

ON THE NUMBER OF EXCEPTIONAL POINTS OF
HOLOMORPHIC CURVES AND THE DEFECT
RELATION FOR HOLOMORPHIC CURVES

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Abstract

Let $X^n(2)$ be a subset of $\mathbf{C}^{n+1} \setminus \{0\}$ any two elements of which are not proportional. We estimate the number of exceptional points in $X^n(2)$ for several holomorphic curves and we consider the defect relation for holomorphic curves. We shall give an example for which the defect relation is extremal and then give some holomorphic curves for which the defect relation is not extremal over $X^n(2)$. Another defect relation is also considered.

1. Introduction

Let $f = [f_1, \dots, f_{n+1}]$ be a holomorphic curve from \mathbf{C} into the n -dimensional complex projective space $P^n(\mathbf{C})$ with a reduced representation

$$(f_1, \dots, f_{n+1}) : \mathbf{C} \rightarrow \mathbf{C}^{n+1} \setminus \{0\},$$

where n is a positive integer. We put

$$\|f(z)\| = (|f_1(z)|^2 + \dots + |f_{n+1}(z)|^2)^{1/2}$$

and the characteristic function of f is given as follows (see [14]):

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta - \log \|f(0)\|.$$

It is known ([1]) that for $U(z) = \max_{1 \leq j \leq n+1} |f_j(z)|$

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log U(re^{i\theta}) d\theta + O(1).$$

We suppose throughout the paper that f is transcendental; that is to say,

$$\lim_{r \rightarrow \infty} T(r, f) / \log r = \infty.$$

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The order ρ_f of f is given as follows:

$$\rho_f = \limsup_{r \rightarrow \infty} \log T(r, f) / \log r.$$

For $\mathbf{a} = (a_1, \dots, a_{n+1}) \in \mathbf{C}^{n+1} \setminus \{\mathbf{0}\}$, we put

$$(\mathbf{a}, f) = a_1 f_1 + \dots + a_{n+1} f_{n+1}$$

and we use the standard notations $N(r, \mathbf{a}, f)$, $N_n(r, \mathbf{a}, f)$, $\delta(\mathbf{a}, f)$, $\delta_n(\mathbf{a}, f)$ as usual (see e.g. [12, Introduction]).

Let $S(r, f)$ be any quantity satisfying

$$S(r, f) = o(T(r, f)) \quad (r \rightarrow \infty; r \notin E),$$

where E is a subset of $(0, \infty)$ of finite linear measure.

We suppose throughout the paper that f is linearly non-degenerate over \mathbf{C} . Namely, the Wronskian of the functions f_1, \dots, f_{n+1} is not identically equal to zero. Let X be a subset of $\mathbf{C}^{n+1} \setminus \{\mathbf{0}\}$ in N -subgeneral position and we put

- (a) $D^1 = \{\mathbf{a} \in X \mid \delta_n(\mathbf{a}, f) = 1\}$;
- (b) $P = \{\mathbf{a} \in X \mid (\mathbf{a}, f) \text{ has at most a finite number of zeros in } |z| < \infty\}$;
- (c) $M^1 = \{\mathbf{a} \in X \mid (\mathbf{a}, f) \text{ has no zeros in } |z| < \infty\}$;
- (d) $B = \{\mathbf{a} \in X \mid \limsup_{r \rightarrow \infty} \log N_n(r, \mathbf{a}, f) / \log r < \rho_f\}$,

where N is an integer such that $N \geq n$. We say that $\mathbf{a} \in B$ is Borel exceptional for f . Then, $M^1 \subset P \subset D^1$ and it is known that

$$\#P \leq \#D^1 \leq 2N - n + 1, \quad B \not\subset D^1, \quad D^1 \not\subset B \quad \text{and} \quad \#M^1 \leq N + N/n$$

(see e.g. [3], [9], [13]).

In this paper, with the set X in N -subgeneral position, we consider the following subsets of $\mathbf{C}^{n+1} \setminus \{\mathbf{0}\}$, which are not always in subgeneral position. For $k = 1, \dots, n+1$, let $X^n(k)$ be a subset of $\mathbf{C}^{n+1} \setminus \{\mathbf{0}\}$, satisfying that any k elements of which are linearly independent. In particular, our main one is $X^n(2)$ in this paper. From the definition it satisfies

- (*) For any $\mathbf{a}, \mathbf{b} \in X^n(2)$, $\mathbf{a} \neq \alpha \mathbf{b}$ ($\alpha \in \mathbf{C}$).

The set $X^n(n+1)$ is in general position.

The first purpose of this paper is to estimate the number of elements of the following subsets of $X^n(2)$.

- DEFINITION 1.A. (1) $\mathcal{D}^1 = \{\mathbf{a} \in X^n(2) \mid \delta_n(\mathbf{a}, f) = 1\}$
 (2) $\mathcal{P} = \{\mathbf{a} \in X^n(2) \mid (\mathbf{a}, f) \text{ has at most a finite number of zeros in } |z| < \infty\}$;
 (3) $\mathcal{M}^1 = \{\mathbf{a} \in X^n(2) \mid (\mathbf{a}, f) \text{ has no zeros in } |z| < \infty\}$;
 (4) $\mathcal{B} = \{\mathbf{a} \in X^n(2) \mid \limsup_{r \rightarrow \infty} \log N_n(r, \mathbf{a}, f) / \log r < \rho_f\}$.

The main tool is the fundamental inequality of H. Cartan ([1]). Some applications of our results are given. We estimate them when f is linearly non-degenerate over a function field

Note 1.1. An interesting result is given in a more delicate situation ([2, Theorem 5.2]).

The second purpose is to estimate $\#\{D^1 \cup B\}$ in X and to estimate $\#\{\mathcal{D}^1 \cup \mathcal{B}\}$ in $X^n(2)$ under a condition.

The last purpose is to give the defect relation of holomorphic curves, one of which is extremal and others are not extremal.

As is cited above the main tool in this paper is the fundamental inequality of H. Cartan ([1]) for holomorphic curves. We also use its generalization by Nochka ([7]).

2. Number of exceptional points of holomorphic curves

Let f , X , $X^n(2)$, N , n etc. be as in Section 1. First of all we give the main tool of this paper, the fundamental inequality by H. Cartan ([1]).

2-1. Lemma. We shall give some lemmas for later use in this section.

LEMMA 2.1 ([1]). *For any q elements \mathbf{a}_j ($j = 1, \dots, q$) of $\mathbf{C}^{n+1} \setminus \{\mathbf{0}\}$ in general position ($n+2 \leq q < \infty$), we have the following inequalities:*

- (I) $(q - n - 1)T(r, f) \leq \sum_{j=1}^q N_n(r, \mathbf{a}_j, f) + S(r, f)$.
 (II) $\sum_{j=1}^q \delta_n(\mathbf{a}_j, f) \leq n + 1$.

Note 2.1. When ρ_f is finite, the error term $S(r, f)$ can be replaced by $O(\log r)$ without exceptional intervals (see the proof of (I) and [4, Theorem 2.2(i)]).

We put

$$\mathcal{M}_f = \{\varphi \mid \text{meromorphic in } |z| < \infty; T(r, \varphi) = S(r, f)\}$$

and when $0 < \rho_f < \infty$

$$\mathcal{M}_o = \{\varphi \mid \text{meromorphic in } |z| < \infty; \rho_\varphi < \rho_f\},$$

where ρ_φ is the order of φ .

We know that the sets \mathcal{M}_f and \mathcal{M}_o are fields. We denote by \mathcal{M} the field \mathcal{M}_f , any subfield of \mathcal{M}_f or \mathcal{M}_o and from now on we suppose that the curve $f = [f_1, \dots, f_{n+1}]$ is linearly non-degenerate over \mathcal{M} .

We note that $\varphi' \in \mathcal{M}_f$ for $\varphi \in \mathcal{M}_f$ because $T(r, \varphi') \leq 2T(r, \varphi) + S(r, \varphi)$.

LEMMA 2.2. *Let $\mathbf{a}_j = (a_{j1}, \dots, a_{jn+1})$ ($j = 1, \dots, v$; $2 \leq v \leq n+1$) be v linearly independent vectors in \mathbf{C}^{n+1} , let \mathbf{a} be a vector in \mathbf{C}^{n+1} such that*

$$\mathbf{a} = \sum_{j=1}^v c_j \mathbf{a}_j \quad (c_j \neq 0 \in \mathbf{C})$$

and we put

$$F_j = (\mathbf{a}_j, f) \quad (j = 1, \dots, v), \quad F_o = (\mathbf{a}, f).$$

Let g be an entire function such that the functions $F_1/g, \dots, F_v/g$ are entire functions without common zeros and put $F = [F_1/g, \dots, F_v/g]$.

Then, we have the followings:

(I) The functions F_1, \dots, F_v are linearly independent over \mathcal{M} .
In particular,

$$F_{j_1}/F_{j_2} \notin \mathcal{M} \quad (1 \leq j_1 \neq j_2 \leq v).$$

(II) (i) $T(r, F) \leq T(r, f) + O(1)$ ($r \geq 1$).

(ii) $T(r, F_{j_1}/F_{j_2}) \leq T(r, F) + O(1)$ ($1 \leq j_1 \neq j_2 \leq v$).

(III) If F is transcendental,

$$T(r, F) \leq \sum_{j=1}^v N_n(r, \mathbf{a}_j, f) + N_n(r, \mathbf{a}, f) + S(r, F).$$

Proof. (I) Suppose that $\sum_{j=1}^v g_j F_j = 0$ ($g_j \in \mathcal{M}$). Then, as f_1, \dots, f_{n+1} are linearly independent over \mathcal{M} , we obtain the equation

$$\sum_{j=1}^v a_{jk} g_j = 0 \quad (k = 1, \dots, n+1).$$

We put $\mathbf{b}_k = (a_{1k}, a_{2k}, \dots, a_{vk})$ ($1 \leq k \leq n+1$). Then,

$$({}^t \mathbf{a}_1, \dots, {}^t \mathbf{a}_v) = \begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_{n+1} \end{pmatrix} \quad \text{and} \quad \text{rank} = \begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_{n+1} \end{pmatrix} = v$$

since $\mathbf{a}_1, \dots, \mathbf{a}_v$ are linearly independent, so that there are v linearly independent vectors $\mathbf{b}_{k_1}, \dots, \mathbf{b}_{k_v}$ in $\{\mathbf{b}_1, \dots, \mathbf{b}_{n+1}\}$.

From the equation

$$\sum_{j=1}^v a_{jk_i} g_j = 0 \quad (i = 1, \dots, v),$$

we obtain that $g_1 = 0, \dots, g_v = 0$, so that F_1, \dots, F_v are linearly independent over \mathcal{M} .

(II) (i) From the definition we have the relation

$$\begin{aligned} \log \|F\| &= \log \left(\sum_{j=1}^v |F_j/g|^2 \right)^{1/2} = \log \left(\sum_{j=1}^v |F_j|^2 \right)^{1/2} - \log |g| \\ &\leq \log \|f\| - \log |g| + O(1), \end{aligned}$$

so that

$$T(r, F) \leq T(r, f) - N(r, 1/g) + O(1) \leq T(r, f) + (1) \quad (r \geq 1).$$

(ii) From [1, p. 10], we have the inequality

$$T(r, F) + O(1) \geq T\left(r, \frac{F_{j_1}/g}{F_{j_2}/g}\right) = T(r, F_{j_1}/F_{j_2})$$

(III) We put $\mathbf{c} = (c_1, \dots, c_v)$. Then, we have that $\mathbf{e}_1, \dots, \mathbf{e}_v, \mathbf{c} \in \mathbf{C}^v$ are in general position since $c_j \neq 0$ ($j = 1, \dots, v$) and that for $F_o = (\mathbf{a}, f)$

$$(2.1) \quad F_o = \sum_{j=1}^v c_j F_j.$$

We apply Lemma 2.1(I) to (2.1) for $n = v - 1, F$ and $\{\mathbf{e}_1, \dots, \mathbf{e}_v, \mathbf{c}\}$. Then, we obtain the inequality

$$T(r, F) \leq \sum_{j=1}^v N_{v-1}(r, \mathbf{e}_j, F) + N_{v-1}(r, \mathbf{c}, F) + S(r, F).$$

Here, from the definition of counting function we have the relations for $r \geq 1$

$$N_{v-1}(r, \mathbf{e}_j, F) \leq N_n(r, \mathbf{a}_j, f) \quad (j = 1, \dots, v); \quad N_{v-1}(r, \mathbf{c}, F) \leq N_n(r, \mathbf{a}, f)$$

since $v - 1 \leq n$, so that we obtain (III). \square

We denote by \mathcal{K} any one of $\mathcal{D}^1, \mathcal{P}, \mathcal{M}^1$ and \mathcal{B} . We suppose that f is linearly non-degenerate over \mathcal{M} , where $\mathcal{M} = \mathcal{M}_f$ when $\mathcal{K} = \mathcal{D}^1$, $\mathcal{M} = \mathbf{C}(z)$ when $\mathcal{K} = \mathcal{P}$, $\mathcal{M} = \mathbf{C}$ when $\mathcal{K} = \mathcal{M}^1$ and $\mathcal{M} = \mathcal{M}_o$ when $\mathcal{K} = \mathcal{B}$. Then, we obtain the following.

