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GLOBAL EXPONENTIAL STABILITY OF POSITIVE ALMOST PERIODIC SOLUTIONS FOR A MODEL OF HEMATOPOIESIS*

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Abstract

In this paper, we study the existence and global exponential stability of positive almost periodic solutions for the generalized model of hematopoiesis with multiple time-varying delays. Under proper conditions, we employ a novel proof to establish some criteria to ensure that all solutions of this model converge exponentially to the positive almost periodic solution.

1. Introduction

In the real-world phenomena, the variation of the environment plays an important role. As pointed out in [4, 6], periodically varying environment and almost periodically varying environment are foundations for the theory of nature selection. Compared with periodic effects, almost periodic effects are more frequent. Hence, the effects of the almost periodic environment on the evolutionary theory have been the object of intensive analysis by numerous authors and some of these results can be found in [1, 3, 7, 13]. In a classic study of population dynamics, the following delay differential equation model

$$x'(t) = -a(t)x(t) + \sum_{i=1}^{m} \frac{b_i(t)}{1 + x^n(t - \tau_i(t))}, \tag{1.1}$$

where n is a positive constant, and

$$a, b_i, \tau_i : R \to (0, +\infty) \text{ are continuous functions for } i = 1, 2, \dots, m,$$

has been used by [5, 9] to describe the dynamics of hematopoiesis (blood cell production). As we known, equation (1.1) belongs to a class of biological systems and it (or its analogue equation) has been attracted more attention on

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problem of almost periodic solutions because of its extensively realistic significance. For example, some criteria ensuring the existence and stability of positive almost periodic solutions were established in [2, 12, 15] and the references cited therein. However, after a careful examination on the above references, we find that the sufficient conditions for the attractivity of almost periodic solutions in [12] are delay-dependent; In the argument of Theorem 3.1 in [2], the zero function belongs to the set D, and the possible almost periodic may be identically vanishing; The exponential stability results on almost periodic solutions obtained in [15] only holds in a locally bounded region. Moreover, to the best of our knowledge, the global exponential stability of positive almost periodic solutions of (1.1) have not been sufficiently researched. On the other hand, since the exponential convergent rate can be unveiled, the global exponential stability of positive almost periodic solutions plays a key role in characterizing the behavior of dynamical system (see [8, 10, 14]). Thus, it is worth while to continue to investigate the existence and global exponential stability of positive almost periodic solutions of (1.1).

Motivated by the above discussions, in this paper, we consider the existence, uniqueness and global exponential stability of positive almost periodic solutions of (1.1). Here in this present paper, a new approach will be developed to obtain a delay-independent condition for the global exponential stability of the positive almost periodic solutions of (1.1), and the exponential convergent rate can be unveiled.

Throughout this paper, for $i = 1, 2, \dots, m$, it will be assumed that $a, b_i, \tau_i : R \to (0, +\infty)$ are almost periodic functions, and

$$(1.2) \quad a^- = \inf_{t\in R} a(t), \quad a^+ = \sup_{t\in R} a(t), \quad b_i^- = \inf_{t\in R} b_i(t) > 0, \quad b_i^+ = \sup_{t\in R} b_i(t),$$

$$(1.3) \qquad r = \max_{1\le i\le m}\left\{\sup_{t\in R} \tau_i(t)\right\} > 0.$$

Let R_+ denote a nonnegative real number space, $C = C([-r, 0], R)$ be the continuous function space equipped with the usual supremum norm $\|\cdot\|$, and let $C_+ = C([-r, 0], R_+)$. If $x(t)$ is defined on $[-r + t_0, \sigma)$ with $t_0, \sigma \in R$, then we define $x_t \in C$ where $x_t(\theta) = x(t + \theta)$ for all $\theta \in [-r, 0]$.

Due to the biological interpretation of model (1.1), only positive solutions are meaningful and therefore admissible. Thus we just consider admissible initial conditions

$$(1.4) \qquad x_{t_0} = \varphi, \quad \varphi \in C_+ \quad \text{and} \quad \varphi(0) > 0.$$

We write by $x_t(t_0, \varphi)(x(t; t_0, \varphi))$ an admissible solution of admissible initial value problem (1.1) and (1.4). Also, let $[t_0, \eta(\varphi))$ be the maximal right-interval of the existence of $x_t(t_0, \varphi)$.

2. Preliminary results

In this section, some lemmas and definitions will be presented, which are of importance in proving our main results in Section 3.

