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POSITIVE PERIODIC SOLUTIONS FOR A NONLINEAR DENSITY-DEPENDENT MORTALITY NICHOLSON'S BLOWFLIES MODEL*

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Abstract

This paper is concerned with a class of Nicholson's blowflies model with a nonlinear density-dependent mortality term. Under appropriate conditions, we establish some criteria to ensure that the solutions of this model converge globally exponentially to a positive periodic solution. Moreover, we give an example and its numerical simulation to illustrate our main results.

1. Introduction

In the classic study of biological and ecological dynamics, Nicholson's blowflies equation was introduced by Nicholson [8] to model laboratory fly population. Its dynamics was later studied in [5] and [9], where this model was referred to as the Nicholsons blowflies equation [5]. Recently, as pointed out by L. Berezansky et al. [2], a linear model of density-dependent mortality will be most accurate for populations at low densities, and marine ecologists are currently constructing new fishery models with nonlinear density-dependent mortality rates. Therefore, L. Berezansky et al. [2] and Wang [13] proposed the following Nicholson's blowflies model with a nonlinear density-dependent mortality term

$$x'(t) = -\frac{a(t)x(t)}{b(t)+x(t)} + P(t)x(t-\tau(t))e^{-\gamma(t)x(t-\tau(t))}, \tag{1.1}$$

where the variable coefficients and delays are continuous functions. More details on biological explanation to coefficients and delays of model (1.1) can be found in [2, 13]. Moreover, the periodical variation of the environment plays an

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important role in many biological and ecological dynamical systems. Consequently, there have been extensive results on existence of positive periodic solutions for Nicholson's blowflies model with nonlinear density-dependent mortality terms. We refer the reader to [2, 3, 4, 7, 13, 14] and the references cited therein. However, to the best of our knowledge, there is no much work on the global exponential stability of the positive periodic solution for model (1.1). On the other hand, the real biological applications of Nicholson's blowflies model heavily depend on the global exponential convergence behaviors, because the exponential convergent rate can be unveiled. Hence, it is worthwhile continuing to investigate the existence and global exponential stability of positive periodic solutions of (1.1).

Motivated by the above discussions, in this paper, we are devoted to the global exponential stability of positive periodic solutions for a general nonlinear density-dependent mortality Nicholson's blowflies model given by

$$x'(t) = -\frac{a(t)x(t)}{b(t)+x(t)} + \sum_{j=1}^{m} \beta_j(t)x(t-\tau_j(t))e^{-\gamma_j(t)x(t-\tau_j(t))}, \tag{1.2}$$

where $a, b, \beta_j, \gamma_j : R \to (0, +\infty)$ and $\tau_j : R \to [0, +\infty)$ are continuous T-periodic functions for $j = 1, 2, \dots, m$ and $T > 0$. Obviously, (1.1) is a special case of (1.2) with $m = 1$.

For convenience, we introduce some notations. In the following part of this paper, given a bounded continuous function g defined on R, let g^+ and g^- be defined as

$$g^+ = \sup_{t\in R} g(t), \quad g^- = \inf_{t\in R} g(t). \tag{1.3}$$

It will be assumed that

$$r = \max_{1\le j\le m} \tau_j^+, \quad \gamma_j^- \ge 1, \quad j = 1, 2, \dots, m. \tag{1.4}$$

Throughout this paper, let R_+ denote nonnegative real number space, $C = C([-r, 0], R)$ be the continuous functions space equipped with the usual supremun norm $\|\cdot\|$, and let $C_+ = C([-r, 0], R_+)$. If $x(t)$ is continuous and defined on $[-r + t_0, \sigma)$ with $t_0, \sigma \in R$, then we define $x_t \in C$ where $x_t(\theta) = x(t + \theta)$ for all $\theta \in [-r, 0]$.

It is biologically reasonable to assume that only positive solutions of model (1.2) are meaningful and therefore admissible. Much can be learned by considering admissible initial conditions

$$x_{t_0} = \varphi, \quad \varphi \in C_+ \quad \text{and} \quad \varphi(0) > 0. \tag{1.5}$$

Define a continuous map $f : R \times C_+ \to R$ by setting

$$f(t, \varphi) = -\frac{a(t)\varphi(0)}{b(t)+\varphi(0)} + \sum_{j=1}^{m} \beta_j(t)\varphi(-\tau_j(t))e^{-\gamma_j(t)\varphi(-\tau_j(t))}.$$

Then, f is a locally Lipschitz map with respect to $\varphi \in C_+$, which ensures the existence and uniqueness of the solution of (1.2) with admissible initial conditions (1.5).

We denote $x_t(t_0, \varphi)(x(t; t_0, \varphi))$ for an admissible solution of the admissible initial value problem (1.2) and (1.5). Also, let $[t_0, \eta(\varphi))$ be the maximal right-interval of existence of $x_t(t_0, \varphi)$.

Since the function $\dfrac{1-x}{e^x}$ is decreasing with the range $[0, 1]$, it follows easily that there exists a unique $\kappa \in (0, 1)$ such that

$$\frac{1-\kappa}{e^{\kappa}} = \frac{1}{e^2}. \tag{1.6}$$

Obviously,

$$\sup_{x \geq \kappa} \left| \frac{1-x}{e^x} \right| = \frac{1}{e^2}. \tag{1.7}$$

Moreover, since xe^{-x} increases on $[0, 1]$ and decreases on $[1, +\infty)$, let $\tilde{\kappa}$ be the unique number in $(1, +\infty)$ such that

$$\kappa e^{-\kappa} = \tilde{\kappa} e^{-\tilde{\kappa}}. \tag{1.8}$$

The remaining of this paper is organized as follows. In Section 2, we give some lemmas, which tell us some kinds of solutions to (1.2) are bounded. These results play an important role in Section 3 to establish the existence of positive periodic solutions of (1.2). Here we also study the global exponential stability of positive periodic solutions. The paper concludes with an example to illustrate the effectiveness of the obtained results by numerical simulation.

