case. Hence, $\{r_i\}$ are bounded, and then $\phi_c'(r) \leq 0$ for large r. Thus, $\phi_c(r)$ is bounded.

For $\delta > 0$, we denote

$$\phi_{c+\delta}(r) = (\phi_1 + c\phi_2 + \delta\phi_2)(r) \quad \text{and} \quad \psi_{c+\delta}(r) = (\psi_1 + c\psi_2 + \delta\psi_2)(r).$$

Claim. *If δ is small, then both $\phi_{c+\delta}$ and $\psi_{c+\delta}$ changes sign once and only once.*

Suppose $\phi_{c+\delta}$ changes sign more than once. Since $\phi_{c+\delta}(r) < 0$ for large r, we may assume that $\phi_{c+\delta}$ attains local positive maximum at $r_2(\delta)$. By the maximum principle,

$$0 \leq -\Delta \phi_{c+\delta}(r_2(\delta)) + e^{(u+v)(r_2(\delta))} \phi_{c+\delta}(r_2(\delta)) = e^{v(r_2(\delta))}(1 - e^{u(r_2(\delta))}) \psi_{c+\delta}(r_2(\delta)),$$

which implies $\psi_{c+\delta}(r_2(\delta)) > 0$. Hence, $r_2(\delta) \le C$ for some constant C as δ is close to 0.

Since $\phi_{c+\delta}$ changes sign more than once, there is a $r_3(\delta) < r_2(\delta)$ such that

$$\phi_{c+\delta} \text{ attains negative local minimum at } r_3(\delta).$$

Denote $r_2^0 = \lim_{\delta\to 0} r_2(\delta)$ and $r_3^0 = \lim_{\delta\to 0} r_3(\delta)$. By passing $\delta \to 0$, we have $\phi_c(r_3^0) = 0$ and $\phi_c'(r_3^0) = 0$. As in step 2, we have either $\psi_c(r_3^0) = 0$ and $\psi_c'(r_3^0) < 0$, or $\psi_c(r_3^0) < 0$. But it yields a contradiction to

$$\psi_c(r_2^0) = \lim_{\delta\to 0} \psi_{c+\delta}(r_2(\delta)) \ge 0.$$

Similarly, we can prove $\psi_{c+\delta}$ changes sign once and only once. We conclude that

$$\widehat{\alpha_2} \in S.$$

Since S is an open set, $\widehat{\alpha_2} + \varepsilon \in S$ for some $\varepsilon > 0$ provided that $\widehat{\alpha_2} < \alpha_2^*$. Hence, $\widehat{\alpha_2} = \alpha_2^*$.

STEP 4. For any $\beta \in S$, then

$$A_1A_2 - B_1B_2 < 0.$$

Let $\phi_c(r;\alpha_1^*,\beta)$ and $\psi_c(r;\alpha_1^*,\beta)$ change sign once and only once. Then

$$A_1 - cB_1 \le 0$$

and

$$B_2 - cA_2 \le 0.$$

Because the two inequalities cannot be equality simultaneously,

$$A_1A_2 - B_1B_2 < 0.$$

Hence, this theorem is proved. □

3. The Proof of Theorem 1.2

Recall that

$$\beta_1(\alpha_1,\alpha_2) = \int_0^\infty e^{v(r;\alpha_1,\alpha_2)}(1 - e^{u(r;\alpha_1,\alpha_2)})r\,dr,$$

$$\beta_2(\alpha_1,\alpha_2) = \int_0^\infty e^{u(r;\alpha_1,\alpha_2)}(1 - e^{v(r;\alpha_1,\alpha_2)})r\,dr.$$

Hence,

$$\frac{\partial\beta_1(\alpha_1,\alpha_2)}{\partial\alpha_i} = \int_0^\infty [\psi_i(r;\alpha_1,\alpha_2)e^{v(r;\alpha_1,\alpha_2)}(1 - e^{u(r;\alpha_1,\alpha_2)}) - \phi_i(r;\alpha_1,\alpha_2)e^{(u+v)(r;\alpha_1,\alpha_2)}]r\,dr$$

and

$$\frac{\partial \beta_2(\alpha_1, \alpha_2)}{\partial \alpha_i} = \int_0^\infty [\phi_i(r; \alpha_1, \alpha_2) e^{u(r; \alpha_1, \alpha_2)}(1 - e^{v(r; \alpha_1, \alpha_2)}) - \psi_i(r; \alpha_1, \alpha_2) e^{(u+v)(r; \alpha_1, \alpha_2)}] r \, dr,$$

$i = 1, 2$.

By Theorem 2.4, we have

$$det \begin{pmatrix} \frac{\partial \beta_1}{\partial \alpha_1} & \frac{\partial \beta_1}{\partial \alpha_2} \\ \frac{\partial \beta_2}{\partial \alpha_1} & \frac{\partial \beta_2}{\partial \alpha_2} \end{pmatrix} = det \begin{pmatrix} A_1 & -B_1 \\ -B_2 & A_2 \end{pmatrix} \neq 0$$

for $(\alpha_1, \alpha_2) \in \Omega$. Hence this theorem is proved.

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