LEMMA 2.3. *Suppose that $\mathcal{K} \neq \phi$. Then, for $1 \leq k \leq \min\{\#\mathcal{K}, n + 1\}$, any k vectors in \mathcal{K} are linearly independent over \mathbf{C} .*

Proof. (i) When $k = 1$. This is trivial since $\mathcal{K} \subset X^n(2) \subset \mathbf{C}^{n+1} \setminus \{\mathbf{0}\}$.

(ii) When $k = 2$. This is also trivial since \mathcal{K} satisfies (*) in Introduction.

(iii) For $2 \leq k \leq \min\{\#\mathcal{K}, n + 1\} - 1$, suppose that any k vectors in \mathcal{K} are linearly independent over \mathbf{C} . We prove that any $k + 1$ vectors in \mathcal{K} are linearly independent over \mathbf{C} .

Suppose to the contrary that there are $k + 1$ vectors in $\mathcal{K} : \mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{a}$ which are linearly dependent over \mathbf{C} . Then, from the hypothesis of the induction we can write

$$\mathbf{a} = \sum_{j=1}^k c_j \mathbf{a}_j \quad (c_j \neq 0, 1 \leq j \leq k),$$

from which we obtain the relation

$$(2.2) \quad (\mathbf{a}, f) = \sum_{j=1}^k c_j (\mathbf{a}_j, f).$$

We put

$$F_j = (\mathbf{a}_j, f) \quad (j = 1, \dots, k) \quad \text{and} \quad F_o = (\mathbf{a}, f).$$

Then, we obtain that from Lemma 2.2(I) F_1, \dots, F_k are linearly independent over \mathcal{M} and that from (2.2)

$$(2.3) \quad F_o = \sum_{j=1}^k c_j F_j \quad (c_j \neq 0).$$

Let g be an entire function such that the functions $F_1/g, \dots, F_k/g$ are entire functions without common zeros, where g is polynomial when $\mathcal{K} = \mathcal{P}$ and $g = 1$ when $\mathcal{K} = \mathcal{M}^1$. We put

$$F = [F_1/g, \dots, F_k/g].$$

Then, F is transcendental from Lemma 2.2(I) and (II)(ii) when $\mathcal{K} = \mathcal{D}^1, \mathcal{P}$ or \mathcal{B} . When $\mathcal{K} = \mathcal{M}^1$, from Lemma 2.2(I) F_{j_1}/F_{j_2} is not constant ($1 \leq j_1 \neq j_2 \leq k$). As F_{j_1}, F_{j_2} have no zero, F_{j_1}/F_{j_2} has neither zero nor pole. This implies that F_{j_1}/F_{j_2} is transcendental, so that F is transcendental from Lemma 2.2(II)(ii). Then, from (2.3) and Lemma 2.2(III) we obtain the inequality

$$(2.4) \quad T(r, F) \leq \sum_{j=1}^k N_n(r, \mathbf{a}_j, f) + N_n(r, \mathbf{a}, f) + S(r, F).$$

(a) When $\mathcal{K} = \mathcal{D}^1$, from Lemma 2.2(I), (II)(i),(ii) and (2.4) we obtain the inequality

$$\begin{aligned} 0 < \limsup_{r \rightarrow \infty, r \notin E} \frac{T(r, F_2/F_1)}{T(r, f)} &\leq \sum_{j=1}^k \limsup_{r \rightarrow \infty} \frac{N_n(r, \mathbf{a}_j, f)}{T(r, f)} \\ &+ \limsup_{r \rightarrow \infty} \frac{N_n(r, \mathbf{a}, f)}{T(r, f)} + \limsup_{r \rightarrow \infty, r \notin E} \frac{S(r, F)}{T(r, f)} = 0 \end{aligned}$$

since $\mathbf{a}_j, \mathbf{a} \in \mathcal{D}^1$. This is a contradiction. This means that any $k+1$ vectors in \mathcal{D}^1 must be linearly independent over \mathbf{C} .

(b) When $\mathcal{K} = \mathcal{P}$, from (2.4) we obtain the inequality

$$T(r, F) \leq O(\log r) + S(r, F) = S(r, F)$$

since $\mathbf{a}_j, \mathbf{a} \in \mathcal{P}$ and F is transcendental. This is a contradiction. This means that any $k+1$ vectors in \mathcal{P} must be linearly independent over \mathbf{C} .

(c) When $\mathcal{K} = \mathcal{M}^1$, from (2.4) we obtain the inequality $T(r, F) \leq S(r, F)$ since $\mathbf{a}_j, \mathbf{a} \in \mathcal{M}^1$. This is a contradiction since F is transcendental. This means that any $k+1$ vectors in \mathcal{M}^1 must be linearly independent over \mathbf{C} .

(d) When $\mathcal{H} = \mathcal{B}$, from (2.4) and Note 2.1 we obtain the inequality

$$T(r, F) \leq \sum_{j=1}^k N_n(r, \mathbf{a}_j, f) + N_n(r, \mathbf{a}, f) + O(\log r),$$

since $\rho_F \leq \rho_f < \infty$. As $\rho_F = \rho_f$ from Lemma 2.2(I) and (II)(i),(ii), we have the inequality

$$\begin{aligned} \rho_f &= \limsup_{r \rightarrow \infty} \frac{\log T(r, F)}{\log r} \\ &\leq \limsup_{r \rightarrow \infty} \frac{\log(\sum_{j=1}^k N_n(r, \mathbf{a}_j, f) + N_n(r, \mathbf{a}, f) + O(\log r))}{\log r} < \rho_f \end{aligned}$$

since $\mathbf{a}_j, \mathbf{a} \in \mathcal{B}$. This is a contradiction. This means that any $k+1$ vectors in \mathcal{B} must be linearly independent over \mathcal{C} .

From (i), (ii) and (iii) we obtain our lemma. \square

LEMMA 2.4 (see [4, p. 24]). *Let $\varphi(z)$ be transcendental meromorphic with at most a finite number of zeros and poles in $|z| < \infty$. Then,*

$$\liminf_{r \rightarrow \infty} T(r, \varphi)/r > 0.$$

Last of all in this section we shall give the following curve for later use.

LEMMA 2.5. *Let $f_o = [1, e^z, \dots, e^{nz}]$. Then, f_o is linearly independent over \mathcal{M}_{f_o} .*

Proof. We first note that f_o is transcendental since $T(r, f_o) \sim (n/\pi)r$ and that

$$T(r, e^{(i-j)z}) \sim \frac{|i-j|}{\pi} r \quad (i \neq j).$$

(i) When $n = 1$. We put

$$g_1 + g_2 e^z = 0 \quad (g_1, g_2 \in \mathcal{M}_{f_o}).$$

Suppose that $g_2 \neq 0$. Then, $e^z = -g_1/g_2$, and we have $T(r, e^z) = T(r, g_1/g_2)$, so that we have the relation

$$0 < \limsup_{r \rightarrow \infty, r \notin E} \frac{T(r, e^z)}{T(r, f_o)} = \limsup_{r \rightarrow \infty, r \notin E} \frac{T(r, g_1/g_2)}{T(r, f_o)} = 0,$$

which is a contradiction. This means that g_2 must be equal to zero and we have that $g_1 = 0$.

(ii) We suppose that this lemma holds for an integer $k \geq 2$:

$$\sum_{j=1}^k g_j e^{(j-1)z} = 0 \quad (g_j \in \mathcal{M}_{f_o}) \Rightarrow g_j = 0 \quad (j = 1, \dots, k)$$

and we prove that if

$$(2.5) \quad \sum_{j=1}^{k+1} g_j e^{(j-1)z} = 0 \quad (g_j \in \mathcal{M}_{f_o}, j = 1, \dots, k+1),$$

then $g_j = 0$ ($j = 1, \dots, k+1$).

Proof. (a) When $g_1 = 0$. In this case, from (2.5) we have the relation

$$\sum_{j=2}^{k+1} g_j e^{(j-2)z} = 0,$$

so that from the hypothesis of the induction $g_2 = 0, \dots, g_{k+1} = 0$.

(b) Suppose that $g_1 \neq 0$. Then, there exists at least one j ($2 \leq j \leq k+1$) such that $g_j \neq 0$. If $g_j = 0$ for all j ($2 \leq j \leq k+1$), $g_1 = 0$ from (2.5).

From (2.5), we obtain

$$(2.6) \quad 1 + \sum_{j=2}^{k+1} \frac{g_j}{g_1} e^{(j-1)z} = 0.$$

Here, $g_j/g_1 \in \mathcal{M}_{f_o}$ ($j = 2, \dots, k+1$). Differentiating (2.6), we obtain

$$\sum_{j=2}^{k+1} \left\{ \left(\frac{g_j}{g_1} \right)' + (j-1) \frac{g_j}{g_1} \right\} e^{(j-1)z} = 0.$$

Now, $g_j/g_1 \in \mathcal{M}_{f_o}$, $(g_j/g_1)' \in \mathcal{M}_{f_o}$ ($j = 2, \dots, k+1$) so that from the hypothesis of the induction we have the relations

$$\left(\frac{g_j}{g_1} \right)' + (j-1) \frac{g_j}{g_1} = 0 \quad (j = 2, \dots, k+1).$$

For j ($2 \leq j \leq k+1$) such that $g_j \neq 0$, the function g_j/g_1 satisfies the differential equation

$$(g_j/g_1)' / (g_j/g_1) = -(j-1)$$

from which we obtain

$$g_j/g_1 = ce^{-(j-1)z} \quad (c \neq 0) \quad \text{and} \quad T\left(r, \frac{g_j}{g_1}\right) = T(r, ce^{-(j-1)z}).$$

From this relation we have that

$$0 = \limsup_{r \rightarrow \infty, r \notin E} \frac{T(r, g_j/g_1)}{T(r, f_o)} = \limsup_{r \rightarrow \infty, r \notin E} \frac{T(r, ce^{-(j-1)z})}{T(r, f_o)} > 0,$$

which is a contradiction. This implies that g_1 must be equal to 0. Then, we obtain that $g_2 = 0, \dots, g_{k+1} = 0$ as in (a).

This means that f_o is linearly non-degenerate over \mathcal{M}_{f_o} . \square

2-2. Theorem. Let f , n , X , $X^n(2)$, \mathcal{D}^1 etc. be as in Section 1.

THEOREM 2.1.

(I) Suppose that f is linearly non-degenerate over \mathcal{M}_f . Then, $\#\mathcal{D}^1 \leq n+1$.

(II) Suppose that f is linearly non-degenerate over $\mathbf{C}(z)$. Then, $\#\mathcal{P} \leq n+1$.

(III) Suppose that f is linearly non-degenerate over \mathbf{C} . Then, $\#\mathcal{M}^1 \leq n+1$.

(IV) Suppose that f is linearly non-degenerate over \mathcal{M}_o and $0 < \rho_f < \infty$.

Then, $\#\mathcal{B} \leq n+1$.

Proof. We denote by \mathcal{K} any one of \mathcal{D}^1 , \mathcal{P} , \mathcal{M}^1 and \mathcal{B} . We suppose that f is linearly non-degenerate over \mathcal{M} , where $\mathcal{M} = \mathcal{M}_f$ when $\mathcal{K} = \mathcal{D}^1$, $\mathcal{M} = \mathbf{C}(z)$ when $\mathcal{K} = \mathcal{P}$, $\mathcal{M} = \mathbf{C}$ when $\mathcal{K} = \mathcal{M}^1$ and $\mathcal{M} = \mathcal{M}_o$ when $\mathcal{K} = \mathcal{B}$.

Suppose that $\#\mathcal{K} > n+1$. Then, \mathcal{K} contains at least $n+2$ elements. Let $\mathbf{a}_1, \dots, \mathbf{a}_{n+2}$ be in \mathcal{K} . Then, from Lemma 2.3, any $n+1$ vectors of $\mathbf{a}_1, \dots, \mathbf{a}_{n+2}$ are linearly independent over \mathbf{C} . This means that $\mathbf{a}_1, \dots, \mathbf{a}_{n+2}$ are in general position in \mathbf{C}^{n+1} .

(i) When $\mathcal{K} = \mathcal{D}^1, \mathcal{P}$ or \mathcal{M}^1 . As f is transcendental in this paper, from lemma 2.1(II) we have the inequality

$$\sum_{j=1}^{n+2} \delta_n(\mathbf{a}_j, f) \leq n+1.$$

On the other hand as $\delta_n(\mathbf{a}_j, f) = 1$ ($j = 1, \dots, n+2$) in any case, we have the equality

$$\sum_{j=1}^{n+2} \delta_n(\mathbf{a}_j, f) = n+2,$$

which is a contradiction. This means that $\#\mathcal{K} \leq n+1$.

(ii) When $\mathcal{K} = \mathcal{B}$. From Lemma 2.1 and Note 2.1, we have the inequality

$$T(r, f) \leq \sum_{j=1}^{n+2} N_n(r, \mathbf{a}_j, f) + O(\log r),$$

from which we obtain the inequality

$$\rho_f \leq \limsup_{r \rightarrow \infty} \frac{\log(\sum_{j=1}^{n+2} N_n(r, \mathbf{a}_j, f) + O(\log r))}{\log r} < \rho_f$$

since $\mathbf{a}_j \in \mathcal{B}$ ($j = 1, \dots, n+2$), which is a contradiction.