2. Preliminary results

In this section, some lemmas will be presented, which are of importance in proving our main results in Section 3.

Lemma 2.1. *Suppose that there exists a positive constant M such that*

$$\frac{a(t)M}{b(t)+M} > \frac{1}{e} \sum_{j=1}^{m} \frac{\beta_j(t)}{\gamma_j(t)}, \quad 0 < \min_{t \in [0,T], s \in [0,\kappa]} \left\{ -\frac{a(t)}{b(t)+s} + \sum_{j=1}^{m} \frac{\beta_j(t)}{\gamma_j(t)} e^{-s} \right\}, \tag{2.1}$$

$$\gamma_j^+ \leq \frac{\tilde{\kappa}}{M},$$

where $j = 1, 2, \ldots, m$. Let

$$C^0 = \{\varphi \mid \varphi \in C, \kappa < \varphi(t) < M, \text{ for all } t \in [-r, 0]\}.$$

Then, the set of $\{x_t(t_0, \varphi) : t \in [t_0, \eta(\varphi))\}$ with $\varphi \in C^0$ is bounded, and $\eta(\varphi) = +\infty$. Moreover, $\kappa < x(t; t_0, \varphi) < M$ for all $t \geq t_0$.

Proof. Let $x(t) = x(t; t_0, \varphi)$, where $\varphi \in C^0$. We first claim:

$$x(t) < M \quad \text{for all } t \in [t_0, \eta(\varphi)). \tag{2.2}$$

Suppose, for the sake of contradiction, there exists $t_1 \in (t_0, \eta(\varphi))$ such that

$$x(t_1) = M, \quad x(t) < M \quad \text{for all } t \in [t_0 - r, t_1). \tag{2.3}$$

Calculating the derivative of $x(t)$, together with the fact that $\sup_{x \in R} xe^{-x} = \frac{1}{e}$, (1.2), (2.1) and (2.3) imply that

$$\begin{aligned} 0 &\le x'(t_1) \\ &= -\frac{a(t_1)M}{b(t_1) + M} + \sum_{j=1}^{m} \frac{\beta_j(t_1)}{\gamma_j(t_1)} \gamma_j(t_1) x(t_1 - \tau_j(t_1)) e^{-\gamma_j(t_1) x(t_1 - \tau_j(t_1))} \\ &\le -\frac{a(t_1)M}{b(t_1) + M} + \frac{1}{e} \sum_{j=1}^{m} \frac{\beta_j(t_1)}{\gamma_j(t_1)} \\ &< 0, \end{aligned}$$

which is a contradiction and implies that (2.2) holds.

We next show that

$$x(t) > \kappa, \quad \text{for all } t \in [t_0, \eta(\varphi)). \tag{2.4}$$

Assume, by way of contradiction, that (2.4) does not hold. Then, there exists $t_2 \in (t_0, \eta(\varphi))$ such that

$$x(t_2) = \kappa \quad \text{and} \quad x(t) > \kappa \quad \text{for all } t \in [t_0 - r, t_2). \tag{2.5}$$

Hence,

$$\kappa \le \gamma_j(t_2) x(t_2 - \tau_j(t_2)) \le \gamma_j(t_2) M \le \tilde{\kappa},$$

which, together with $\kappa e^{-\kappa} = \tilde{\kappa} e^{-\tilde{\kappa}}$, (1.2), (2.1), (2.2) and (2.5) imply that

$$\begin{aligned} 0 &\ge x'(t_2) \\ &= -\frac{a(t_2)\kappa}{b(t_2) + \kappa} + \sum_{j=1}^{m} \frac{\beta_j(t_2)}{\gamma_j(t_2)} \gamma_j(t_2) x(t_2 - \tau_j(t_2)) e^{-\gamma_j(t_2) x(t_2 - \tau_j(t_2))} \\ &\ge -\frac{a(t_2)\kappa}{b(t_2) + \kappa} + \sum_{j=1}^{m} \frac{\beta_j(t_2)}{\gamma_j(t_2)} \kappa e^{-\kappa} \\ &\ge \kappa \min_{t \in [0, T], s \in [0, \kappa]} \left\{ -\frac{a(t)}{b(t) + s} + \sum_{j=1}^{m} \frac{\beta_j(t)}{\gamma_j(t)} e^{-s} \right\} \\ &> 0, \end{aligned}$$

which is a contradiction and implies that (2.4) holds. This implies that $x(t)$ is bounded on $[t_0, \eta(\varphi))$. From Theorem 2.3.1 in [6], we easily obtain $\eta(\varphi) = +\infty$. This ends the proof of Lemma 2.1.

LEMMA 2.2. *Suppose* (2.1) *holds, and*

$$\frac{a(t)b(t)}{(b(t)+M)^2} > \sum_{j=1}^{m} \beta_j(t)\frac{1}{e^2}, \quad \textit{for all } t \in R. \tag{2.6}$$

Moreover, let

$$x^*(t) = x(t; t_0, \varphi^*), \quad x(t) = x(t; t_0, \varphi), \quad \textit{where } \varphi, \varphi^* \in C^0.$$

Then, there exists a positive constant λ *such that*

$$x(t) - x^*(t) = O(e^{-\lambda t}). \tag{2.7}$$

Proof. Define a continuous function $\Gamma(u)$ by setting

$$\Gamma(u) = \sup_{t \in R}\left\{-\left[\frac{a(t)b(t)}{(b(t)+M)^2} - u\right] + \sum_{j=1}^{m}\beta_j(t)\frac{1}{e^2}e^{ur}\right\}, \quad u \in [0,1]. \tag{2.8}$$