DEFINITION 2.1 (see [4, 6]). Let $u(t): R \to R$ be continuous in t. $u(t)$ is said to be almost periodic on R if, for any $\varepsilon > 0$, the set $T(u, \varepsilon) = \{\delta : |u(t+\delta) - u(t)| < \varepsilon$ for all $t \in R\}$ is relatively dense, i.e., for any $\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$, for any interval with length $l(\varepsilon)$, there exists a number $\delta = \delta(\varepsilon)$ in this interval such that $|u(t+\delta) - u(t)| < \varepsilon$, for all $t \in R$.

From the theory of almost periodic functions in [4, 6], it follows that for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\delta = \delta(\epsilon)$ in this interval such that

$$(2.1) \quad |a(t+\delta) - a(t)| < \epsilon, \quad |b_i(t+\delta) - b_i(t)| < \epsilon, \quad |\tau_i(t+\delta) - \tau_i(t)| < \epsilon,$$

for all $t \in R$ and $i = 1, 2, \ldots, m$.

LEMMA 2.1 (see [15, Lemma 2.3]). *Every solution $x(t; t_0, \varphi)$ of* (1.1) *and* (1.4) *is positive and bounded on $[t_0, \eta(\varphi))$, and $\eta(\varphi) = +\infty$.*

LEMMA 2.2. *Suppose that there exists two positive constants κ and M such that*

$$(2.2) \quad M > \kappa, \quad \sup_{t \in R}\left\{-a(t)M + \sum_{i=1}^{m} b_i(t)\right\} < 0, \quad \inf_{t \in R}\left\{-a(t)\kappa + \sum_{i=1}^{m} \frac{b_i(t)}{1+M^n}\right\} > 0.$$

Then, there exists $t_\varphi > t_0$ such that

$$(2.3) \qquad \kappa < x(t; t_0, \varphi) < M \quad \text{for all } t \geq t_\varphi.$$

Proof. Let $x(t) = x(t; t_0, \varphi)$. We first claim that there exists $t^{\#} \in [t_0, +\infty)$ such that

$$(2.4) \qquad x(t^{\#}) < M.$$

Otherwise,

$$x(t) \geq M \quad \text{for all } t \in [t_0, +\infty),$$

which together with (2.2), implies that

$$\begin{aligned} x'(t) &= -a(t)x(t) + \sum_{i=1}^{m} \frac{b_i(t)}{1 + x^n(t - \tau_i(t))} \\ &\leq -a(t)M + \sum_{i=1}^{m} \frac{b_i(t)}{1+M^n} \\ &\leq -a(t)M + \sum_{i=1}^{m} b_i(t) \end{aligned}$$

$$\begin{aligned} &\le \sup_{t\in R}\left\{-a(t)M + \sum_{i=1}^{m} b_i(t)\right\} \\ &< 0, \quad \text{for all } t \ge t_0. \end{aligned}$$

This yields that

$$x(t) = x(t_0) + \int_{t_0}^{t} x'(s)\, ds \le x(t_0) + \sup_{t\in R}\left\{-a(t)M + \sum_{i=1}^{m} b_i(t)\right\}(t - t_0), \quad \forall t \ge t_0.$$

Thus

$$\lim_{t\to+\infty} x(t) = -\infty,$$

which contradicts the fact that $x(t)$ is positive and bounded on $[t_0, +\infty)$. Hence, (2.4) holds. In the sequel, we prove that

$$x(t) < M \quad \text{for all } t \in [t^{\#}, +\infty). \tag{2.5}$$

Suppose, for the sake of contradiction, that there exists $\tilde{t} \in (t^{\#}, +\infty)$ such that

$$x(\tilde{t}) = M, \quad x(t) < M \quad \text{for all } t \in [t^{\#}, \tilde{t}). \tag{2.6}$$

Calculating the derivative of $x(t)$, together with (2.2), (1.1) and (2.6) imply that

$$\begin{aligned} 0 &\le x'(\tilde{t}) \\ &= -a(\tilde{t})x(\tilde{t}) + \sum_{i=1}^{m} \frac{b_i(\tilde{t})}{1 + x^n(\tilde{t} - \tau_i(\tilde{t}))} \\ &\le -a(\tilde{t})M + \sum_{i=1}^{m} b_i(\tilde{t}) \\ &< 0, \end{aligned}$$

which is a contradiction and hence (2.5) holds.