From (i) and (ii) we have our theorem. □

COROLLARY 2.1. Let $\mathbf{a}_1, \dots, \mathbf{a}_{n+2}$ be any $n+2$ elements in $X^n(2)$.

(I) If f is linearly non-degenerate over \mathcal{M}_f , then

$$(2.7) \quad \limsup_{r \rightarrow \infty} \frac{\sum_{j=1}^{n+2} N_n(r, \mathbf{a}_j, f)}{T(r, f)} > 0.$$

(II) If f is linearly non-degenerate over $\mathbf{C}(z)$, then

$$(2.8) \quad \lim_{r \rightarrow \infty} \frac{\sum_{j=1}^{n+2} N_n(r, \mathbf{a}_j, f)}{\log r} = \infty.$$

(a) In particular, if $\mathbf{a}_1, \dots, \mathbf{a}_{n+1} \in \mathcal{P}$ and if f is of order finite (resp. infinite),

$$(2.9) \quad \liminf_{r \rightarrow \infty} \frac{N_n(r, \mathbf{a}_{n+2}, f)}{r} > 0 \quad (\text{resp. } \limsup_{r \rightarrow \infty, r \notin E} \frac{N_n(r, \mathbf{a}_{n+2}, f)}{r} > 0).$$

(III) If f is linearly non-degenerate over \mathbf{C} , then

$$(2.10) \quad \bigcup_{j=1}^{n+2} \{z \mid (\mathbf{a}_j, f(z)) = 0\} \neq \emptyset.$$

(b) In particular, if $\mathbf{a}_1, \dots, \mathbf{a}_{n+1} \in \mathcal{M}^1$,

$$(2.11) \quad \lim_{r \rightarrow \infty} N_n(r, \mathbf{a}_{n+2}, f) / \log r = \infty.$$

(IV) If f is linearly non-degenerate over \mathcal{M}_o and $0 < \rho_f < \infty$, then

$$\limsup_{r \rightarrow \infty} \frac{\log(\sum_{j=1}^{n+2} N_n(r, \mathbf{a}_j, f))}{\log r} = \rho_f.$$

Proof. (I) Suppose that there exist $n+2$ vectors $\mathbf{a}_1, \dots, \mathbf{a}_{n+2} \in X^n(2)$ such that

$$\limsup_{r \rightarrow \infty} \frac{\sum_{j=1}^{n+2} N_n(r, \mathbf{a}_j, f)}{T(r, f)} = 0.$$

Then, for \mathbf{a}_j ($j = 1, \dots, n+2$) we obtain that

$$\limsup_{r \rightarrow \infty} N_n(r, \mathbf{a}_j, f) / T(r, f) = 0.$$

This implies that

$$\delta_n(\mathbf{a}_j, f) = 1 \quad (j = 1, \dots, n+2),$$

so that $\#\mathcal{D}^1 \geq n+2$, which contradicts Theorem 2.1(I). (2.7) must hold.

(II) Suppose that there exist $n+2$ vectors $\mathbf{a}_1, \dots, \mathbf{a}_{n+2} \in X^n(2)$ such that

$$\lim_{r \rightarrow \infty} \frac{\sum_{j=1}^{n+2} N_n(r, \mathbf{a}_j, f)}{\log r} < \infty.$$

Then, for any $j = 1, \dots, n+2$ $\lim_{r \rightarrow \infty} N_n(r, \mathbf{a}_j, f) / \log r < \infty$. This means that $\mathbf{a}_j \in \mathcal{P}$ ($j = 1, \dots, n+2$), so that $\#\mathcal{P} \geq n+2$, which contradicts Theorem 2.1(II). (2.8) must hold.

(III) Suppose that there exist $n+2$ vectors $\mathbf{a}_1, \dots, \mathbf{a}_{n+2} \in X^n(2)$ such that (\mathbf{a}_j, f) has no zeros ($j = 1, \dots, n+2$). Then $\mathbf{a}_j \in \mathcal{M}^1$, so that $\#\mathcal{M}^1 \geq n+2$, which contradicts Theorem 2.1(III). (2.10) must hold.

(IV) Suppose that there exist $n+2$ vectors $\mathbf{a}_1, \dots, \mathbf{a}_{n+2} \in X^n(2)$ such that

$$\limsup_{r \rightarrow \infty} \frac{\log(\sum_{j=1}^{n+2} N_n(r, \mathbf{a}_j, f))}{\log r} < \rho_f.$$

then, for any $j = 1, \dots, n+2$

$$\limsup_{r \rightarrow \infty} \frac{\log N_n(r, \mathbf{a}_j, f)}{\log r} < \rho_f.$$

This means that $\mathbf{a}_j \in \mathcal{B}$ ($j = 1, \dots, n+2$), so that $\#\mathcal{B} \geq n+2$, which contradicts Theorem 2.1(IV). (IV) must hold.

(II)(a) and (III)(b) We denote by \mathcal{K} any one of \mathcal{P} and \mathcal{M}^1 . We suppose that f is linearly non-degenerate over \mathcal{M} , where $\mathcal{M} = \mathbf{C}(z)$ when $\mathcal{K} = \mathcal{P}$ and $\mathcal{M} = \mathbf{C}$ when $\mathcal{K} = \mathcal{M}^1$.

As $\mathbf{a}_1, \dots, \mathbf{a}_{n+1} \in \mathcal{K}$, from Lemma 2.3, $\mathbf{a}_1, \dots, \mathbf{a}_{n+1}$ are linearly independent, and so \mathbf{a}_{n+2} can be represented as a linear combination of $\mathbf{a}_1, \dots, \mathbf{a}_{n+1}$ with constant coefficients. We suppose without loss of generality that

$$\mathbf{a}_{n+2} = \sum_{j=1}^k c_j \mathbf{a}_j \quad (c_j \neq 0, 2 \leq k \leq n+1),$$

from which we obtain the relation

$$(2.12) \quad (\mathbf{a}, f) = \sum_{j=1}^k c_j (\mathbf{a}_j, f).$$

We put $F_j = (\mathbf{a}_j, f)$ ($j = 1, \dots, k$) and $F_o = (\mathbf{a}, f)$. Then, from Lemma 2.2(I) F_1, \dots, F_k are linearly independent over \mathcal{M} and from (2.12)

$$(2.13) \quad F_o = \sum_{j=1}^k c_j F_j \quad (c_j \neq 0).$$

Let g be a polynomial such that the functions $F_1/g, \dots, F_k/g$ are entire functions without common zeros and put $F = [F_1/g, \dots, F_k/g]$, where $g = 1$ when $\mathcal{K} = \mathcal{M}^1$.

Then, from Lemma 2.2(I) and (II)(ii), F is transcendental. We note that the functions F_1, \dots, F_k have no zeros when $\mathcal{K} = \mathcal{M}^1$. From Lemma 2.2(III) applying to (2.13) we have the inequality

$$(2.14) \quad \begin{aligned} T(r, F) &\leq \sum_{j=1}^k N_n(r, \mathbf{a}_j, f) + N_n(r, \mathbf{a}_{n+2}, f) + S(r, F) \\ &\leq N_n(r, \mathbf{a}_{n+2}, f) + S(r, F), \end{aligned}$$

since $\mathbf{a}_j \in \mathcal{K}$ ($j = 1, \dots, k$).

(a) When $\mathcal{K} = \mathcal{P}$. If the order of f is finite, then that of F is also finite from Lemma 2.2(II)(i). From Note 2.1 and (2.14) we have that

$$\liminf_{r \rightarrow \infty} \frac{N_n(r, \mathbf{a}_{n+2}, f)}{T(r, F)} \geq 1,$$

and so from Lemma 2.2(I), (II)(ii) and Lemma 2.4 we have that

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{N_n(r, \mathbf{a}_{n+2}, f)}{r} &\geq \liminf_{r \rightarrow \infty} \frac{N_n(r, \mathbf{a}_{n+2}, f)}{T(r, F)} \cdot \liminf_{r \rightarrow \infty} \frac{T(r, F)}{r} \\ &\geq \liminf_{r \rightarrow \infty} \frac{T(r, F_1/F_2)}{r} > 0. \end{aligned}$$

If the order of f is infinite, from (2.14) we have

$$\limsup_{r \rightarrow \infty, r \notin E} N_n(r, \mathbf{a}_{n+2}, f)/T(r, F) \geq 1,$$

and so from Lemma 2.2(I), (II)(ii) and Lemma 2.4 we have that

$$\begin{aligned} \limsup_{r \rightarrow \infty, r \notin E} \frac{N_n(r, \mathbf{a}_{n+2}, f)}{r} &\geq \limsup_{r \rightarrow \infty, r \notin E} \frac{N_n(r, \mathbf{a}_{n+2}, f)}{T(r, F)} \cdot \liminf_{r \rightarrow \infty, r \notin E} \frac{T(r, F)}{r} \\ &\geq \liminf_{r \rightarrow \infty, r \notin E} \frac{T(r, F_1/F_2)}{r} > 0 \end{aligned}$$

We obtain (2.9).

(b) When $\mathcal{K} = \mathcal{M}^1$. Suppose that $N_n(r, \mathbf{a}_{n+2}, f) = O(\log r)$. Then, from (2.14) we have that $T(r, F) = S(r, F)$, which is absurd. As $N_n(r, \mathbf{a}_{n+2}, f)$ is convex with respect to $\log r$, (2.11) must hold. \square

Note 2.2. In (III)(b), we can prove the same conclusion as in (2.9).

Note 2.3. A more delicate result is given in [2, Theorem 5.2].

2-3. Example. Let $f, X^n(2), \mathcal{D}^1, \mathcal{P}, \mathcal{M}^1, \mathcal{B}, n$ etc. be as in Introduction and let e_j ($j = 1, \dots, n + 1$) be the standard basis of \mathbf{C}^{n+1} .

Example 2.1. Let $f = [1, e^z, \dots, e^{(n-1)z}, e^{nz}]$. Then, from [9, (5.3) p. 95],

$$T(r, f) = (n/\pi)r + O(1),$$

so that f is transcendental. Let $X^n(2)$ be any subset of $\mathbf{C}^{n+1} \setminus \{0\}$ containing e_j ($j = 1, \dots, n + 1$), which satisfies (*) in Introduction. From Lemma 2.5 f is linearly non-degenerate over \mathcal{M}_f , over $\mathbf{C}(z)$, over \mathbf{C} and over \mathcal{M}_o . Further (e_j, f) has no zero for $j = 1, \dots, n + 1$ and so

$$\#\mathcal{D}^1 = \#\mathcal{P} = \#\mathcal{M}^1 = \#\mathcal{B} = n + 1.$$

This shows that Theorem 2.1 is sharp.

Further, let $\mathbf{a} = (a_1, \dots, a_{n+1}) \in X^n(2)$. Then, at least two elements of a_1, \dots, a_{n+1} are different from 0. Let $a_{j_1} \neq 0, \dots, a_{j_k} \neq 0$ ($1 \leq j_1 < j_2 < \dots < j_k \leq n+1, 2 \leq k \leq n+1$), $a_j = 0$ ($j \neq j_1, \dots, j_k$) and put

$$F = [e^{j_1 z}, \dots, e^{j_k z}].$$

Then, $T(r, F) = (j_k - j_1)r/\pi + O(1)$ so that F is transcendental and of order 1. From Lemma 2.2(III) and Note 2.1 we have the inequality

$$T(r, F) \leq N_n(r, \mathbf{a}, f) + O(\log r).$$

From this inequality we obtain that

$$\lim_{r \rightarrow \infty} \frac{\log N_n(r, \mathbf{a}, f)}{\log r} = 1, \quad \liminf_{r \rightarrow \infty} \frac{N_n(r, \mathbf{a}, f)}{T(r, f)} \geq \frac{1}{n}, \quad \liminf_{r \rightarrow \infty} \frac{N_n(r, \mathbf{a}, f)}{r} \geq \frac{1}{\pi}.$$

Example 2.2. Let $g(z)$ be a transcendental entire function such that $T(r, g) = O((\log r)^2)$ and let $f = [1, e^z, \dots, e^{(n-1)z}, g(z)]$. Then, it is easy to see from Example 2.1 that there are positive constants $a < b < \infty$ satisfying

$$ar \leq T(r, f) \leq br,$$

so that f is transcendental and of order 1. Let

$$X^n(2) = \{\mathbf{e}_1, \dots, \mathbf{e}_{n+1}\} \cup \{\mathbf{e}_1 + \alpha \mathbf{e}_{n+1} \mid \alpha \in \mathbf{C}, \alpha \neq 0\}.$$

Then, $X^n(2)$ satisfies (*) in Introduction and

$$N_n(r, \mathbf{e}_j, f) = 0 \quad (j = 1, \dots, n), \quad N_n(r, \mathbf{a}, f) = O((\log r)^2),$$

where $\mathbf{a} = \mathbf{e}_1 + \alpha \mathbf{e}_{n+1}$.

1) f is linearly degenerate over $\mathcal{M}_f, \mathcal{M}_o$ and $\#\mathcal{D}^1 = \#\mathcal{B} = \infty$.