Then, from (2.6), we have

$$\Gamma(0) = \sup_{t \in R}\left\{-\frac{a(t)b(t)}{(b(t)+M)^2} + \sum_{j=1}^{m}\beta_j(t)\frac{1}{e^2}\right\} < 0,$$

which, together with the periodicity of coefficient functions, implies that there exist two constants $\eta > 0$ and $\lambda \in (0,1]$ such that

$$\Gamma(\lambda) = \sup_{t \in R}\left\{-\left[\frac{a(t)b(t)}{(b(t)+M)^2} - \lambda\right] + \sum_{j=1}^{m}\beta_j(t)\frac{1}{e^2}e^{\lambda r}\right\} < -\eta < 0, \quad \forall t \in R. \tag{2.9}$$

Set $y(t) = x(t) - x^*(t)$, where $t \in [t_0 - r, +\infty)$. Then

$$\begin{aligned} y'(t) = &-\left[\frac{a(t)x(t)}{b(t)+x(t)} - \frac{a(t)x^*(t)}{b(t)+x^*(t)}\right] \\ &+ \sum_{j=1}^{m}\beta_j(t)[x(t-\tau_j(t))e^{-\gamma_j(t)x(t-\tau_j(t))} - x^*(t-\tau_j(t))e^{-\gamma_j(t)x^*(t-\tau_j(t))}]. \end{aligned} \tag{2.10}$$

It follows from Lemma 2.1 that

$$\kappa < x(t), \quad x^*(t) < M, \quad \text{for all } t \in [t_0 - r, +\infty). \tag{2.11}$$

We consider the Lyapunov functional

$$V(t) = |y(t)|e^{\lambda t}. \tag{2.12}$$

Calculating the upper left derivative of $V(t)$ along the solution $y(t)$ of (2.10), we have

$$\begin{aligned} D^{-}(V(t)) \le & -\left[\frac{a(t)x(t)}{b(t)+x(t)} - \frac{a(t)x^*(t)}{b(t)+x^*(t)}\right] \operatorname{sgn}(x(t)-x^*(t))e^{\lambda t} \\ & + \sum_{j=1}^{m} \beta_j(t)|x(t-\tau_j(t))e^{-\gamma_j(t)x(t-\tau_j(t))} \\ & - x^*(t-\tau_j(t))e^{-\gamma_j(t)x^*(t-\tau_j(t))}|e^{\lambda t} + \lambda|y(t)|e^{\lambda t}, \quad \text{for all } t > t_0. \end{aligned} \tag{2.13}$$

We claim that

$$\begin{aligned} V(t) &= |y(t)|e^{\lambda t} \\ &< e^{\lambda t_0}\left(\max_{t\in[t_0-r,\,t_0]}|x(t)-x^*(t)|+1\right) \\ &= e^{\lambda t_0}\left(\max_{\theta\in[-r,0]}|\varphi(\theta)-\varphi^*(\theta)|+1\right) \\ &:= K, \quad \text{for all } t > t_0. \end{aligned} \tag{2.14}$$

Contrarily, there must exist $t_* > t_0$ such that

$$V(t_*) = K \quad \text{and} \quad V(t) < K \quad \text{for all } t \in [t_0 - r, t_*). \tag{2.15}$$

Since $x(t) \ge \kappa$ and $x^*(t) \ge \kappa$ for all $t \ge t_0 - r$. Together with (1.7), (1.8), (2.13), (2.15) and the inequalities

$$\begin{aligned} & -\left(\frac{a(t)A}{b(t)+A} - \frac{a(t)B}{b(t)+B}\right)\operatorname{sgn}(A-B) \\ & \quad = -\frac{a(t)b(t)}{(b(t)+A+\theta(A-B))^2}|A-B| \\ & \quad \le -\frac{a(t)b(t)}{(b(t)+M)^2}|A-B|, \quad \text{where } A, B \in [\kappa, M],\ 0<\theta<1, \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} |se^{-s} - te^{-t}| &= \left|\frac{1-(s+\theta(t-s))}{e^{s+\theta(t-s)}}\right| |s-t| \\ &\le \frac{1}{e^2}|s-t|, \quad \text{where } s, t \in [\kappa, +\infty),\ 0<\theta<1, \end{aligned} \tag{2.17}$$

we obtain

$$
\begin{aligned}
(2.18)\qquad 0 &\le D^-(V(t_*)) \\
&\le -\left[\frac{a(t_*)x(t_*)}{b(t_*)+x(t_*)} - \frac{a(t_*)x^*(t_*)}{b(t_*)+x^*(t_*)}\right] \operatorname{sgn}(x(t_*)-x^*(t_*))e^{\lambda t_*} \\
&\quad + \sum_{j=1}^m \beta_j(t_*)|x(t_*-\tau_j(t_*))e^{-\gamma_j(t_*)x(t_*-\tau_j(t_*))} \\
&\quad - x^*(t_*-\tau_j(t_*))e^{-\gamma_j(t_*)x^*(t_*-\tau_j(t_*))}|e^{\lambda t_*} + \lambda|y(t_*)|e^{\lambda t_*} \\
&\le -\frac{a(t_*)b(t_*)}{(b(t_*)+M)^2}|y(t_*)|e^{\lambda t_*} + \sum_{j=1}^m \beta_j(t_*)|x(t_*-\tau_j(t_*))e^{-\gamma_j(t_*)x(t_*-\tau_j(t_*))} \\
&\quad - x^*(t_*-\tau_j(t_*))e^{-\gamma_j(t_*)x^*(t_*-\tau_j(t_*))}|e^{\lambda t_*} + \lambda|y(t_*)|e^{\lambda t_*} \\
&= -\left[\frac{a(t_*)b(t_*)}{(b(t_*)+M)^2} - \lambda\right]|y(t_*)|e^{\lambda t_*} \\
&\quad + \sum_{j=1}^m \frac{\beta_j(t_*)}{\gamma_j(t_*)}|\gamma_j(t_*)x(t_*-\tau_j(t_*))e^{-\gamma_j(t_*)x(t_*-\tau_j(t_*))} \\
&\quad - \gamma_j(t_*)x^*(t_*-\tau_j(t_*))e^{-\gamma_j(t_*)x^*(t_*-\tau_j(t_*))}|e^{\lambda t_*} \\
&\le -\left[\frac{a(t_*)b(t_*)}{(b(t_*)+M)^2} - \lambda\right]|y(t_*)|e^{\lambda t_*} \\
&\quad + \sum_{j=1}^m \beta_j(t_*)\frac{1}{e^2}|y(t_*-\tau_j(t_*))|e^{\lambda(t_*-\tau_j(t_*))}e^{\lambda\tau_j(t_*)} \\
&\le \left\{-\left[\frac{a(t_*)b(t_*)}{(b(t_*)+M)^2} - \lambda\right] + \sum_{j=1}^m \beta_j(t_*)\frac{1}{e^2}e^{\lambda r}\right\}K.
\end{aligned}
$$