We finally show that $l = \liminf_{t\to\infty} x(t) > \kappa$. By way of contradiction, we assume that $0 \le l \le \kappa$. By the fluctuation lemma [11, Lemma A.1.], there exists a sequence $\{t_k\}_{k\ge 1}$ such that

$$t_k \to +\infty, \quad x(t_k) \to \liminf_{t\to+\infty} x(t), \quad x'(t_k) \to 0 \quad \text{as } k \to +\infty.$$

Since $\{x_{t_k}\}$ is bounded and equicontinuous, by the Ascoli-Arzelá theorem, there exists a subsequence, still denoted by itself for simplicity of notation, such that

$$x_{t_k} \to \varphi^*(k \to +\infty) \quad \text{for some } \varphi^* \in C_+.$$

Moreover,

$$\varphi^*(0) = l \le \varphi^*(\theta) \le M \quad \text{for } \theta \in [-r, 0).$$

Without loss of generality, we assume that all $a(t_k)$, $b_i(t_k)$ and $\tau_i(t_k)$ are convergent to a^*, b_i^* and τ_i^*, respectively. This can be achieved because of almost periodicity. It follows from

$$x'(t_k) = -a(t_k)x(t_k) + \sum_{i=1}^{m} \frac{b_i(t_k)}{1 + x^n(t_k - \tau_i(t_k))}$$

that (taking limits)

$$\begin{aligned} 0 &= -a^* l + \sum_{i=1}^{m} \frac{b_i^*}{1 + (\varphi^*(-\tau_i^*))^n} \\ &\geq -a^* l + \sum_{i=1}^{m} \frac{b_i^*}{1 + M^n} \\ &\geq -a^* \kappa + \sum_{i=1}^{m} \frac{b_i^*}{1 + M^n} \\ &\geq \inf_{t \in R} \left\{ -a(t)\kappa + \sum_{i=1}^{m} \frac{b_i(t)}{1 + M^n} \right\} \\ &> 0, \end{aligned}$$

a contradiction. This proves that $l > \kappa$. Thus, from (2.5), we can choose $t_\varphi > t_0$ such that

$$\kappa < x(t; t_0, \varphi) < M \quad \text{for all } t \geq t_\varphi.$$

This ends the proof of Lemma 2.2.

Lemma 2.3. *Suppose that* (2.2) *holds, and*

$$\sup_{t \in R} \left\{ -a(t) + \sum_{i=1}^{m} b_i(t) \frac{n}{4\kappa} \right\} < 0. \tag{2.7}$$

Moreover, assume that $x(t) = x(t; t_0, \varphi)$ is a solution of equation (1.1) with initial condition (1.4) and φ' is bounded continuous on $[-r, 0]$. Then for any $\epsilon > 0$, there exists $l = l(\epsilon) > 0$, such that every interval $[\alpha, \alpha + l]$ contains at least one number δ for which there exists $N > 0$ satisfing

$$|x(t + \delta) - x(t)| \leq \epsilon, \quad \text{for all } t > N. \tag{2.8}$$

Proof. Define a continuous function $\Gamma(u)$ by setting

$$\Gamma(u) = \sup_{t \in R} \left\{ -[a(t) - u] + \sum_{i=1}^{m} b_i(t) \frac{n}{4\kappa} e^{ru} \right\}, \quad u \in [0, 1]. \tag{2.9}$$

Then, we have

$$\Gamma(0) = \sup_{t \in R} \left\{ -a(t) + \sum_{i=1}^{m} b_i(t) \frac{n}{4\kappa} \right\} < 0,$$

which implies that there exist two constants $\eta > 0$ and $\lambda \in (0, 1]$ such that

$$\Gamma(\lambda) = \sup_{t \in R} \left\{ -[a(t) - \lambda] + \sum_{i=1}^{m} b_i(t) \frac{n}{4\kappa} e^{\lambda r} \right\} < -\eta < 0. \tag{2.10}$$

For $t \in (-\infty, t_0 - r]$, we add the definition of $x(t)$ with $x(t) \equiv x(t_0 - r)$. Set

$$\begin{aligned} \epsilon(\delta, t) &= -[a(t+\delta) - a(t)]x(t+\delta) \\ &\quad + \sum_{i=1}^{m} [b_i(t+\delta) - b_i(t)] \frac{1}{1 + x^n(t + \delta - \tau_i(t+\delta))} \\ &\quad + \sum_{i=1}^{m} b_i(t) \left[\frac{1}{1 + x^n(t + \delta - \tau_i(t+\delta))} - \frac{1}{1 + x^n(t + \delta - \tau_i(t))} \right], \\ &\quad t \in R. \end{aligned} \tag{2.11}$$