2) f is linearly non-degenerate over $\mathbf{C}(z)$, $\#\mathcal{P} = n$ and (2.8) holds, but (2.9) does not hold since this example does not satisfy the condition of Corollary 2.1(II)(a).

Example 2.3. Let $f = [1, e^z, \dots, e^{(n-1)z}, z]$ ($n \geq 2$). Then, f is transcendental and of order 1 as in Example 2.2. Let $X^n(2)$ be the set given in Example 2.2. Then, $X^n(2)$ satisfies (*) in Introduction and

$$N_n(r, \mathbf{e}_j, f) = 0 \quad (j = 1, \dots, n), \quad N_n(r, \mathbf{a}, f) = O(\log r)$$

where $\mathbf{a} = \mathbf{e}_1 + \alpha \mathbf{e}_{n+1}$.

1) f is linearly degenerate over $\mathbf{C}(z)$ and $\#\mathcal{P} = \infty$.

2) f is linearly non-degenerate over \mathbf{C} , $\#\mathcal{M}^1 = n$ and (2.10) holds:

Let $\mathbf{a}_1, \dots, \mathbf{a}_{n+2}$ be any $n+2$ elements in $X^n(2)$. Then,

$$2 \leq \#\left\{ \bigcup_{j=1}^{n+2} \{z \mid (\mathbf{a}_j, f(z)) = 0\} \right\} \leq n+2.$$

The formula (2.11) does not hold as this example does not satisfy the condition of Corollary 2.1(III)(b):

$$\#\left\{\bigcup_{j=1}^{n+1}\{z \mid (e_j, f(z)) = 0\}\right\} = 1 \neq 0; \quad \#\{z \mid (\mathbf{a}, f(z)) = 0\} = 1,$$

where $\mathbf{a} = e_1 + \alpha e_{n+1}$.

3. Relation between \mathcal{D}^1 and \mathcal{B}

Let $f, X^n(2), n$ etc. be as in Section 1 or 2.

3-1. Lemma. First of all we shall give the main tool in this section, the fundamental inequality of H. Cartan ([1]) when $N = n$ (Lemma 2.1) and that in general case by E. I. Nochka (see [3]) and then several lemmas to prove theorems.

LEMMA 3.1 ([1], see [3]). *For any q elements \mathbf{a}_j ($j = 1, \dots, q$) of X ($2N - n + 2 \leq q < \infty$), we have the following inequalities:*

- (I) $(q - 2N + n - 1)T(r, f) \leq \sum_{j=1}^q N_n(r, \mathbf{a}_j, f) + S(r, f)$.
- (II) $\sum_{j=1}^q \delta_n(\mathbf{a}_j, f) \leq 2N - n + 1$.

Note 3.1. When ρ_f is finite, the error term $S(r, f)$ can be replaced by $O(\log r)$ without exceptional intervals (see the proof of (I) and [4, Theorem 2.2(i)]).

DEFINITION 3.1. For $0 \leq \alpha < \rho_f$, we put

$$T_\alpha(r, f) = \int_1^r \frac{T(t, f)}{t^{1+\alpha}} dt; \quad T_\alpha(r, \varphi) = \int_1^r \frac{T(t, \varphi)}{t^{1+\alpha}} dt;$$

and

$$N_{n,\alpha}(r, \mathbf{a}, f) = \int_1^r \frac{N_n(t, \mathbf{0}, F)}{t^{1+\alpha}} dt; \quad \delta_{n,\alpha}(\mathbf{a}, f) = 1 - \limsup_{r \rightarrow \infty} \frac{N_{n,\alpha}(r, \mathbf{a}, f)}{T_\alpha(r, f)},$$

where φ is a meromorphic function in the plane and $F = (\mathbf{a}, f)$ for a vector $\mathbf{a} \in \mathbf{C}^{n+1} \setminus \{\mathbf{0}\}$.

PROPOSITION 3.1 ([8], Proposition 1). *For $0 \leq \alpha < \rho_f$, $T_\alpha(r, f)$ tends to $+\infty$ when $r \rightarrow \infty$ and we have the followings:*

- (1) $\limsup_{r \rightarrow \infty} \frac{\log T_\alpha(r, f)}{\log r} = \rho_f - \alpha$; (2) $\lim_{r \rightarrow \infty} \frac{T_0(r, f)}{(\log r)^2} = +\infty$.
- (3) For $0 \leq \alpha < \beta < \rho_f$,

$$\limsup_{r \rightarrow \infty} T_\alpha(r, \varphi) / T_\alpha(r, f) \geq \limsup_{r \rightarrow \infty} T_\beta(r, \varphi) / T_\beta(r, f).$$

- (4) For any $\mathbf{a} \in \mathbf{C}^{n+1} \setminus \{\mathbf{0}\}$ and $0 \leq \alpha < \beta < \rho_f$, $\delta_{n,\alpha}(\mathbf{a}, f) \leq \delta_{n,\beta}(\mathbf{a}, f)$.

In particular, if $\mathbf{a} \in D^1$ or if

$$\limsup_{r \rightarrow \infty} \frac{\log N_n(r, 0, F)}{\log r} = \alpha < \rho_f,$$

then for $\beta > \alpha$, $\delta_{n,\beta}(\mathbf{a}, f) = 1$.

DEFINITION 3.2.

- 1) $C(f) = \{\varphi \mid \text{meromorphic in } |z| < \infty \text{ and } T(r, \varphi) = o(T(r, f)) \text{ (} r \rightarrow \infty)\}$;
- 2) for $0 \leq \alpha < \rho_f$

$$C_\alpha(f) = \{\varphi \mid \text{meromorphic in } |z| < \infty \text{ and } T_\alpha(r, \varphi) = o(T_\alpha(r, f)) \text{ (} r \rightarrow \infty)\}.$$

PROPOSITION 3.2 ([8], Proposition 2). For $0 \leq \alpha < \beta < \rho_f$, we have the following:

$$C(f) \subset C_\alpha(f) \subset C_\beta(f).$$

Proof. (i) Proof of $C(f) \subset C_\alpha(f)$. Let $\varphi \in C(f)$. Then, for any positive number ε there is a positive number r_0 such that for any $r \geq r_0$ we have the inequality

$$T(r, \varphi) \leq \varepsilon T(r, f),$$

so that we have the inequality

$$\frac{T_\alpha(r, \varphi)}{T_\alpha(r, f)} \leq \frac{T_\alpha(r_0, \varphi) / \{T_\alpha(r, f) - T_\alpha(r_0, f)\} + \varepsilon}{T_\alpha(r_0, f) / \{T_\alpha(r, f) - T_\alpha(r_0, f)\} + 1},$$

from which we obtain

$$\limsup_{r \rightarrow \infty} T_\alpha(r, \varphi) / T_\alpha(r, f) \leq \varepsilon.$$

As ε is arbitrary, we have that

$$\lim_{r \rightarrow \infty} T_\alpha(r, \varphi) / T_\alpha(r, f) = 0.$$

This means that φ belongs to $C_\alpha(f)$.

(ii) Proof of $C_\alpha(f) \subset C_\beta(f)$. Let $\varphi \in C_\alpha(f)$. Then, for any positive number ε there is a positive number r_0 such that for any $r \geq r_0$ we have the inequality

$$(3.1) \quad T_\alpha(r, \varphi) \leq \varepsilon T_\alpha(r, f).$$

On the other hand the integration by parts gives us the following relations for $\gamma = \beta - \alpha > 0$:

$$T_\beta(r, \varphi) = \gamma \int_1^r \frac{T_\alpha(t, \varphi)}{t^{1+\gamma}} dt + \frac{T_\alpha(r, \varphi)}{r^\gamma}$$

and

$$T_\beta(r, f) = \gamma \int_1^r \frac{T_\alpha(t, f)}{t^{1+\gamma}} dt + \frac{T_\alpha(r, f)}{r^\gamma}.$$

From the above two equations with (3.1) we obtain

$$\frac{T_\beta(r, \varphi)}{T_\beta(r, f)} \leq \varepsilon + \gamma \frac{\int_1^{r_0} \frac{T_\alpha(t, \varphi) - \varepsilon T_\alpha(t, f)}{t^{1+\gamma}} dt}{\gamma \int_1^r \frac{T_\alpha(t, f)}{t^{1+\gamma}} dt + \frac{T_\alpha(r, f)}{r^\gamma}}.$$

As the denominator of the right-hand side tends to ∞ for $r \rightarrow \infty$, we obtain that

$$\limsup_{r \rightarrow \infty} T_\beta(r, \varphi)/T_\beta(r, f) \leq \varepsilon,$$

from which we obtain that

$$\lim_{r \rightarrow \infty} T_\beta(r, \varphi)/T_\beta(r, f) = 0,$$

since ε is any positive number. This means that $\varphi \in C_\beta(f)$.

DEFINITION 3.3. We put

$$\mathcal{M}(f) = C(f) \cup \left\{ \bigcup_{0 \leq \alpha < \rho_f} C_\alpha(f) \right\}$$

LEMMA 3.2. *The set $\mathcal{M}(f)$ is a field.*

Proof. Let φ_1, φ_2 be in $\mathcal{M}(f)$. Without loss of generality we suppose that $\varphi_1, \varphi_2 \in C_\beta$ for some positive number $\beta < \rho_f$.

i) We have the relation from the Definition 3.1:

$$T_\beta(r, \varphi_1 \pm \varphi_2) \leq T_\beta(r, \varphi_1) + T_\beta(r, \varphi_2) + O(1) = o(T_\beta(r, f)) \quad (r \rightarrow \infty),$$

which means that $\varphi_1 \pm \varphi_2 \in C_\beta(f) \subset \mathcal{M}(f)$.

ii) From a property of the characteristic function, we obtain the following inequality

$$T_\beta(r, \varphi_1 \cdot \varphi_2^{\pm 1}) \leq T_\beta(r, \varphi_1) + T_\beta(r, \varphi_2) + O(1) = o(T_\beta(r, f)) \quad (r \rightarrow \infty),$$

where $\varphi_2 \neq 0$ when $\pm 1 = -1$.

From i) and ii) we have this lemma.

LEMMA 3.3. (i) ([6], pp. 62–63) *For any meromorphic function h in the plane,*

$$\int_1^r \frac{m(t, h'/h)}{t^{1+\alpha}} dt = O\left(\int_1^r \frac{\log^+ T(t, h)}{t^{1+\alpha}} dt\right) \quad (r \rightarrow \infty, 0 < \alpha).$$

(ii) (see [6], (21), p. 69) *For $0 < \alpha < \rho_f$*

$$S_\alpha(r, f) \equiv \int_1^r \frac{S(t, f)}{t^{1+\alpha}} dt = O\left(\int_1^r \frac{\log^+ T(t, f)}{t^{1+\alpha}} dt\right) = o(T_\alpha(r, f)) \quad (r \rightarrow \infty).$$

Proof (ii). We obtain this relation from the proof of Lemma 2.1(1) for the error term, from the inequality (6') in [1] and (i) of this lemma.

From now on in this section we suppose that the curve $f = [f_1, \dots, f_{n+1}]$ is of order $0 < \rho_f \leq \infty$ and is linearly non-degenerate over $\mathcal{M}(f)$. In this case, Lemma 2.2 holds for $\mathcal{M} = \mathcal{M}(f)$ since we can apply the proof of Lemma 2.2 for $\mathcal{M} = \mathcal{M}(f)$ so that we apply Lemma 2.2 for $\mathcal{M} = \mathcal{M}(f)$ in Lemma 3.4.

LEMMA 3.4. *Suppose that $\{\mathcal{D}^1 \cup \mathcal{B}\} \neq \emptyset$. Then, for $1 \leq k \leq \min\{\#\{\mathcal{D}^1 \cup \mathcal{B}\}, n+1\}$, any k vectors in $\{\mathcal{D}^1 \cup \mathcal{B}\}$ are linearly independent over \mathbf{C} .*

Proof. (i) When $k = 1$. This is trivial since $\mathcal{D}^1 \cup \mathcal{B} \subset X^n(2) \subset \mathbf{C}^{n+1} \setminus \{\mathbf{0}\}$.

(ii) When $k = 2$. This is also trivial since $\mathcal{D}^1 \cup \mathcal{B}$ satisfies (*) in Introduction.

(iii) For $2 \leq k \leq \min\{\#\{\mathcal{D}^1 \cup \mathcal{B}\} - 1, n\}$, suppose that any k vectors in $\mathcal{D}^1 \cup \mathcal{B}$ are linearly independent over \mathbf{C} . We prove that any $k+1$ vectors in $\mathcal{D}^1 \cup \mathcal{B}$ are linearly independent over \mathbf{C} .