Thus,

$$
0 \le -\left[\frac{a(t_*)b(t_*)}{(b(t_*)+M)^2} - \lambda\right] + \sum_{j=1}^m \beta_j(t_*)\frac{1}{e^2}e^{\lambda r},
$$

which contradicts with (2.9). Hence, (2.14) holds. It follows that

$$
(2.19)\qquad |y(t)| < Ke^{-\lambda t} \quad \text{for all } t > t_0.
$$

This completes the proof.

3. Existence and exponential stability of positive periodic solutions

In this section, we establish sufficient conditions on the existence and global exponential stability of positive T-periodic solutions of equation (1.2).

Theorem 3.1. *Suppose that all conditions in Lemma* 2.2 *are satisfied. Then equation* (1.2) *has at least one positive* T*-periodic solution* $x^*(t)$.

Proof. Let $x(t) = x(t; t_0, \varphi)$, where $\varphi \in C^0$. By Lemma 2.1, we get

$$\kappa < x(t) < M, \quad \text{for all } t \geq t_0 - r.$$

By the periodicity of coefficients and delays for (1.2), we have, for any natural number h,

$$\begin{aligned}
(3.1) \qquad & [x(t+(h+1)T)]' \\
&= -\frac{a(t+(h+1)T)x(t+(h+1)T)}{b(t+(h+1)T)+x(t+(h+1)T)} \\
&\quad + \sum_{j=1}^{m} \beta_j(t+(h+1)T)x(t+(h+1)T-\tau_j(t+(h+1)T)) \\
&\quad \times e^{-\gamma_j(t+(h+1)T)x(t+(h+1)T-\tau_j(t+(h+1)T))} \\
&= -\frac{a(t)x(t+(h+1)T)}{b(t)+x(t+(h+1)T)} \\
&\quad + \sum_{j=1}^{m} \beta_j(t)x(t+(h+1)T-\tau_j(t))e^{-\gamma_j(t)x(t+(h+1)T-\tau_j(t))}, \\
&\quad t+(h+1)T \in [t_0, +\infty).
\end{aligned}$$

Thus, for any natural number h, we obtain that $x(t+(h+1)T)$ is a solution of system (1.2) for all $t+(h+1)T \geq t_0$. Hence, $x(t+T)$ ($t \in [t_0 - r, +\infty)$) is also a solution of (1.2) with initial values

$$\psi(s) = x(s + t_0 + T), \quad s \in [-r, 0].$$

Then, by the proof of Lemma 2.2, there exists a constant

$$K = e^{\lambda t_0}\left(\max_{\theta \in [-r,0]} |\varphi(\theta) - \psi(\theta)| + 1\right)$$

such that for any natural number h,

$$\begin{aligned}
(3.2) \qquad & |x(t+(h+1)T) - x(t+hT)| \\
&= |x(t+hT+T) - x(t+hT)| \\
&\leq Ke^{-\lambda(t+hT)} \\
&= Ke^{-\lambda t}\left(\frac{1}{e^{\lambda T}}\right)^h, \quad t+hT \geq t_0.
\end{aligned}$$

Now, we show that $x(t+qT)$ is convergent on any compact interval as $q \to \infty$.

Let $[a,b] \subset R$ be an arbitrary subset of R. Choose a nonnegative integer q_0 such that $t + q_0 T \geq t_0$ for $t \in [a,b]$. Then for $t \in [a,b]$ and $q > q_0$ we have

$$x(t+qT) = x(t+q_0T) + \sum_{h=q_0}^{q-1} [x(t+(h+1)T) - x(t+hT)]. \tag{3.3}$$

Then $x(t+qT)$ will converge uniformly to a continuous function, say $x^*(t)$, on $[a,b]$. Because of arbitrariness of $[a,b]$, we see that $x(t+qT) \to x^*(t)$ as $q \to \infty$ for $t \in R$. Then, (3.1) leads to

$$\kappa \leq x^*(t) \leq M, \quad \text{for all } t \in R. \tag{3.4}$$

It remains to show that x^* is a T-periodic solution of (1.2). The periodicity is obvious since

$$x^*(t+T) = \lim_{q\to\infty} x((t+T)+qT) = \lim_{q+1\to\infty} x(t+(q+1)T) = x^*(t)$$

for all $t \in R$.

Noting that the right side of (1.2) is continuous, together with (3.1) and (3.4), we know that $\{x'(t+(h+1)T)\}$ converges uniformly to a continuous function on any compact set of R. Therefore, letting $h \to +\infty$ on both sides of (3.1), we get

$$\frac{d}{dt}\{x^*(t)\} = -\frac{a(t)x^*(t)}{b(t)+x^*(t)} + \sum_{j=1}^{m} \beta_j(t) x^*(t-\tau_j(t)) e^{-\gamma_j(t)x^*(t-\tau_j(t))}. \tag{3.5}$$

Therefore, $x^*(t)$ is a solution of is a positive T-periodic solution of (1.2). This completes the proof.