By Lemma 2.2, the solution $x(t)$ is bounded and

$$\kappa < x(t) < M, \quad \text{for all } t \geq t_\varphi. \tag{2.12}$$

which implies that the right-hand side of (1.1) is also bounded, and $x'(t)$ is a bounded function on $[t_0 - r, +\infty)$. Thus, in view of the fact that $x(t) \equiv x(t_0 - r)$ for $t \in (-\infty, t_0 - r]$, we obtain that $x(t)$ is uniformly continuous on R. From (2.1), for any $\epsilon > 0$, there exists $l = l(\epsilon) > 0$, such that every interval $[\alpha, \alpha + l]$, $\alpha \in R$, contains a δ for which

$$|\epsilon(\delta, t)| \leq \frac{1}{2} \eta \epsilon, \quad \text{for all } t \in R. \tag{2.13}$$

Let $N_0 \geq \max\{t_0, t_0 - \delta, t_\varphi + r, t_\varphi + r - \delta\}$. For $t \in R$, denote

$$u(t) = x(t + \delta) - x(t).$$

Then, for all $t \geq N_0$, we get

$$\begin{aligned} \frac{du(t)}{dt} &= -a(t)[x(t+\delta) - x(t)] \\ &\quad + \sum_{i=1}^{m} b_i(t) \left[\frac{1}{1 + x^n(t + \delta - \tau_i(t))} - \frac{1}{1 + x^n(t - \tau_i(t))} \right] + \epsilon(\delta, t). \end{aligned} \tag{2.14}$$

From (2.14) and the inequalities

$$\left|\frac{1}{1+x^n}-\frac{1}{1+y^n}\right|=\left|\frac{-n\theta^{n-1}}{(1+\theta^n)^2}\right||x-y|\le\frac{n\theta^{n-1}}{(2\sqrt{\theta^n})^2}|x-y|\le\frac{n}{4\kappa}|x-y|, \tag{2.15}$$

where $x, y \in [\kappa, M]$, θ lies between x and y, we obtain

$$\begin{aligned} &D^-(e^{\lambda s}|u(s)|)|_{s=t} \\ &\quad\le \lambda e^{\lambda t}|u(t)| + e^{\lambda t}\Bigg\{-a(t)|x(t+\delta)-x(t)| \\ &\qquad + \left|\sum_{i=1}^{m} b_i(t)\left[\frac{1}{1+x^n(t+\delta-\tau_i(t))}-\frac{1}{1+x^n(t-\tau_i(t))}\right]+\epsilon(\delta,t)\right|\Bigg\} \\ &\quad\le \lambda e^{\lambda t}|u(t)| + e^{\lambda t}\left\{-a(t)|u(t)| + \sum_{i=1}^{m} b_i(t)\frac{n}{4\kappa}|u(t-\tau_i(t))| + |\epsilon(\delta,t)|\right\} \\ &\quad= -[a(t)-\lambda]e^{\lambda t}|u(t)| + \sum_{i=1}^{m} b_i(t)\frac{n}{4\kappa}e^{\lambda\tau_i(t)}e^{\lambda(t-\tau_i(t))}|u(t-\tau_i(t))| \\ &\qquad + e^{\lambda t}|\epsilon(\delta,t)|, \quad \text{for all } t \ge N_0. \end{aligned} \tag{2.16}$$

Let

$$U(t) = \sup_{-\infty<s\le t}\{e^{\lambda s}|u(s)|\}. \tag{2.17}$$

It is obvious that $e^{\lambda t}|u(t)| \le U(t)$, and $U(t)$ is non-decreasing.

Now, we distinguish two cases to finish the proof.