Suppose to the contrary that there are $k+1$ vectors in $\mathcal{D}^1 \cup \mathcal{B}$, linearly dependent over \mathbf{C} . Let $\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{a}$ be in $\mathcal{D}^1 \cup \mathcal{B}$ and be linearly dependent over \mathbf{C} . Then, from the hypothesis of induction we can write

$$\mathbf{a} = \sum_{j=1}^k c_j \mathbf{a}_j \quad (c_j \neq 0, 1 \leq j \leq k),$$

from which we obtain the relation

$$(3.2) \quad (\mathbf{a}, f) = \sum_{j=1}^k c_j (\mathbf{a}_j, f).$$

We put

$$F_j = (\mathbf{a}_j, f) \quad (j = 1, \dots, k) \quad \text{and} \quad F_o = (\mathbf{a}, f).$$

Then, we obtain that from Lemma 2.2(I) F_1, \dots, F_k are linearly independent over $\mathcal{M}(f)$ and that from (3.2)

$$(3.3) \quad F_o = \sum_{j=1}^k c_j F_j \quad (c_j \neq 0).$$

Let g be an entire function such that the functions $F_1/g, \dots, F_k/g$ are entire functions without common zeros. We put $F = [F_1/g, \dots, F_k/g]$. Then, F is transcendental from Lemma 2.2(I) and (II)(i),(ii). From (3.3) and Lemma 2.2(III) we obtain the inequality

$$(3.4) \quad T(r, F) \leq \sum_{j=1}^k N_n(r, \mathbf{a}_j, f) + N_n(r, \mathbf{a}, f) + S(r, F).$$

(a) When $\mathcal{D}^1 \cup \mathcal{B} = \mathcal{D}^1$. From Lemma 2.2(I), (II)(ii) and (3.4) we obtain the inequality

$$0 < \limsup_{r \rightarrow \infty} \frac{T_\alpha(r, F_2/F_1)}{T_\alpha(r, f)} \leq \sum_{j=1}^k \limsup_{r \rightarrow \infty} \frac{N_{n,\alpha}(r, \mathbf{a}_j, f)}{T_\alpha(r, f)} \\ + \limsup_{r \rightarrow \infty} \frac{N_{n,\alpha}(r, \mathbf{a}, f)}{T_\alpha(r, f)} + \limsup_{r \rightarrow \infty} \frac{S_\alpha(r, f)}{T_\alpha(r, f)} = 0$$

from Lemma 3.3(ii) and Proposition 3.1(4) since $\mathbf{a}_j, \mathbf{a} \in \mathcal{D}^1$. This is a contradiction. This means that any $k+1$ vectors in \mathcal{D}^1 must be linearly independent over \mathbf{C} .

(b) When $\mathcal{D}^1 \cup \mathcal{B} = \mathcal{B}$. Suppose that

$$\limsup_{r \rightarrow \infty} \frac{\log N_n(r, \mathbf{a}, f)}{\log r} = \alpha, \quad \text{and} \quad \limsup_{r \rightarrow \infty} \frac{\log N_n(r, \mathbf{a}_j, f)}{\log r} = \alpha_j \quad (j = 1, \dots, k).$$

Then, there is a number $\rho_f > \beta > \max\{\alpha, \alpha_j \ (j = 1, \dots, k)\}$ as $\mathbf{a}, \mathbf{a}_j \ (j = 1, \dots, k)$ are in \mathcal{B} .

From Lemma 2.2(I), (II)(ii) and (3.4) we obtain the inequality

$$0 < \limsup_{r \rightarrow \infty} \frac{T_\beta(r, F_2/F_1)}{T_\beta(r, f)} \leq \sum_{j=1}^k \limsup_{r \rightarrow \infty} \frac{N_{n,\beta}(r, \mathbf{a}_j, f)}{T_\beta(r, f)} \\ + \limsup_{r \rightarrow \infty} \frac{N_{n,\beta}(r, \mathbf{a}, f)}{T_\beta(r, f)} + \limsup_{r \rightarrow \infty} \frac{S_\beta(r, f)}{T_\beta(r, f)} = 0$$

from Lemma 3.3(ii) and Proposition 3.1(4) since \mathbf{a}_j, \mathbf{a} are of order less than β . This is a contradiction. This means that any $k+1$ vectors in \mathcal{B} must be linearly independent over \mathbf{C} .

(c) When $\mathcal{D}^1 \neq \phi$ and $\mathcal{B} \neq \phi$. We may suppose without loss of generality that $\mathcal{D}^1 = \{\mathbf{a}, \mathbf{a}_1, \dots, \mathbf{a}_{k_1}\}$ and $\mathcal{B} = \{\mathbf{a}_{k_1+1}, \dots, \mathbf{a}_k\}$, where $0 \leq k_1 \leq k-1$.

Suppose that

$$\limsup_{r \rightarrow \infty} \frac{\log N_n(r, \mathbf{a}_j, f)}{\log r} = \alpha_j \quad (j = k_1 + 1, \dots, k).$$

Let $\alpha_j \ (j = k_1 + 1, \dots, k) < \beta < \rho_f$. From Lemma 2.4(I), (II)(ii) and (3.4) we obtain the inequality

$$0 < \limsup_{r \rightarrow \infty} \frac{T_\beta(r, F_2/F_1)}{T_\beta(r, f)} \leq \sum_{j=1}^k \limsup_{r \rightarrow \infty} \frac{N_{n,\beta}(r, \mathbf{a}_j, f)}{T_\beta(r, f)} \\ + \limsup_{r \rightarrow \infty} \frac{N_{n,\beta}(r, \mathbf{a}, f)}{T_\beta(r, f)} + \limsup_{r \rightarrow \infty} \frac{S_\beta(r, f)}{T_\beta(r, f)} = 0$$

from Lemma 3.3(ii) and Proposition 3.1(4) since $\mathbf{a}, \mathbf{a}_j \ (1 \leq j \leq k_1)$ are in \mathcal{D}^1 and $\mathbf{a}_j \ (k_1 + 1 \leq j \leq k)$ are of order less than β . This is a contradiction.

This means that any $k+1$ vectors in $\mathcal{D}^1 \cup \mathcal{B}$ must be linearly independent over \mathcal{C} .

3-3. Estimates of $\#\{D^1 \cup B\}$ and of $\#\{\mathcal{D}^1 \cup \mathcal{B}\}$. We shall give two theorems on D^1 , \mathcal{D}^1 , B and \mathcal{B}

THEOREM 3.1. *Suppose that $\rho_f > 0$. Then, $\#\{D^1 \cup B\} \leq 2N - n + 1$.*

Proof. (a) When $\{D^1 \cup B\} = D^1$. It is trivial from Lemma 3.1(II) that $\#D^1 \leq 2N - n + 1$ in this case.

(b) When $\{D^1 \cup B\} = B$. Let $B = \{\mathbf{a}_j \mid j = 1, \dots, q\} (\subset X)$,

$$\limsup_{r \rightarrow \infty} N_n(r, \mathbf{a}_j, f) / \log r = \alpha_j \quad (j = 1, \dots, q)$$

and $\alpha_j < \beta < \rho_f$ ($j = 1, \dots, q$). Suppose that $q > 2N - n + 1$. Then from Lemma 3.1(I) we obtain the inequality

$$(q - 2N + n - 1)T_\beta(r, f) \leq \sum_{j=1}^q N_{n,\beta}(r, \mathbf{a}_j, f) + S_\beta(r, f),$$

from which we obtain that

$$0 < q - 2N + n - 1 \leq \sum_{j=1}^q \limsup_{r \rightarrow \infty} \frac{N_{n,\beta}(r, \mathbf{a}_j, f)}{T_\beta(r, f)} + \limsup_{r \rightarrow \infty} \frac{S_\beta(r, f)}{T_\beta(r, f)} = 0$$

from Proposition 3.1(4) and Lemma 3.3(ii). This is a contradiction. This means that

$$\#B \leq 2N - n + 1.$$

(c) When $D^1 = \{\mathbf{a}_j \mid j = 1, \dots, k\}$ and $B = \{\mathbf{a}_j \mid j = k+1, \dots, q\}$ ($1 \leq k \leq q-1$). Let

$$\limsup_{r \rightarrow \infty} \log N_n(r, \mathbf{a}_j, f) / \log r = \alpha_j \quad (j = k+1, \dots, q)$$

and $\alpha_j < \beta < \rho_f$ ($j = k+1, \dots, q$).

Suppose that $q > 2N - n + 1$. Then from Lemma 3.1(I) we obtain the inequality

$$(q - 2N + n - 1)T_\beta(r, f) \leq \sum_{j=1}^q N_{n,\beta}(r, \mathbf{a}_j, f) + S_\beta(r, f),$$

from which we obtain that

$$0 < q - 2N + n - 1 \leq \sum_{j=1}^q \limsup_{r \rightarrow \infty} \frac{N_{n,\beta}(r, \mathbf{a}_j, f)}{T_\beta(r, f)} + \limsup_{r \rightarrow \infty} \frac{S_\beta(r, f)}{T_\beta(r, f)} = 0$$

from Proposition 3.1(4) and Lemma 3.3(ii). This is a contradiction. This means that

$$\#\{D^1 \cup B\} \leq 2N - n + 1.$$

THEOREM 3.2. *Suppose that $\rho_f > 0$ and f is linearly non-degenerate over $\mathcal{M}(f)$. Then, $\#\{\mathcal{D}^1 \cup \mathcal{B}\} \leq n + 1$.*

Proof. Suppose that $\#\{\mathcal{D}^1 \cup \mathcal{B}\} \geq n + 2$. Let $\mathbf{a}_1, \dots, \mathbf{a}_{n+2}$ be in $\mathcal{D}^1 \cup \mathcal{B}$. Then, from Lemma 3.4 any $n + 1$ vectors in $\{\mathbf{a}_1, \dots, \mathbf{a}_{n+2}\}$ are linearly independent over \mathcal{C} . This means that $\mathbf{a}_1, \dots, \mathbf{a}_{n+2}$ are in general position in \mathcal{C}^{n+1} . We can apply Lemma 2.1(I) for $N = n$ and $q = n + 2$, and from Proposition 3.1(4) we obtain a contradiction as follows:

$$1 \leq \sum_{j=1}^{n+2} \limsup_{r \rightarrow \infty} \frac{N_{n,\beta}(r, \mathbf{a}_j, f)}{T_\beta(r, f)} + \limsup_{r \rightarrow \infty} \frac{S_\beta(r, f)}{T_\beta(r, f)} = 0$$

for some positive number $\beta < \rho_f$. This is a contradiction. We obtain our theorem.

COROLLARY 3.1. *Under the same condition as in Theorem 3.2, for any vectors $\{\mathbf{a}_1, \dots, \mathbf{a}_{n+2}\} \subset X^n(2)$,*

$$(3.5) \quad \limsup_{r \rightarrow \infty} \frac{\sum_{j=1}^{n+2} N_n(r, \mathbf{a}_j, f)}{T(r, f)} > 0$$

and at least one of the following $n + 2$ numbers

$$(3.6) \quad \limsup_{r \rightarrow \infty} \log N_n(r, \mathbf{a}_j, f) / \log r$$

is equal to ρ_f .

Proof. Suppose that (3.5) does not hold. Then, for $j = 1, \dots, n + 2$

$$\limsup_{r \rightarrow \infty} N_n(r, \mathbf{a}_j, f) / T(r, f) = 0.$$

This means that $\{\mathbf{a}_1, \dots, \mathbf{a}_{n+2}\} \subset \mathcal{D}^1$. Namely, $\#\mathcal{D}^1 \geq n + 2$, which contradicts Theorem 3.2.

Next, suppose that none of the $n + 2$ numbers of (3.6) is not equal to ρ_f . Then, the $n + 2$ numbers

$$\limsup_{r \rightarrow \infty} \log N_n(r, \mathbf{a}_j, f) / \log r \quad (j = 1, \dots, n + 2)$$

are smaller than ρ_f , which means that $\#\mathcal{B} \geq n + 2$. This contradicts Theorem 3.2. We obtain our corollary.

3-4. Example. The purpose of this section is to give a holomorphic curve. It is a holomorphic curve f_o of positive order and linearly non-degenerate over

$\mathcal{M}(f_o)$ with a vector \mathbf{a} such that $\delta_n(\mathbf{a}, f_o) = 1$ but \mathbf{a} is not Borel exceptional for f_o .

LEMMA 3.5. *Let $h = g_1/g_2$ be a non-constant meromorphic function, where g_1 and g_2 are entire functions without common zero and we put*

$$f = [g_1^n, g_1^{n-1}g_2, \dots, g_1g_2^{n-1}, g_2^n].$$

Then, $T(r, f) = nT(r, h) + O(1)$.

Proof. From the definition of $T(r, f)$ by H. Cartan ([1]),

$$\begin{aligned} (3.7) \quad T(r, f) &= \frac{1}{2\pi} \int_0^{2\pi} \log \max\{|g_1|^n, |g_1|^{n-1}|g_2|, \dots, |g_1||g_2|^{n-1}, |g_2|^n\} d\theta + O(1) \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \log \max\{|g_1|^n, |g_2|^n\} d\theta + O(1) \\ &\leq \frac{n}{2\pi} \int_0^{2\pi} \log \max\{|g_1|, |g_2|\} d\theta + O(1) = nT(r, h) + O(1). \end{aligned}$$

On the other hand

$$\begin{aligned} (3.8) \quad T(r, f) &\geq \frac{1}{2\pi} \int_0^{2\pi} \log \max\{|g_1|^n, |g_2|^n\} d\theta + O(1) \\ &= \frac{n}{2\pi} \int_0^{2\pi} \log \max\{|g_1|, |g_2|\} d\theta + O(1) = nT(r, h) + O(1). \end{aligned}$$

From (3.7) and (3.8) we have our lemma.