THEOREM 3.2. *Suppose that all conditions in Lemma* 2.2 *are satisfied. Then equation* (1.2) *has exactly one positive T-periodic solution $x^*(t)$. Moreover, $x^*(t)$ is globally exponentially stable. That is*

$$x(t) - x^*(t) = O(e^{-\lambda t}), \quad \textit{where } x(t) = x(t; t_0, \varphi).$$

Proof. From Theorem 3.1, we should show the global exponential stability for positive T-periodic solution $x^*(t)$ of equation (1.2). Since $\varphi \in C_+$, using Theorem 5.2.1 in [10, p. 81], we have $x_t(t_0,\varphi) \in C_+$ for all $t \in [t_0, \eta(\varphi))$. Let $x(t) = x(t; t_0, \varphi)$. From (1.2) and the fact that $\frac{a(t)x}{b(t)+x} \leq \frac{a(t)x}{b(t)}$ for all $t \in R$, $x \geq 0$, we get

$$\begin{aligned} x'(t) &= -\frac{a(t)x(t)}{b(t)+x(t)} + \sum_{j=1}^{m} \beta_j(t) x(t-\tau_j(t)) e^{-\gamma_j(t)x(t-\tau_j(t))} \\ &\geq -\frac{a(t)}{b(t)} x(t) + \sum_{j=1}^{m} \beta_j(t) x(t-\tau_j(t)) e^{-\gamma_j(t)x(t-\tau_j(t))}. \end{aligned} \tag{3.6}$$

In view of $x(t_0) = \varphi(0) > 0$, integrating (3.6) from t_0 to t, we have

$$(3.7)\quad x(t) \ge e^{-\int_{t_0}^{t}(a(u)/b(u))\,du}x(t_0) + e^{-\int_{t_0}^{t}(a(u)/b(u))\,du}\int_{t_0}^{t} e^{\int_{t_0}^{s}(a(v)/b(v))\,dv}\sum_{j=1}^{m}\beta_j(s)x(s-\tau_j(s))e^{-\gamma_j(s)x(s-\tau_j(s))}\,ds > 0, \quad \text{for all } t \in [t_0, \eta(\varphi)).$$

We next show that there is $t_3 \in [t_0, \eta(\varphi))$ such that

$$(3.8)\quad \kappa < x(t) < M \quad \text{for all } t \in [t_3, \eta(\varphi)), \quad \text{and} \quad \eta(\varphi) = +\infty.$$

We first prove that there exists $t_4 \in [t_0, \eta(\varphi))$ such that

$$(3.9)\quad x(t_4) < M.$$

Otherwise,

$$(3.10)\quad x(t) \ge M \quad \text{for all } t \in [t_0, \eta(\varphi)),$$

which together with (2.1), implies that

$$(3.11)\quad x'(t) = -\frac{a(t)x(t)}{b(t)+x(t)} + \sum_{j=1}^{m}\frac{\beta_j(t)}{\gamma_j(t)}\gamma_j(t)x(t-\tau_j(t))e^{-\gamma_j(t)x(t-\tau_j(t))} \le -\frac{a(t)M}{b(t)+M} + \frac{1}{e}\sum_{j=1}^{m}\frac{\beta_j(t)}{\gamma_j(t)} < 0, \quad \text{for all } t \in [t_0, \eta(\varphi)).$$

This yields that $x(t)$ is bounded and monotone decreasing on $[t_0, \eta(\varphi))$. Again from Theorem 2.3.1 in [6], we easily obtain $\eta(\varphi) = +\infty$. Then, (3.11) leads to

$$x(t) = x(t_0) + \int_{t_0}^{t} x'(s)\,ds \le x(t_0) + \max_{t\in R}\left\{-\frac{a(t)M}{b(t)+M} + \frac{1}{e}\sum_{j=1}^{m}\frac{\beta_j(t)}{\gamma_j(t)}\right\}(t-t_0), \quad \forall t \ge t_0,$$

and

$$\lim_{t\to+\infty} x(t) = -\infty,$$

which contradicts with (3.7). Hence, (3.9) holds. We claim:

$$(3.12)\quad x(t) < M \quad \text{for all } t \in [t_4, \eta(\varphi)), \quad \text{and} \quad \eta(\varphi) = +\infty.$$

Suppose, for the sake of contradiction, there exists $t_5 \in (t_4, \eta(\varphi))$ such that

$$(3.13)\quad x(t_5) = M, \quad x(t) < M \quad \text{for all } t \in [t_4, t_5).$$

Calculating the derivative of $x(t)$, together with the fact that $\sup_{x\in R} xe^{-x} = \frac{1}{e}$, (1.2), (2.1) and (3.13) imply that

$$
\begin{aligned}
0 &\le x'(t_5) \\
&= -\frac{a(t_5)M}{b(t_5)+M} + \sum_{j=1}^{m} \frac{\beta_j(t_5)}{\gamma_j(t_5)} \gamma_j(t_5) x(t_5 - \tau_j(t_5)) e^{-\gamma_j(t_5)x(t_5-\tau_j(t_5))} \\
&\le -\frac{a(t_5)M}{b(t_5)+M} + \frac{1}{e}\sum_{j=1}^{m} \frac{\beta_j(t_5)}{\gamma_j(t_5)} \\
&< 0,
\end{aligned}
$$

which is a contradiction and implies that (3.12) holds.

Furthermore, we prove that there exists a positive constant l such that

$$\liminf_{t\to+\infty} x(t) = l. \tag{3.14}$$

Otherwise, we assume that $\liminf_{t\to+\infty} x(t) = 0$. For each $t \ge t_0$, we define

$$m(t) = \max\left\{\xi : \xi \le t, x(\xi) = \min_{t_0\le s\le t} x(s)\right\}.$$