Case one.

$$U(t) > e^{\lambda t}|u(t)| \quad \text{for all } t \ge N_0. \tag{2.18}$$

We claim that

$$U(t) \equiv U(N_0) \text{ is a constant for all } t \ge N_0. \tag{2.19}$$

Assume, by way of contradiction, that (2.19) does not hold. Then, there exists $t_1 > N_0$ such that $U(t_1) > U(N_0)$. Since

$$e^{\lambda t}|u(t)| \le U(N_0) \quad \text{for all } t \le N_0.$$

There must exist $\beta \in (N_0, t_1)$ such that

$$e^{\lambda\beta}|u(\beta)| = U(t_1) \ge U(\beta),$$

which contradicts (2.18). This contradiction implies that (2.19) holds. It follows that there exists $t_2 > N_0$ such that

$$|u(t)| \le e^{-\lambda t}U(t) = e^{-\lambda t}U(N_0) < \epsilon \quad \text{for all } t \ge t_2. \tag{2.20}$$

Case two. There is a $t_0^* \geq N_0$ that $U(t_0^*) = e^{\lambda t_0^*}|u(t_0^*)|$. Then, in view of (2.10), (2.13) and (2.16), we get

$$
\begin{aligned}
(2.21)\qquad 0 &\leq D^-(e^{\lambda s}|u(s)|)|_{s=t_0^*} \\
&\leq -[a(t_0^*) - \lambda]e^{\lambda t_0^*}|u(t_0^*)| \\
&\quad + \sum_{i=1}^{m} b_i(t_0^*)\frac{n}{4\kappa}e^{\lambda\tau_i(t_0^*)}e^{\lambda(t_0^*-\tau_i(t_0^*))}|u(t_0^* - \tau_i(t_0^*))| + e^{\lambda t_0^*}|\epsilon(\delta, t_0^*)| \\
&\leq \left\{-[a(t_0^*) - \lambda] + \sum_{i=1}^{m} b_i(t_0^*)\frac{n}{4\kappa}e^{\lambda r}\right\}U(t_0^*) + \frac{1}{2}\eta\epsilon e^{\lambda t_0^*} \\
&< -\eta U(t_0^*) + \eta\epsilon e^{\lambda t_0^*},
\end{aligned}
$$

which yields that

$$(2.22)\qquad e^{\lambda t_0^*}|u(t_0^*)| = U(t_0^*) < \epsilon e^{\lambda t_0^*}, \quad \text{and} \quad |u(t_0^*)| < \epsilon.$$

For any $t > t_0^*$, with the same approach as that in deriving of (2.22), we can show

$$(2.23)\qquad e^{\lambda t}|u(t)| < \epsilon e^{\lambda t}, \quad \text{and} \quad |u(t)| < \epsilon,$$

if $U(t) = e^{\lambda t}|u(t)|$.

On the other hand, if $U(t) > e^{\lambda t}|u(t)|$ and $t > t_0^*$. We can choose $t_0^* \leq t_3 < t$ such that

$$U(t_3) = e^{\lambda t_3}|u(t_3)| \quad \text{and} \quad U(s) > e^{\lambda s}|u(s)| \quad \text{for all } s \in (t_3, t],$$

which, together with (2.23), yields

$$|u(t_3)| < \epsilon.$$

With a similar argument as that in the proof of Case one, we can show that

$$(2.24)\qquad U(s) \equiv U(t_3) \text{ is a constant for all } s \in (t_3, t],$$

which implies that

$$|u(t)| < e^{-\lambda t}U(t) = e^{-\lambda t}U(t_3) = |u(t_3)|e^{-\lambda(t-t_3)} < \epsilon.$$

In summary, there must exist $N > \max\{t_0^*, N_0, t_2\}$ such that $|u(t)| \leq \epsilon$ holds for all $t > N$. The proof of Lemma 2.3 is now complete.

3. Main results

In this section, we establish sufficient conditions on the existence and global exponential stability of almost periodic solutions of (1.1).

Theorem 3.1. *Under the assumptions of Lemma* 2.3, *equation* (1.1) *has at least one positive almost periodic solution* $x^*(t)$. *Moreover*, $x^*(t)$ *is globally exponentially stable*, *i.e.*, *there exist constants* K_{φ,x^*} *and* t_{φ,x^*} *such that*

$$|x(t;t_0,\varphi) - x^*(t)| < K_{\varphi,x^*}e^{-\lambda t} \quad \text{for all } t > t_{\varphi,x^*},$$

where λ *has been defined in* (2.10).