LEMMA 3.6. *Let h, g_1, \dots, g_{n+1} be meromorphic functions in $|z| < \infty$. Then,*

$$(3.9) \quad T\left(r, \sum_{j=1}^{n+1} g_j h^{n+1-j}\right) \leq nT(r, h) + \sum_{j=1}^{n+1} T(r, g_j) + O(1).$$

Proof. (a) When $n = 1$. From fundamental properties of the characteristic function for meromorphic functions we have the inequality

$$T(r, g_1h + g_2) \leq T(r, g_1h) + T(r, g_2) + O(1) \leq T(r, h) + T(r, g_1) + T(r, g_2) + O(1).$$

(b) We suppose that (3.9) holds when $n = k$, where k is a positive integer. We shall prove that (3.9) holds when $n = k + 1$. From the hypothesis of induction we have the inequality:

$$\begin{aligned} T\left(r, \sum_{j=1}^{k+2} g_j h^{k+2-j}\right) &= T\left(r, \sum_{j=1}^{k+1} g_j h^{k+2-j} + g_{k+2}\right) \\ &\leq T\left(r, h \sum_{j=1}^{k+1} g_j h^{k+2-j}\right) + T(r, g_{k+2}) + O(1) \end{aligned}$$

$$\begin{aligned}
&\leq T(r, h) + T\left(r, \sum_{j=1}^{k+1} g_j h^{k+2-j}\right) + T(r, g_{k+2}) + O(1) \\
&\leq T(r, h) + kT(r, h) + \sum_{j=1}^{k+1} T(r, g_j) + T(r, g_{k+2}) + O(1) \\
&= (k+1)T(r, h) + \sum_{j=1}^{k+2} T(r, g_j) + O(1).
\end{aligned}$$

From (a) and (b) we see that this lemma holds for any positive integer n .

LEMMA 3.7. *Let $h_1(z) = e^z$, $h_2(z) = \prod_{n=2}^{\infty} (1 + z/n(\log n)^2)$ and we put $h(z) = h_1(z)/h_2(z)$. Then, $h(z)$ is meromorphic in $|z| < \infty$, of order 1, $N(r, h)$ is of order 1 and $\delta(\infty, h) = 1$.*

Proof. It is known that

$$(3.10) \quad T(r, h_1) = r/\pi$$

(see [4, p. 7]) and that

$$(3.11) \quad n(r, 1/h_2) \sim r/(\log r)^2.$$

(see [4, p. 29]). We estimate $\log M(r, h_2)$. From (3.11) let $r_o \geq 1$ be a positive number such that for any $r \geq r_o$

$$(3.12) \quad \frac{9}{10} \frac{r}{(\log r)^2} \leq n\left(r, \frac{1}{h_2}\right) \leq \frac{10}{9} \frac{r}{(\log r)^2}.$$

Since $\sum_{n=2}^{\infty} 1/(n(\log n)^2) < \infty$, from an inequality in [4, p. 28] and from (3.12) we obtain the inequality for any $r \geq r_o$

$$\begin{aligned}
\log M(r, h_2) &\leq r \int_0^{\infty} \frac{n(t, 1/h_2)}{t(t+r)} dt \\
&= r \int_{r_o}^r \frac{n(t, 1/h_2)}{t(t+r)} dt + r \int_r^{\infty} \frac{n(t, 1/h_2)}{t(t+r)} dt + O(1) \\
&\leq \frac{10}{9} \left\{ r \int_{r_o}^r \frac{dt}{(t+r)(\log t)^2} + r \int_r^{\infty} \frac{dt}{(t+r)(\log t)^2} \right\} + O(1) \\
&\leq \frac{10}{9} \left\{ \frac{r}{2} \int_{r_o}^r \frac{dt}{t(\log t)^2} + r \int_r^{\infty} \frac{dt}{t(\log t)^2} \right\} + O(1) \\
&= \frac{10}{9} r \left\{ \frac{1}{2} \int_{\log r_o}^{\log r} \frac{du}{u^2} + \int_{\log r}^{\infty} \frac{du}{u^2} \right\} + O(1) \\
&= \frac{5r}{9} \left(\frac{1}{\log r} + \frac{1}{\log r_o} \right) + O(1) \quad (r \geq r_o),
\end{aligned}$$

so that we have the inequality

$$(3.13) \quad N(r, 1/h_2) + O(1) \leq T(r, h_2) \leq \log M(r, h_2) \\ \leq \frac{5r}{9} \left(\frac{1}{\log r} + \frac{1}{\log r_o} \right) + O(1) \quad (r \geq r_o)$$

On the other hand, for $r \geq r_o$

$$(3.14) \quad N(r, 1/h_2) = \int_0^r \frac{n(t, 1/h_2)}{t} dt = \int_{r_o}^r \frac{n(t, 1/h_2)}{t} dt + O(1) \\ \geq \frac{9}{10} \int_{r_o}^r \frac{dt}{(\log t)^2} + O(1) \geq \frac{9}{10} \frac{(r - r_o)}{(\log r)^2} + O(1)$$

From (3.10) and (3.13),

$$(3.15) \quad T(r, h) \leq T(r, h_1) + T(r, h_2) + O(1) \leq \frac{r}{\pi} + \frac{5r}{9} \left(\frac{1}{\log r} + \frac{1}{\log r_o} \right) + O(1)$$

$$(3.16) \quad T(r, h) \geq T(r, h_1) - T(r, h_2) + O(1) \geq \frac{r}{\pi} - \frac{5r}{9} \left(\frac{1}{\log r} + \frac{1}{\log r_o} \right) + O(1).$$

From (3.13), (3.14), (3.15) and (3.16) we obtain that h is of order 1, $N(r, h)$ is also of order 1 and $\lim_{r \rightarrow \infty} N(r, h)/T(r, h) = 0$, so that $\delta(\infty, h) = 1$, but ∞ is not Borel exceptional for h .

Example 3.1. Let

$$f_o = [h_1^n, h_1^{n-1}h_2, \dots, h_1h_2^{n-1}, h_2^n] : \mathbf{C} \rightarrow P^n(\mathbf{C}),$$

where h_1, h_2 are those given in Lemma 3.7. Then f_o is a holomorphic curve

- (i) of order 1;
- (ii) linearly non-degenerate over $\mathcal{M}(f_o)$;
- (iii) $\delta_n(\mathbf{e}_j, f_o) = 1$ ($j = 2, \dots, n+1$) and
- (iv) \mathbf{e}_j ($j = 2, \dots, n+1$) are not Borel exceptional for f_o .

Proof. (i) From Lemma 3.5, $T(r, f_o) = nT(r, h_1/h_2) + O(1)$ and from Lemma 3.7 h_1/h_2 is of order 1, so is f_o .

(ii) Suppose that there are $g_1, \dots, g_{n+1} \in \mathcal{M}(f_o)$ satisfying

$$(3.17) \quad \sum_{j=1}^{n+1} g_j h_1^{n+1-j} h_2^{j-1} = h_2^n \left\{ \sum_{j=1}^{n+1} g_j h^{n+1-j} \right\} = 0,$$

where $h = h_1/h_2$.

(a) When $n = 1$. (3.17) reduces to

$$(3.18) \quad g_1 h + g_2 = 0.$$

Suppose that $g_1 \neq 0$. Then, from (3.18) $h = -g_2/g_1$ so that we obtain the inequality

$$T(r, h) \leq T(r, g_1) + T(r, g_2) + O(1).$$

The left-hand side is of order 1, but the right-hand side is of order less than 1 since $g_1, g_2 \in \mathcal{M}(f_o)$, which is absurd. This means that g_1 must be equal to 0, so that $g_2 = 0$ from (3.18).

(b) When $n = k$, we suppose that $g_1 = \dots = g_{k+1} = 0$ in (3.17). We shall show that when $n = k + 1$, $g_1 = \dots = g_{k+2} = 0$ in (3.17).

Suppose that $g_1 \neq 0$ in (3.17) for $n = k + 1$. Then, from (3.17) we obtain that

$$h^{k+1} = -\frac{1}{g_1} \left\{ \sum_{j=2}^{k+2} g_j h^{k+2-j} \right\}.$$

Applying Lemma 3.6 to this equality we obtain that

$$T(r, h) \leq \sum_{j=1}^{k+2} T(r, g_j) + O(1),$$

which is absurd since left-hand side is of order 1 but the right-hand side is of order less than 1 as $g_1, \dots, g_{k+2} \in \mathcal{M}(f_o)$. This means that g_1 must be equal to 0. From (3.17) for $n = k + 1$, we obtain that

$$h \sum_{j=1}^{k+1} g_{j+1} h^{k+1-j} = 0.$$

From the hypothesis of induction, $g_2 = \dots = g_{k+2} = 0$.

From (a) and (b) we obtain (ii).

(iii) Since

$$N(r, e_j, f_o) = (j-1)N(r, 1/h_2) = (j-1)N(r, h) \quad (j = 2, \dots, n+1)$$

and from Lemma 3.5, $T(r, f_o) = nT(r, h) + O(1)$, we obtain from Lemma 3.7 for $j = 2, \dots, n+1$

$$\delta_n(e_j, f_o) = 1 - \limsup_{r \rightarrow \infty} \frac{(j-1)N(r, 1/h_2)}{T(r, f_o)} = 1 - \frac{j-1}{n} \limsup_{r \rightarrow \infty} \frac{N(r, h)}{T(r, h)} = 1.$$

(iv) As in (iii) for $j = 2, \dots, n+1$,

$$N(r, e_j, f_o) = (j-1)N(r, h)$$

so that from Lemma 3.7, $N(r, e_j, f)$ is of order 1, namely, e_j ($j = 2, \dots, n+1$) are not Borel exceptional for f_o .

Note 3.1. G. Valiron ([11, p. 73]) writes that it is evident that a Borel exceptional value is not necessary a deficient value since there are meromorphic functions of irregular growth.

4. Defect relation

Let $f, X, X^n(k)$ ($k = 1, \dots, n+1$), n etc. be as in Section 1, 2 or 3 such that $2N - n + 1 \leq \min\{\#X^n(k), \#X\}$.

4-1. Defect relation. The fundamental tools are Lemma 2.1 and Lemma 3.1. From Lemma 3.1 we obtain the following

Defect relation. Let $D^+ = \{\mathbf{a} \in X \mid \delta_n(\mathbf{a}, f) > 0\}$. Then, D^+ is at most countable and we have the inequality

$$(4.1) \quad \sum_{\mathbf{a} \in D^+} \delta_n(\{\mathbf{a}, f\}) \leq 2N - n + 1.$$

We are interested in a holomorphic curve f for which the defect relation is extremal:

$$(4.2) \quad \sum_{\mathbf{a} \in D^+} \delta_n(\mathbf{a}, f) = 2N - n + 1.$$

As in Introduction, we set $D^1 = \{\mathbf{a} \in X \mid \delta_n(\mathbf{a}, f) = 1\}$. Then, from (4.1), we have that $\#D^1 \leq 2N - n + 1$.

For any finite subset S of X , we denote by $d(S)$ the dimension of the subspace generated by elements of S .

THEOREM 4.A. *Suppose that the relation (4.2) holds.*

(I) ([9, Theorem 3.2]) *If $d(D^1) = n + 1$, then $\#D^1 = 2N - n + 1$.*

(II) ([12, Theorem 6.1]) *If (i) $N > n = 2m$ ($m \in \mathbb{N}$) and (ii) $d(D^1) \leq n$, then*

$$\#D^1 = d(D^1) + N - n.$$

The last purpose of this paper is to give an example of holomorphic curve for which the defect relation is extremal and to give the defect relation of holomorphic curves which are not extremal in several cases applying these results.

4-2. Preliminaries and lemma. For a non-empty finite subset S of X , we denote by $V(S)$ the vector space spanned by elements of S and by $d(S)$ the dimension of $V(S)$. We put

$$\mathcal{O} = \{S \subset X \mid 0 < \#S \leq N + 1\}.$$

DEFINITION 4.1 ([12, Definition 2.1]). (I) We put $\lambda = \min_{S \in \mathcal{O}} \frac{d(S)}{\#S}$.

(II) For $R \subsetneq S$ ($R, S \in \mathcal{O}$), we put

$$A(R; S) = \frac{d(S) - d(R)}{\#S - \#R}.$$

Then, it is known that $0 \leq A(R; S) \leq 1$ ([3, p. 67]).

LEMMA 4.1. *Suppose that $\lambda < (n+1)/(2N-n+1)$.*

(I)(see [12, Proposition 2.3(II)(i)]) *There exist an integer p ($1 \leq p < (n+1)/2$) and a subfamily $\{T_i \mid 1 \leq i \leq p\}$ of \mathcal{O} such that*

- (i) $\phi = T_0 \not\subseteq T_1 \not\subseteq \cdots \not\subseteq T_p$, $d(T_p) < (n+1)/2$.
(ii) $\Lambda(T_0; T_1) < \Lambda(T_1; T_2) < \cdots < \Lambda(T_{p-1}; T_p) < \frac{n+1-d(T_p)}{2N-n+1-\#T_p}$.