Observe that $m(t) \to +\infty$ as $t \to +\infty$ and that

$$\lim_{t\to+\infty} x(m(t)) = 0. \tag{3.15}$$

However, $x(m(t)) = \min_{t_0\le s\le t} x(s)$, and so $x'(m(t)) \le 0$ for all $m(t) > t_0$. According to (1.2), we have

$$
\begin{aligned}
0 &\ge x'(m(t)) \\
&= -\frac{a(m(t))x(m(t))}{b(m(t))+x(m(t))} + \sum_{j=1}^{m} \beta_j(m(t)) x(m(t) - \tau_j(m(t))) e^{-\gamma_j(m(t))x(m(t)-\tau_j(m(t)))} \\
&\ge -\frac{a(m(t))x(m(t))}{b(m(t))} + \sum_{j=1}^{m} \beta_j(m(t)) x(m(t) - \tau_j(m(t))) e^{-\gamma_j(m(t))x(m(t)-\tau_j(m(t)))},
\end{aligned}
$$

and consequently,

$$
\begin{aligned}
\frac{a(m(t))x(m(t))}{b(m(t))} &\ge \sum_{j=1}^{m} \beta_j(m(t)) x(m(t) - \tau_j(m(t))) e^{-\gamma_j(m(t))x(m(t)-\tau_j(m(t)))} \\
&\ge \beta_j(m(t)) x(m(t) - \tau_j(m(t))) e^{-\gamma_j(m(t))x(m(t)-\tau_j(m(t)))},
\end{aligned} \tag{3.16}
$$

where $m(t) > t_0$, $j = 1, 2, \ldots, m$. This, together with (3.15), implies that

$$\lim_{t\to+\infty} x(m(t) - \tau_j(m(t))) = 0, \quad j = 1, 2, \ldots, m. \tag{3.17}$$

Noting that the continuities and boundedness of the functions $a(t)$, $b(t)$ and $\beta_j(t)$, we can select a sequence $\{t_n\}_{n=1}^{+\infty}$ such that

$$\begin{aligned} &\lim_{n\to+\infty} t_n = +\infty, \quad \lim_{n\to+\infty} x(m(t_n)) = 0, \\ &\lim_{n\to+\infty} \frac{\beta_j(m(t_n))b(m(t_n))}{a(m(t_n))} = a_j^*, \quad j = 1, 2, \ldots, m. \end{aligned} \tag{3.18}$$

In view of (3.16), we get

$$\begin{aligned} \frac{a(m(t_n))}{b(m(t_n))} &\geq \sum_{j=1}^{m} \beta_j(m(t_n)) \frac{x(m(t_n) - \tau_j(m(t_n)))e^{-\gamma_j(m(t_n))x(m(t_n)-\tau_j(m(t_n)))}}{x(m(t_n))} \\ &\geq \sum_{j=1}^{m} \beta_j(m(t_n)) \frac{x(m(t_n) - \tau_j(m(t_n)))e^{-\gamma_j^+ x(m(t_n)-\tau_j(m(t_n)))}}{x(m(t_n) - \tau_j(m(t_n)))} \\ &= \sum_{j=1}^{m} \beta_j(m(t_n))e^{-\gamma_j^+ x(m(t_n)-\tau_j(m(t_n)))}, \end{aligned}$$

and

$$1 \geq \sum_{j=1}^{m} \frac{\beta_j(m(t_n))b(m(t_n))}{a(m(t_n))} e^{-\gamma_j^+ x(m(t_n)-\tau_j(m(t_n)))}. \tag{3.19}$$

Letting $n \to +\infty$, (3.17), (3.18) and (3.19) imply that

$$\begin{aligned} 1 &\geq \sum_{j=1}^{m} \lim_{n\to+\infty} \frac{\beta_j(m(t_n))b(m(t_n))}{a(m(t_n))} \lim_{n\to+\infty} e^{-\gamma_j^+ x(m(t_n)-\tau_j(m(t_n)))} \\ &= \lim_{n\to+\infty} \sum_{j=1}^{m} \frac{\beta_j(m(t_n))b(m(t_n))}{a(m(t_n))} \\ &\geq \liminf_{t\to+\infty} \sum_{j=1}^{m} \frac{\beta_j(t)b(t)}{a(t)}. \end{aligned} \tag{3.20}$$

From (2.1), we get

$$\begin{aligned} 0 &< \min_{t\in[0,T], s\in[0,\kappa]} \left\{ -\frac{a(t)}{b(t)+s} + \sum_{j=1}^{m} \frac{\beta_j(t)}{\gamma_j(t)} e^{-s} \right\} \\ &\leq \min_{t\in[0,T]} \left\{ -\frac{a(t)}{b(t)} + \sum_{j=1}^{m} \frac{\beta_j(t)}{\gamma_j(t)} \right\} \end{aligned}$$

$$= \min_{t \in R}\left\{-\frac{a(t)}{b(t)} + \sum_{j=1}^{m}\frac{\beta_j(t)}{\gamma_j(t)}\right\}$$

$$\leq \min_{t \in R}\left\{-\frac{a(t)}{b(t)} + \sum_{j=1}^{m}\beta_j(t)\right\},$$

and

$$1 < \sum_{j=1}^{m}\frac{\beta_j(t)b(t)}{a(t)}, \quad \text{for all } t \in R,$$

which contradicts to (3.20). Hence, (3.14) holds.

To prove (3.8), it is sufficiently to show $l > \kappa$. If not, we assume that $l \leq \kappa$. Then, for fixed $\varepsilon > 0$, there is $J = J(\varepsilon)$ such that

$$x_t(t_0, \varphi) > l_\varepsilon := l - \varepsilon \quad \text{for all } t \geq J. \tag{3.21}$$

By the fluctuation lemma [11, Lemma A.1.], there exists a sequence $\{t_k\}_{k=1}^{+\infty}$ such that

$$\begin{aligned} &t_k \nearrow +\infty, \quad x(t_k; t_0, \varphi) \to l, \quad \text{and} \\ &x'(t_k) = f(t_k, x_{t_k}(t_0, \varphi)) \to 0, \quad \text{as } k \to +\infty. \end{aligned} \tag{3.22}$$

For $t \geq J$, (3.7) and (3.12) leads to that there is a constant $B > 0$ such that

$$\begin{aligned} |x'(t; t_0, \varphi)| &= |f(t, x_t(t_0, \varphi))| \\ &= \left| -\frac{a(t)x(t; t_0, \varphi)}{b(t) + x(t; t_0, \varphi)} + \sum_{j=1}^{m}\beta_j(t)x(t - \tau_{ij}(t); t_0, \varphi)e^{-\gamma_j(t)x(t-\tau_{ij}(t); t_0, \varphi)} \right| \\ &< B. \end{aligned}$$