Proof. Let $v(t) = v(t;t_0,\varphi^v)$ be a solution of equation (1.1) with initial conditions satisfying the assumptions in Lemma 2.3. We also add the definition of $v(t)$ with $v(t) \equiv v(t_0 - r)$ for all $t \in (-\infty, t_0 - r]$. Set

$$\begin{aligned}\epsilon(k,t) = {} & -[a(t+t_k) - a(t)]v(t+t_k) \\ & + \sum_{i=1}^{m}[b_i(t+t_k) - b_i(t)]\frac{1}{1+v^n(t+t_k-\tau_i(t+t_k))} \\ & + \sum_{i=1}^{m} b_i(t)\left[\frac{1}{1+v^n(t+t_k-\tau_i(t+t_k))} - \frac{1}{1+v^n(t+t_k-\tau_i(t))}\right], \\ & t \in R,\end{aligned} \tag{3.1}$$

where $\{t_k\}$ is any sequence of real numbers. By Lemma 2.2, the solution $v(t)$ is bounded and

$$\kappa < v(t) < M, \quad \text{for all } t \geq t_{\varphi^v}, \tag{3.2}$$

which implies that the right side of (1.1) is also bounded, and $v'(t)$ is a bounded function on $[t_0 - r, +\infty)$. Thus, in view of the fact that $v(t) \equiv v(t_0 - r)$ for $t \in (-\infty, t_0 - r]$, we obtain that $v(t)$ is uniformly continuous on R. Then, from the almost periodicity of a, b_i and τ_i, we can select a sequence $\{t_k\} \to +\infty$ such that

$$\begin{aligned}& |a(t+t_k) - a(t)| \leq \frac{1}{k}, \quad |b_i(t+t_k) - b_i(t)| \leq \frac{1}{k}, \\ & |\tau_i(t+t_k) - \tau_i(t)| \leq \frac{1}{k}, \quad |\epsilon(k,t)| \leq \frac{1}{k},\end{aligned} \tag{3.3}$$

for all i, t.

Since $\{v(t+t_k)\}_{k=1}^{+\infty}$ is uniformly bounded and equiuniformly continuous, by Arzala-Ascoli Lemma and diagonal selection principle, we can choose a subsequence $\{t_{k_j}\}$ of $\{t_k\}$, such that $v(t+t_{k_j})$ (for convenience, we still denote by $v(t+t_k)$) uniformly converges to a continuous function $x^*(t)$ on any compact set of R, and

$$\kappa \leq x^*(t) \leq M, \quad \text{for all } t \in R. \tag{3.4}$$

Now, we prove that $x^*(t)$ is a solution of (1.1). In fact, for any $t \geq t_0$ and $\Delta t \in R$, from (3.3), we have

$$
\begin{aligned}
(3.5)\quad & x^*(t+\Delta t) - x^*(t) \\
&= \lim_{k\to+\infty} [v(t+\Delta t + t_k) - v(t+t_k)] \\
&= \lim_{k\to+\infty} \int_t^{t+\Delta t} \Bigg\{ -a(\mu+t_k)v(\mu+t_k) \\
&\qquad + \sum_{i=1}^{m} b_i(\mu+t_k) \frac{1}{1+v^n(\mu+t_k-\tau_i(\mu+t_k))} \Bigg\}\, d\mu \\
&= \lim_{k\to+\infty} \int_t^{t+\Delta t} \Bigg\{ -a(\mu)v(\mu+t_k) \\
&\qquad + \sum_{i=1}^{m} b_i(\mu) \frac{1}{1+v^n(\mu+t_k-\tau_i(\mu))} + \epsilon(k,\mu) \Bigg\}\, d\mu \\
&= \int_t^{t+\Delta t} \Bigg\{ -a(\mu)x^*(\mu) + \sum_{i=1}^{m} b_i(\mu) \frac{1}{1+(x^*(\mu-\tau_i(\mu)))^n} \Bigg\}\, d\mu \\
&\quad + \lim_{k\to+\infty} \int_t^{t+\Delta t} \epsilon(k,\mu)\, d\mu \\
&= \int_t^{t+\Delta t} \Bigg\{ -a(\mu)x^*(\mu) + \sum_{i=1}^{m} b_i(\mu) \frac{1}{1+(x^*(\mu-\tau_i(\mu)))^n} \Bigg\}\, d\mu,
\end{aligned}
$$

where $t+\Delta t \geq t_0$. Consequently, (3.5) implies that

$$
(3.6)\qquad \frac{d}{dt}\{x^*(t)\} = -a(t)x^*(t) + \sum_{i=1}^{m} b_i(t) \frac{1}{1+(x^*(t-\tau_i(t)))^n}.
$$

Therefore, $x^*(t)$ is a solution of (1.1).