(II)(see [12, Definition 3.1], [10, Note 2.2], [9, Theorem 3.1]) *We put*

$$w(\mathbf{a}) = \begin{cases} \Lambda(T_{i-1}; T_i) & \text{if } \mathbf{a} \in T_i \setminus T_{i-1} \quad (i = 1, \dots, p), \\ \frac{n+1-d(T_p)}{2N-n+1-\#T_p} & \text{if } \mathbf{a} \in X \setminus T_p \end{cases}$$

and

$$h = \frac{2N-n+1-\#T_p}{n+1-d(T_p)}.$$

Then, the function $w : X \rightarrow (0, 1]$ and the constant h have the following properties:

- (a) For any $\mathbf{a} \in X$, $0 < hw(\mathbf{a}) \leq 1$ and

$$T_p = \{\mathbf{a} \in X \mid hw(\mathbf{a}) < 1\};$$

- (b-1) For any $Q \subset X$ satisfying (i) $Q \supset T_p$ and (ii) $2N-n+1 \leq \#Q < \infty$,

$$\#Q - (2N-n+1) = h \left(\sum_{\mathbf{a} \in Q} w(\mathbf{a}) - n - 1 \right);$$

- (b-2) $\sum_{\mathbf{a} \in X} (1 - hw(\mathbf{a})) = 2N-n+1 - h(n+1)$;

- (c) $N/n \leq h \leq (2N-n+1)/(n+1)$;

- (d) For any $S \in \mathcal{O}$, $\sum_{\mathbf{a} \in S} w(\mathbf{a}) \leq d(S)$.

Suppose that $\lambda < (n+1)/(2N-n+1)$ and we put

$$\mathcal{O}_p = \{S \in \mathcal{O} \mid T_p \not\subseteq S, d(T_p) < d(S)\}.$$

Then, we have

PROPOSITION 4.1 (see [12, Proposition 2.3(II)(iv)]). *For any $S \in \mathcal{O}_p$,*

$$\frac{n+1-d(T_p)}{2N-n+1-\#T_p} \leq \frac{d(S)-d(T_p)}{\#S-\#T_p}.$$

DEFINITION 4.2 ([12, Definition 3.1]). We say that

- (I) X is of type I if for any $S \in \mathcal{O}_p$

$$h^{-1} = \frac{n+1-d(T_p)}{2N-n+1-\#T_p} < \frac{d(S)-d(T_p)}{\#S-\#T_p}.$$

(II) X is of type II if for some $S \in \mathcal{O}_p$

$$h^{-1} = \frac{n+1-d(T_p)}{2N-n+1-\#T_p} = \frac{d(S)-d(T_p)}{\#S-\#T_p}.$$

We know ([3, p. 68]) that for any $S \in \mathcal{O}$, $\#S \leq d(S) + N - n$.

DEFINITION 4.3 ([12, p. 357]). For $S \in \mathcal{O}$, we say that S is maximal when

$$\#S = d(S) + N - n.$$

PROPOSITION 4.2 ([3, p. 68]). Let $R, S \in \mathcal{O}$ such that $R \subseteq S$. If R is maximal, so is S .

As a note, we estimate $1 - hw(\mathbf{a}) > 0$ ($\mathbf{a} \in T_p$).

Note 4.1. Suppose that $\lambda < (n+1)/(2N-n+1)$ and that T_p is maximal. Then,

$$1 - hw(\mathbf{a}) \geq 2/n(2N-n) \quad (\mathbf{a} \in T_p).$$

Proof. From the definition of w (see Lemma 4.1) there is a subset S , which may be empty, of T_p such that for any $\mathbf{a} \in T_p$

$$(4.3) \quad w(\mathbf{a}) \leq \frac{d(T_p) - d(S)}{\#T_p - \#S} < h^{-1} = \frac{n+1-d(T_p)}{N+1-d(T_p)},$$

since T_p is maximal, so that we have the inequality

$$(d(T_p) - d(S))(N+1-d(T_p)) < (n+1-d(T_p))(\#T_p - \#S).$$

As both terms are integers, we have the inequality

$$(d(T_p) - d(S))(N+1-d(T_p)) \leq (n+1-d(T_p))(\#T_p - \#S) - 1,$$

so that we obtain

$$\frac{d(T_p) - d(S)}{\#T_p - \#S} \cdot \frac{N+1-d(T_p)}{n+1-d(T_p)} \leq 1 - \frac{1}{(n+1-d(T_p))(\#T_p - \#S)} \leq 1 - \frac{2}{n(2N-n)}.$$

We obtain this note from this inequality, (4.3) and the inequality

$$\#T_p - \#S \leq \#T_p = d(T_p) + N - n < (n+1)/2 + N - n = (2N - n + 1)/2. \quad \square$$

LEMMA 4.2 ([12, Corollary 4.1]). For f and X as in Section 1, we have the inequality

$$(I) \sum_{\mathbf{a} \in X} w(\mathbf{a})\delta_n(\mathbf{a}, f) \leq n+1. \quad (II) \sum_{\mathbf{a} \in X} \delta_n(\mathbf{a}, f) \leq (n+1)/\lambda.$$

From the equality ([12, Lemma 5.2]):

$$2N - n + 1 - \sum_{\mathbf{a} \in X} \delta_n(\mathbf{a}, f) = \sum_{\mathbf{a} \in X} (1 - hw(\mathbf{a}))(1 - \delta_n(\mathbf{a}, f)) \\ + h \left(n + 1 - \sum_{\mathbf{a} \in X} w(\mathbf{a})\delta_n(\mathbf{a}, f) \right)$$

and Lemma 4.2(I) we have the following

LEMMA 4.3 ([12, Lemma 6.1]). *Suppose that $N > n$. The truncated defect relation for f is extremal:*

$$\sum_{\mathbf{a} \in X} \delta_n(\mathbf{a}, f) = 2N - n + 1$$

if and only if

- (i) $(1 - hw(\mathbf{a}))(1 - \delta_n(\mathbf{a}, f)) = 0$ ($\mathbf{a} \in X$);
- (ii) $\sum_{\mathbf{a} \in X} w(\mathbf{a})\delta_n(\mathbf{a}, f) = n + 1$.

LEMMA 4.4 ([12, Theorem 5.4]). *Suppose that $\lambda < (n + 1)/(2N - n + 1)$ and that (i) X is of type I and T_p is not maximal or (ii) X is of type II. Then*

$$\sum_{\mathbf{a} \in X} \delta_n(\mathbf{a}, f) \leq 2N - n + 1 - \frac{1}{2n}.$$

LEMMA 4.5 ([12, Corollary 5.1]). *Suppose that $N > n = 2m$ ($m \in \mathbf{N}$). If*

$$\sum_{\mathbf{a} \in X} \delta_n(\mathbf{a}, f) > 2N - n + 1 - \frac{1}{2n},$$

then $\lambda < (n + 1)/(2N - n + 1)$.

4-3. Example. In this section we shall give a transcendental holomorphic curve and a set $X^n(2)$ in N -subgeneral position for which the defect relation is extremal.

Example 4.1. For $n \geq 3$, there are a transcendental holomorphic curve f and a set $X^n(2)$ in N -subgeneral position satisfying

$$\sum_{\mathbf{a} \in X^n(2)} \delta(\mathbf{a}, f) = 2N - n + 1.$$

Proof. Let d be an integer satisfying $2 \leq d \leq (n + 1)/2$ and p_1, \dots, p_{n+1} be $n + 1$ polynomials without common zeros and linearly independent over \mathbf{C} . We put

$$f = [p_1, \dots, p_d, p_{d+1}e^z, \dots, p_{n+1}e^z].$$

Then, f is linearly non-degenerate over \mathbf{C} and $T(r, f) \sim T(r, e^z)$, so that f is transcendental. Let $\mathbf{e}'_1 = (1, 0, \dots, 0), \dots, \mathbf{e}'_d = (0, \dots, 0, 1)$ be the standard basis of \mathbf{C}^d and let \mathbf{a}'_j ($j = 1, \dots, N - n$) be vectors in \mathbf{C}^d such that $\mathbf{e}'_1, \dots, \mathbf{e}'_d, \mathbf{a}'_1, \dots, \mathbf{a}'_{N-n}$ are in general position in \mathbf{C}^d . Next, let

$$\mathbf{e}'_{d+1} = (1, 0, \dots, 0), \dots, \mathbf{e}'_{n+1} = (0, \dots, 0, 1)$$

be the standard basis of \mathbf{C}^{n+1-d} and let \mathbf{b}'_j ($j = 1, \dots, N - n$) be vectors in \mathbf{C}^{n+1-d} such that $\mathbf{e}'_{d+1}, \dots, \mathbf{e}'_{n+1-d}, \mathbf{b}'_1, \dots, \mathbf{b}'_{N-n}$ are in general position in \mathbf{C}^{n+1-d} . We put

$$\mathbf{e}_j = (\mathbf{e}'_j, \mathbf{0}) \quad (1 \leq j \leq d); \quad \mathbf{e}_k = (\mathbf{0}, \mathbf{e}'_k) \quad (d+1 \leq k \leq n+1)$$

so that the vectors $\mathbf{e}_1, \dots, \mathbf{e}_{n+1}$ are the standard basis of \mathbf{C}^{n+1} , and

$$\mathbf{a}_j = (\mathbf{a}'_j, \mathbf{0}) \in \mathbf{C}^{n+1} \quad (1 \leq j \leq N - n); \quad \mathbf{b}_k = (\mathbf{0}, \mathbf{b}'_k) \in \mathbf{C}^{n+1} \quad (1 \leq k \leq N - n).$$

Then, the set $X^n(2) = \{\mathbf{e}_1, \dots, \mathbf{e}_{n+1}, \mathbf{a}_1, \dots, \mathbf{a}_{N-n}, \mathbf{b}_1, \dots, \mathbf{b}_{N-n}\}$ is a subset of $\mathbf{C}^{n+1} \setminus \{\mathbf{0}\}$ in N -subgeneral position satisfying (*) in Introduction and

$$\delta(\mathbf{e}_i, f) = 1 \quad (i = 1, \dots, n+1);$$

$$\delta(\mathbf{a}_j, f) = 1 \quad (j = 1, \dots, N - n); \quad \delta(\mathbf{b}_k, f) = 1 \quad (k = 1, \dots, N - n)$$

because

$$(\mathbf{e}_i, f) \quad (i = 1, \dots, d); \quad (\mathbf{a}_j, f) \quad (j = 1, \dots, N - n)$$

are polynomials and

$$(\mathbf{e}_i, f)/e^z \quad (i = d+1, \dots, n+1); \quad (\mathbf{b}_k, f)/e^z \quad (k = 1, \dots, N - n)$$

are polynomials. We have that

$$\sum_{\mathbf{a} \in X^n(2)} \delta(\mathbf{a}, f) = 2N - n + 1. \quad \square$$

4-4. Holomorphic curves with the non-extremal defect relation. Let $f, X, X^n(k)$ ($k = 1, \dots, n+1$), $N > n$ etc. be as in Introduction, Section 1, Section 2 or Section 3.

THEOREM 4.1. *Suppose that the sets $X^n(k)$ ($k = 2, \dots, n+1$) are in N -subgeneral position. When $n \geq k \geq 2$, we have the defect relation*

$$\sum_{\mathbf{a} \in X^k(k)} \delta_k(\mathbf{a}, f) \leq N + N/k.$$

Proof. From the definition of λ (Definition 4.1(I)), we have that $k/N \leq \lambda$ due to the definition of $X^k(k)$. Then from Lemma 4.2(II) we obtain the inequality

$$\sum_{\mathbf{a} \in X^k(k)} \delta_k(\mathbf{a}, f) \leq N + N/k \leq 2N - k + 1. \quad \square$$

We use \mathcal{M}_f as in Section 2.

LEMMA 4.6. *Suppose that f is linearly non-degenerate over \mathcal{M}_f . Let $\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{a}$ be $k+1$ vectors in D^1 such that $\mathbf{a}_1, \dots, \mathbf{a}_k$ are linearly independent over \mathbb{C} and*

$$(4.4) \quad \mathbf{a} = \sum_{j=1}^k c_j \mathbf{a}_j \quad (c_j \neq 0, 1 \leq j \leq k).$$

Then, $\mathbf{a} = c_1 \mathbf{a}_1$.