It follows that $x(t; t_0, \varphi)$ and $x'(t; t_0, \varphi)$ are uniformly bounded on $[t_0, +\infty)$, thus $\{x_{t_k}(t_0, \varphi)\}_{k=1}^{+\infty}$ is bounded and equicontinuous. By Ascoli-Arzelà Theorem, for a subsequence, still denoted by $\{x_{t_k}(t_0, \varphi)\}_{k=1}^{+\infty}$, we have

$$x_{t_k}(t_0, \varphi) \to \varphi^* \quad \text{for some } \varphi^* \in C([-r, 0], (0, +\infty)).$$

Since

$$x_t(t_0, \varphi) > l_\varepsilon \quad \text{for } t \geq J \text{ and } \varepsilon > 0 \text{ is arbitrary},$$

then $\varphi^*(s) \geq l$ for $t \in [-r, 0]$. From (3.12), we get

$$\varphi^*(0) = l \leq \varphi^*(s) \leq M \quad \text{for } t \in [-r, 0]. \tag{3.23}$$

By the boundedness of $\{\tau_j(t_k)\}_{k=1}^{+\infty}$, there is a subsequence of $\{t_k\}_{k=1}^{+\infty}$, still denoted by $\{t_k\}_{k=1}^{+\infty}$, which converges to a point $\tau_j^* \in [\tau_j^-, \tau_j^+]$ with $j = 1, 2, \ldots, m$.

Similarly, we can also suppose that

$$\lim_{k\to+\infty} a(t_k) = a^* \in [a^-, a^+], \quad \lim_{k\to+\infty} b(t_k) = b^* \in [b^-, b^+]$$

and

$$\lim_{k\to+\infty} \beta_j = \beta_j^* \in [\beta_j^-, \beta_j^+], \quad \lim_{k\to+\infty} \gamma_j = \gamma_j^* \in [\gamma_j^-, \gamma_j^+], \quad j = 1, 2, \dots, m.$$

Hence,

$$f(t_k, x_{t_k}(t_0, \varphi)) \to \Lambda, \quad \text{as } k \to +\infty, \tag{3.24}$$

with

$$\Lambda = -\frac{a^*\varphi^*(0)}{b^* + \varphi^*(0)} + \sum_{j=1}^{m} \beta_j^* \varphi^*(-\tau_j^*) e^{-\gamma_j^*\varphi^*(-\tau_j^*)}. \tag{3.25}$$

According to (1.7), (1.8), (1.9), (2.1) and the fact that

$$0 < l \le \kappa, \quad l \le \gamma_j^*\varphi^*(-\tau_j^*) \le \gamma_j^+ M \le \tilde{\kappa}, \quad j = 1, 2, \dots, m,$$

we obtain

$$\begin{aligned}
\Lambda &= -\frac{a^* l}{b^* + l} + \sum_{j=1}^{m} \frac{\beta_j^*}{\gamma_j^*} \gamma_j^* \varphi^*(-\tau_j^*) e^{-\gamma_j^*\varphi^*(-\tau_j^*)} \\
&\ge -\frac{a^* l}{b^* + l} + \sum_{j=1}^{m} \frac{\beta_j^*}{\gamma_j^*} l e^{-l} \\
&= l\left[-\frac{a^*}{b^* + l} + \sum_{j=1}^{m} \frac{\beta_j^*}{\gamma_j^*} e^{-l}\right] \\
&\ge l \min_{t\in[0,T], s\in[0,\kappa]} \left\{-\frac{a(t)}{b(t) + s} + \sum_{j=1}^{m} \frac{\beta_j(t)}{\gamma_j(t)} e^{-s}\right\} \\
&> 0,
\end{aligned}$$

which contradicts (3.22) and implies $l > \kappa$.

The notations in Lemma 2.2 are still used as follows. With a similar argument as that in the proof of Lemma 2.2, we can prove that

$$\begin{aligned}
V(t) &= |y(t)|e^{\lambda t} \\
&= |x(t) - x^*(t)|e^{\lambda t} \\
&< e^{\lambda t_3}\left(\max_{t\in[t_0-r, t_3]} |x(t) - x^*(t)| + 1\right) \\
&:= K, \quad \text{for all } t > t_3,
\end{aligned} \tag{3.26}$$

which yields

$$x(t) - x^*(t) = O(e^{-\lambda t}),$$

and hence the proof is complete.

4. An example

In this section, we present an example to check the validity of our results we obtained in the previous sections.

Example 4.1. Consider the following Nicholson's blowflies model with a nonlinear density-dependent mortality term:

$$x'(t) = -\frac{0.6951934x(t)}{0.7537127 + x(t)} + \frac{100 + \sin t}{100 + \cos t} x(t - 2e^{\sin^4 t}) e^{-x(t-2e^{\sin^4 t})}. \tag{4.1}$$

Obviously, $r = 2e$, $a^- = a^+ = 0.6951934$, $b^- = b^+ = 0.7537127$, $\beta_1^- \geq \frac{99}{101}$, $\beta_j^+ \leq \frac{101}{99}$, $\gamma_1^- = \gamma_1^+ = 1$. From (1.6), (1.8), $\tilde{\kappa} > 1$ and $\kappa e^{-\kappa} = \tilde{\kappa} e^{-\tilde{\kappa}}$, we obtain

$$\kappa \approx 0.7215355, \quad \tilde{\kappa} \approx 1.342276.$$

Let $M = 1.087308$, we get

$$\frac{a^- M}{b^+ + M} = \frac{0.6951934 \times 1.087308}{0.7537127 + 1.087308} \approx 0.4105817,$$

$$\frac{\beta_1^+}{\gamma_1^-} \frac{1}{e} \leq \frac{101}{99} \frac{1}{e} \approx 0.3753113,$$