Secondly, we prove that $x^*(t)$ is an almost periodic solution of (1.1). From Lemma 2.3, for any $\varepsilon > 0$, there exists $l = l(\varepsilon) > 0$, such that every interval $[\alpha, \alpha + l]$ contains at least one number δ for which there exists $N > 0$ satisfies

$$
(3.7)\qquad |v(t+\delta) - v(t)| \leq \varepsilon, \quad \text{for all } t > N.
$$

Then, for any fixed $s \in R$, we can find a sufficient large positive integer $N_1 > N$ such that for any $k > N_1$,

$$
(3.8)\qquad s + t_k > N, \quad |v(s+t_k+\delta) - v(s+t_k)| \leq \varepsilon.
$$

Let $k \to +\infty$, we obtain

$$
|x^*(s+\delta) - x^*(s)| \leq \varepsilon,
$$

which implies that $x^*(t)$ is an almost periodic solution of equation (1.1).

Finally, we prove that $x^*(t)$ is globally exponentially stable.

Let $x(t) = x(t; t_0, \varphi)$ and $y(t) = x(t) - x^*(t)$, where $t \in [t_0 - r, +\infty)$. Then

$$y'(t) = -a(t)y(t) + \sum_{i=1}^{m} b_i(t)\left[\frac{1}{1 + x^n(t - \tau_i(t))} - \frac{1}{1 + x^{*\,n}(t - \tau_i(t))}\right]. \tag{3.9}$$

It follows from Lemma 2.2 that there exists $t_{\varphi,x^*} > t_0$ such that

$$\kappa \le x(t), \quad x^*(t) \le M, \quad \text{for all } t \in [t_{\varphi,x^*} - r, +\infty). \tag{3.10}$$

We consider the Lyapunov functional

$$V(t) = |y(t)|e^{\lambda t}. \tag{3.11}$$

Calculating the upper left derivative of $V(t)$ along the solution $y(t)$ of (3.9), we have

$$\begin{aligned} D^-(V(t)) &\le -a(t)|y(t)|e^{\lambda t} + \sum_{i=1}^{m} b_i(t)\left|\frac{1}{1 + x^n(t - \tau_i(t))} - \frac{1}{1 + x^{*\,n}(t - \tau_i(t))}\right| e^{\lambda t} + \lambda|y(t)|e^{\lambda t} \\ &= \left[-(a(t) - \lambda)|y(t)| + \sum_{i=1}^{m} b_i(t)\left|\frac{1}{1 + x^n(t - \tau_i(t))} - \frac{1}{1 + x^{*\,n}(t - \tau_i(t))}\right|\right] e^{\lambda t}, \quad \text{for all } t > t_{\varphi,x^*}. \end{aligned} \tag{3.12}$$

We claim that

$$\begin{aligned} V(t) &= |y(t)|e^{\lambda t} \\ &< e^{\lambda t_{\varphi,x^*}}\left(\max_{t \in [t_0 - r, t_{\varphi,x^*}]} |x(t) - x^*(t)| + 1\right) \\ &:= K_{\varphi,x^*} \quad \text{for all } t > t_{\varphi,x^*}. \end{aligned} \tag{3.13}$$

Contrarily, there must exist $t_* > t_{\varphi,x^*}$ such that

$$V(t_*) = K_{\varphi,x^*} \quad \text{and} \quad V(t) < K_{\varphi,x^*} \quad \text{for all } t \in [t_0 - r, t_*). \tag{3.14}$$

Together with (2.15), (3.12) and (3.14), we obtain

$$\begin{aligned} 0 &\le D^-(V(t_*)) \\ &\le \Bigg[-(a(t_*) - \lambda)|y(t_*)| \\ &\quad + \sum_{i=1}^{m} b_i(t_*)\left|\frac{1}{1 + x^n(t_* - \tau_i(t_*))} - \frac{1}{1 + x^{*\,n}(t_* - \tau_i(t_*))}\right|\Bigg] e^{\lambda t_*} \end{aligned}$$

$$\leq -(a(t_*) - \lambda)|y(t_*)|e^{\lambda t_*} + \sum_{i=1}^{m} b_i(t_*)\frac{n}{4\kappa}e^{\lambda\tau_i(t_*)}e^{\lambda(t_* - \tau_i(t_*))}|y(t_* - \tau_i(t_*))|$$

$$\leq \left\{-(a(t_*) - \lambda) + \sum_{i=1}^{m} b_i(t_*)\frac{n}{4\kappa}e^{\lambda r}\right\}K_{\varphi, x^*}.$$

Thus,

$$0 \leq -(a(t_*) - \lambda) + \sum_{i=1}^{m} b_i(t_*)\frac{n}{4\kappa}e^{\lambda r},$$

which contradicts with (2.10). Hence, (3.13) holds. It follows that

$$|y(t)| < K_{\varphi, x^*}e^{-\lambda t} \quad \text{for all } t > t_{\varphi, x^*}.$$

This completes the proof of Theorem 3.1.

4. An example

In this section, we present an example to check the validity of our results we obtained in the previous sections.

Example 4.1. Consider the following model of hematopoiesis with multiple time-varying delays:

$$x'(t) = -1.3x(t) + \frac{1}{2}\left(2 + \frac{1}{2}|\cos\sqrt{2}t|\right)\frac{1}{1 + x(t - 2e^{\cos t})} + \frac{1}{2}\left(2 + \frac{1}{2}|\sin\sqrt{3}t|\right)\frac{1}{1 + x(t - 2e^{\sin t})}. \tag{4.1}$$

Obviously,

$$a^+ = a^- = 1.3, \quad b_1^- = b_2^- = 1, \quad b_1^+ = b_2^+ = 1.25, \quad n = 1, \quad r = 2e.$$

Let $\kappa = 0.5$ and $M = 2$. Then

$$-a^- M + b_1^+ + b_2^+ = -0.1 < 0, \quad -a^+\kappa + \frac{b_1^- + b_2^-}{1 + M} = \frac{1}{60} > 0,$$

$$-a^- + (b_1^+ + b_2^+)\frac{n}{4\kappa} = -1.3 + 2.5 \times \frac{1}{2} = -0.05 < 0,$$

which imply that (4.1) satisfies the assumptions of Theorem 3.1. Therefore, equation (4.1) has a unique positive almost periodic solution $x^*(t)$, which is globally exponentially stable with the exponential convergent rate $\lambda \approx 0.0001$. The numerical simulation in Fig. 1 strongly supports the conclusion.

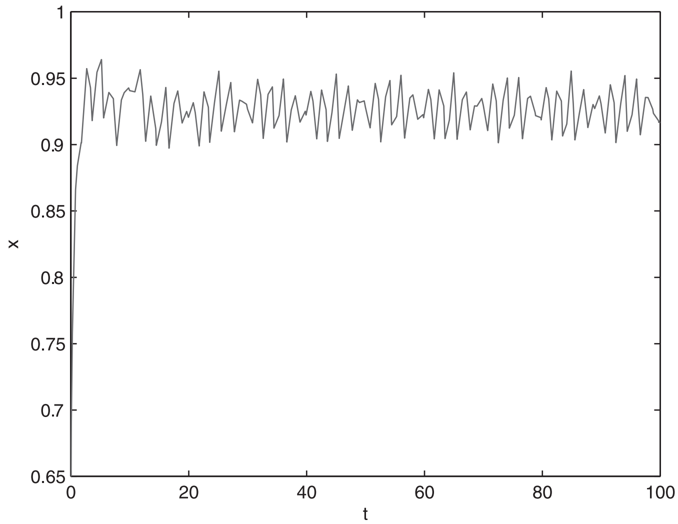

FIGURE 1. Numerical solution $x(t)$ of equation (4.1) for initial value $\varphi(s) \equiv 0.65$, $s \in [-2e, 0]$.

Remark 4.1. We remark that the results in [12] and [15] give no opinions about global exponential convergence for the positive almost periodic solution. Moreover, the authors in [1] considered (1.1) with the following conditions:

$$m = 1, \quad \sup_{t \in R} b_1(t) < \inf_{t \in R} a(t), \quad b_1(0) = 0.$$

Thus, the results in [1, 12, 15] and the references therein cannot be applied to prove the global exponential stability of positive almost periodic solution for (4.1). This implies that the results of this paper are new and they complement previously known results. In particular, in this present paper, we employ a novel proof to establish some criteria to guarantee the global dynamic behaviors of positive almost periodic solutions for non-autonomous model of hematopoiesis with multiple time-varying delays.

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