$$\begin{aligned}
&\min_{t \in [0,T], s \in [0,\kappa]} \left\{ -\frac{a(t)}{b(t) + s} + \frac{\beta_1(t)}{\gamma_1(t)} e^{-s} \right\} \\
&\geq \min_{t \in [0,T], s \in [0,\kappa]} \left\{ -\frac{0.6951934}{0.7537127 + s} + \frac{99}{101} e^{-s} \right\} \\
&= \min_{s \in [0,\kappa]} \left\{ -\frac{0.6951934}{0.7537127 + s} + \frac{99}{101} e^{-s} \right\} \approx 0.005143492,
\end{aligned}$$

$$\frac{a^- b^-}{(b^+ + M)^2} = \frac{6951934 \times 0.7537127}{(0.7537127 + 1.087308)^2} \approx 0.1545945,$$

$$\beta_1^+ \frac{1}{e^2} \leq \frac{101}{99} \frac{1}{e^2} \approx 0.1380693,$$

which implies that the Nicholson's blowflies model (4.1) satisfies (2.1) and (2.6). Hence, from Theorems 3.2, equation (4.1) has exactly one positive 2π-periodic solution $x^*(t)$. Moreover, $x^*(t)$ is globally exponentially stable. This fact is verified by the numerical simulation in Fig. 1.

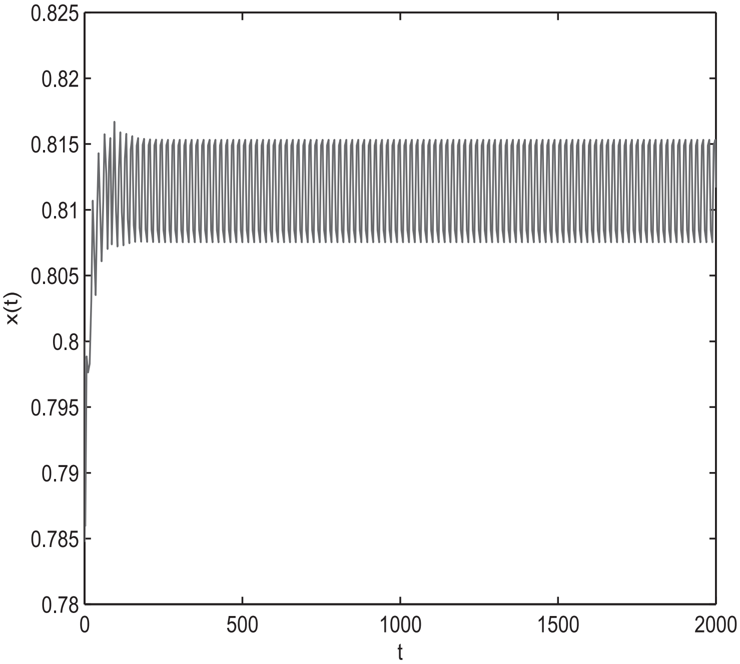

FIGURE 1. Numerical solution $x(t)$ of equation (4.1) for initial value $\varphi(s) \equiv 0.8$, $s \in [-2e, 0]$.

Remark 4.1. As is known to us, there is no literature concerning the global exponential stability of the positive periodic solution of Nicholson's blowflies model with a nonlinear density-dependent mortality term $\frac{a(t)x(t)}{b(t)+x(t)}$. Thus, all the results in the references [3, 4, 7, 13, 14] cannot be applied to prove that all the solutions of (4.1) converge exponentially to the positive 2π-periodic solution. Moreover, in [1, 12], the authors only proved the existence of positive periodic solutions for the first order functional differential equations with no conclusions about the globally exponential stability. This implies that all the results obtained in [13–14] also fail for (4.1).

REFERENCES

[1] P. AMSTER AND L. IDELS, Periodic solutions in general scalar models of population dynamics with delays, Nonlinear Differ. Equ. Appl. **20** (2013), 1577–1596.

[2] L. BEREZANSKY, E. BRAVERMAN AND L. IDELS, Nicholson's blowflies differential equations revisited: Main results and open problems, Appl. Math. Model. **34** (2010), 1405–1417.

[3] W. CHEN, Permanence for Nicholson-type delay systems with patch structure and nonlinear density-dependent mortality terms, Electron. J. Qual. Theory Differ. Equ. **73** (2012), 1–14.

[4] W. Chen and L. Wang, Positive periodic solutions of Nicholson-type delay systems with nonlinear density-dependent mortality terms, Abstr. Appl. Anal. **2012** (2012), 843178.
[5] W. Gurney, S. Blythe and R. Nisbet, Nicholsons blowflies revisited, Nature **287** (1980), 17–21.
[6] J. K. Hale and S. M. Verduyn Lunel, Introduction to functional differential equations, Springer-Verlag, New York, 1993.
[7] X. Hou, L. Duan and Z. Huang, Permanence and periodic solutions for a class of delay Nicholson's blowflies models, Appl. Math. Model. **37** (2013), 1537–1544.
[8] A. Nicholson, An outline of the dynamics of animal populations, Aust. J. Zool. **2** (1954), 9–65.
[9] R. Nisbet and W. Gurney, Modelling fluctuating populations, John Wiley and Sons, NY, 1982.
[10] H. L. Smith, Monotone dynamical systems, Math. Surveys Monogr., Amer. Math. Soc., Providence, RI, 1995.
[11] H. L. Smith, An introduction to delay differential equations with applications to the life sciences, Springer, New York, 2011.
[12] H. Wang, Positive periodic solutions of functional differential equations, J. Diff. Equ. **202** (2004), 354–366.
[13] W. Wang, Positive periodic solutions of delayed Nicholson's blowflies models with a nonlinear density-dependent mortality term, Appl. Math. Model. **36** (2012), 4708–4713.
[14] W. Wang, Exponential extinction of Nicholson's blowflies system with nonlinear density-dependent mortality terms, Abstr. Appl. Anal. **2012** (2012), 302065.

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