Proof. Suppose that $k \geq 2$. From (4.4) we obtain the equality

$$(4.5) \quad (\mathbf{a}, f) = \sum_{j=1}^k c_j (\mathbf{a}_j, f).$$

We put

$$F_j = (\mathbf{a}_j, f) \quad (j = 1, \dots, k); \quad \text{and} \quad F_0 = (\mathbf{a}, f).$$

Then, from Lemma 2.1(I) for $\mathcal{M} = \mathcal{M}_f$, F_1, \dots, F_k are linearly independent over \mathcal{M}_f and from (4.5) we obtain the equality

$$(4.6) \quad F_0 = \sum_{j=1}^k c_j F_j \quad (c_j \neq 0).$$

Let g be an entire function such that the functions $F_1/g, \dots, F_k/g$ are entire functions without common zeros. We put

$$F = [F_1/g, \dots, F_k/g].$$

Then, F is transcendental from Lemma 2.1(I) and Lemma 2.1(II)(ii) for $\mathcal{M} = \mathcal{M}_f$. From (4.6) and Lemma 3.1(I) for $N = n = k-1$ and $q = k+1$ we obtain the following inequality

$$(4.7) \quad \begin{aligned} T(r, F) &\leq \sum_{j=1}^k N_{k-1}(r, \mathbf{a}_j, f/g) + N_{k-1}(r, \mathbf{a}, f/g) + S(r, F) \\ &\leq \sum_{j=1}^k N_n(r, \mathbf{a}_j, f) + N_n(r, \mathbf{a}, f) + S(r, f) \end{aligned}$$

since

$$N_{k-1}(r, \mathbf{a}_j, f/g) \leq N_n(r, \mathbf{a}_j, f) \quad (1 \leq j \leq k); \quad N_{k-1}(r, \mathbf{a}, f/g) \leq N_n(r, \mathbf{a}, f)$$

from the definition of these counting functions and

$$T(r, F) \leq T(r, f) + O(1)$$

from Lemma 2.1(II)(i) for $\mathcal{M} = \mathcal{M}_f$. As

$$T(r, F_1/F_2) \leq T(r, F) + O(1)$$

from Lemma 2.1(II)(ii) for $\mathcal{M} = \mathcal{M}_f$, we have from (4.7) and Lemma 2.2(I) for $\mathcal{M} = \mathcal{M}_f$ that

$$\begin{aligned} 0 < \limsup_{r \rightarrow \infty, r \notin E} \frac{T(r, F_2/F_1)}{T(r, f)} &\leq \sum_{j=1}^k \limsup_{r \rightarrow \infty} \frac{N_n(r, \mathbf{a}_j, f)}{T(r, f)} \\ &+ \limsup_{r \rightarrow \infty} \frac{N_n(r, \mathbf{a}, f)}{T(r, f)} + \limsup_{r \rightarrow \infty, r \notin E} \frac{S(r, f)}{T(r, f)} = 0 \end{aligned}$$

since $\mathbf{a}_j, \mathbf{a} \in D^1$. This is a contradiction. This means that k must be equal to 1. \square

THEOREM 4.2. *Suppose that f is linearly non-degenerate over \mathcal{M}_f . Then, $\#D^1 \leq N + N/n$.*

Proof. We have only to prove this inequality when $\#D^1 \geq N + 1$. As X is in N -subgeneral position, there are $n + 1$ linearly independent vectors $\mathbf{a}_1, \dots, \mathbf{a}_{n+1}$ in D^1 . There is nothing to prove when $D^1 \setminus \{\mathbf{a}_1, \dots, \mathbf{a}_{n+1}\} = \emptyset$. We suppose that $D^1 \setminus \{\mathbf{a}_1, \dots, \mathbf{a}_{n+1}\} \neq \emptyset$. For any $\mathbf{a} \in D^1 \setminus \{\mathbf{a}_1, \dots, \mathbf{a}_{n+1}\}$, there are constants c_j ($1 \leq j \leq n + 1$) at least one of which is not zero such that

$$\mathbf{a} = \sum_{j=1}^{n+1} c_j \mathbf{a}_j \quad (c_j \in \mathbf{C}),$$

so that from Lemma 4.6 there is only one $c_{j_1} \neq 0$ such that $\mathbf{a} = c_{j_1} \mathbf{a}_{j_1}$

Let

$$\#(D^1 \setminus \{\mathbf{a}_1, \dots, \mathbf{a}_{n+1}\}) = x.$$

As X is in N -subgeneral position, we have the following inequality:

$$\{(n+1)n + nx\}/(n+1) \leq N,$$

so that $x \leq N + N/n - (n+1)$ and we have our inequality.

THEOREM 4.3. *Suppose that f is linearly non-degenerate over \mathcal{M}_f , and that $N > n \geq 3$. If $d(D^1) = n + 1$, then*

$$\sum_{\mathbf{a} \in D^+} \delta_n(\mathbf{a}, f) < 2N - n + 1.$$

Proof. Suppose to the contrary that for an f

$$\sum_{\mathbf{a} \in D^+} \delta_n(\mathbf{a}, f) = 2N - n + 1.$$

From Theorem 4.A(I), we have that

$$D^+ = D^1 \quad \text{and} \quad \#D^1 = 2N - n + 1.$$

On the other hand from Theorem 4.2

$$\#D^1 \leq N + N/n < 2N - n + 1$$

as $n \geq 3$. This is a contradiction. We have our theorem. □

THEOREM 4.4. *Suppose that f is linearly non-degenerate over \mathcal{M}_f , that $N > n = 2m$ ($m \in \mathbf{N}$) and that $X^n(2)$ is in N -subgeneral position. Then,*

$$\sum_{\mathbf{a} \in X^n(2)} \delta_n(\mathbf{a}, f) < 2N - n + 1.$$

Proof. (i) When $d(\mathcal{D}^1) = n + 1$. This is a special case of Theorem 4.3.
 (ii) When $d(\mathcal{D}^1) \leq n$. Suppose that

$$(4.8) \quad \sum_{\mathbf{a} \in X^n(2)} \delta_n(\mathbf{a}, f) = 2N - n + 1.$$

Then, from Theorem 4.A(II), we have that

$$(4.9) \quad \#\mathcal{D}^1 = d(\mathcal{D}^1) + N - n.$$

From (4.9) we have that $d(\mathcal{D}^1) \geq 2$ since $X^n(2)$ is in N -subgeneral position satisfying (*) in Introduction. Let $d(\mathcal{D}^1) = k (\geq 2)$ and let $\mathbf{b}_1, \dots, \mathbf{b}_k$ be k linearly independent vectors in \mathcal{D}^1 . Then, from (4.9) $\mathcal{D}^1 \setminus \{\mathbf{b}_1, \dots, \mathbf{b}_k\} \neq \emptyset$. For a vector $\mathbf{b} \in \mathcal{D}^1 \setminus \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$, there exist constants $c_j \neq 0 \in \mathbf{C}$ ($j = 1, \dots, k$) such that $\mathbf{b} = \sum_{j=1}^k c_j \mathbf{b}_j$.

Then from Lemma 4.6, k must be equal to 1. This is a contradiction. This implies that (4.8) does not hold. We have our theorem. □

4-5. The μ_n -defect relation. Let $f, X, X^n(k)$ ($k = 1, \dots, n + 1$), n , etc. be as in the previous sections. Let \mathbf{a} be any vector in $\mathbf{C}^{n+1} \setminus \{\mathbf{0}\}$. We say that

“ \mathbf{a} has multiplicity μ if (\mathbf{a}, f) has at least one zero and the zeros of the function $(\mathbf{a}, f(z))$ have multiplicity at least μ , while at least one zero has multiplicity μ .”

When (\mathbf{a}, f) has no zero, we set $\mu = \infty$.

DEFINITION 4.4 ([11, Definition 1.1]). For $\mathbf{a} \in \mathbf{C}^{n+1} \setminus \{\mathbf{0}\}$ with multiplicity μ we put

$$\mu_n(\mathbf{a}, f) = \left(1 - \frac{n}{\mu}\right)^+ = 1 - \frac{n}{\max(\mu, n)},$$

where $a^+ = \max(a, 0)$ for any real number a .

We call $\mu_n(\mathbf{a}, f)$ the μ_n -defect of \mathbf{a} with respect to f . It is known that

$$(4.10) \quad 0 \leq \mu_n(\mathbf{a}, f) \leq \delta_n(\mathbf{a}, f) \leq 1$$

(see for example [11, Corollary 2.2]).

As a corollary of the Defect relation (4.1), we have the following defect relation for $\mu_n(\mathbf{a}, f)$:

PROPOSITION 4.3 (see [5, Corollary (3.B.46)]). *For any $\mathbf{a}_1, \dots, \mathbf{a}_q \in X$ ($2N - n + 1 \leq q < \infty$), we have the following inequality:*

$$\sum_{j=1}^q \mu_n(\mathbf{a}_j, f) \leq 2N - n + 1.$$

We call this inequality the μ_n -defect relation. Let

$$M^+ = \{\mathbf{a} \in X \mid \mu_n(\mathbf{a}, f) > 0\} \quad \text{and} \quad M^1 = \{\mathbf{a} \in M^+ \mid \mu_n(\mathbf{a}, f) = 1\}.$$

Then, $M^+ \subset D^+$ by (4.10) and we have the defect relation:

$$(4.11) \quad \sum_{\mathbf{a} \in X} \mu_n(\mathbf{a}, f) \leq 2N - n + 1$$

from Proposition 4.3. If the equality holds in (4.11):

$$(4.12) \quad \sum_{\mathbf{a} \in X} \mu_n(\mathbf{a}, f) = 2N - n + 1,$$

then, by (4.1), (4.10) and (4.11) we have

$$(4.13) \quad \mu_n(\mathbf{a}, f) = \delta_n(\mathbf{a}, f) \quad (\mathbf{a} \in M^+)$$

and so $M^+ = D^+$.

Further we put (see Introduction)

$$\mathcal{M}^+ = \{\mathbf{a} \in X^n(2) \mid \mu_n(\mathbf{a}, f) > 0\} \quad \text{and} \quad \mathcal{M}^1 = \{\mathbf{a} \in \mathcal{M}^+ \mid \mu_n(\mathbf{a}, f) = 1\}.$$

The main purpose of this section is to estimate

$$\sum_{\mathbf{a} \in X^n(2)} \mu_n(\mathbf{a}, f).$$

PROPOSITION 4.4. (I)[[9, Theorem 4.2 and Proposition 4.2]] (i) *If $d(M^1) = n + 1$, then, $M^+ = M^1$.* (ii) $\#M^1 \leq N + N/n$.

(II)[[11, Note 3.1)] $\#\mathcal{M}^+ + n\#M^1 \leq (n + 1)(2N - n + 1)$. *In particular, $\#\mathcal{M}^+$ is finite.*

LEMMA 4.7 ([10, in the proof of Theorem 3.2]). *Suppose that $N > n \geq 1$ and that (4.12) holds. If $d(M^1) \geq 1$, then*

$$d(M^1) = \sum_{\mathbf{a} \in M^1} w(\mathbf{a}).$$

LEMMA 4.8. *Suppose that (i) $N > n \geq 2$ and that (ii) (4.12) holds. If $d(M^1) \geq 1$, then*

$$h = (2N - n + 1 - \#M^1)/(n + 1 - d(M^1)).$$

Proof. From Lemma 4.3 and (4.13) we have that

$$(4.14) \quad hw(\mathbf{a}) = 1 \quad (\mathbf{a} \in M^+ - M^1)$$

and

$$(4.15) \quad \sum_{\mathbf{a} \in M^+} w(\mathbf{a})\mu_n(\mathbf{a}, f) = n + 1.$$

From (4.15) and Lemma 4.7,

$$\begin{aligned} n + 1 &= \sum_{\mathbf{a} \in M^+} w(\mathbf{a})\mu_n(\mathbf{a}, f) = \sum_{\mathbf{a} \in M^1} w(\mathbf{a}) + \sum_{\mathbf{a} \in M^+ - M^1} w(\mathbf{a})\mu_n(\mathbf{a}, f) \\ &= d(M^1) + \sum_{\mathbf{a} \in M^+ - M^1} w(\mathbf{a})\mu_n(\mathbf{a}, f). \end{aligned}$$

From (4.14)

$$\begin{aligned} h(n + 1 - d(M^1)) &= \sum_{\mathbf{a} \in M^+ - M^1} hw(\mathbf{a})\mu_n(\mathbf{a}, f) \\ &= \sum_{\mathbf{a} \in M^+ - M^1} \mu_n(\mathbf{a}, f) = 2N - n + 1 - \#M^1. \end{aligned}$$

We have our lemma since $d(M^1) \leq n$ from Proposition 4.4(I),(II) and (4.12). \square

LEMMA 4.9. *Suppose that (i) $N > n = 2m$ ($m \in \mathbf{N}$) and that (ii) (4.12) holds. Then, $\lambda < (n + 1)/(2N - n + 1)$.*

Proof. From (4.13) and Lemma 4.5 we have $\lambda < (n + 1)/(2N - n + 1)$. \square

LEMMA 4.10. *Suppose that (i) $N > n \geq 2$ and that (ii) (4.12) holds. If $\lambda < (n + 1)/(2N - n + 1)$, then X is of type I, T_p is maximal and $M^1 = T_p$ so that*

$$\#M^1 = d(M^1) + N - n, \quad d(M^1) < (n + 1)/2.$$

Proof. First of all we note that $M^1 \in \mathcal{O}$ since $d(M^1) \leq n$ from Proposition 4.4(I), (4.12), (4.13) and Theorem 4.A(I). By Lemma 4.3 and (4.13) we have

$$(4.16) \quad T_p \subset M^1.$$

From (4.12), (4.13) and Lemma 4.4, we have that X is of type I, T_p is maximal and so M^1 is also maximal from (4.16) and Proposition 4.2:

$$(4.17) \quad \#M^1 = d(M^1) + N - n$$

Further, as $d(M^1) \geq 1$ from (4.17), by Lemma 4.8 and Lemma 4.1(II) we have

$$h = \frac{2N - n + 1 - \#T_p}{n + 1 - d(T_p)} = \frac{2N - n + 1 - \#M^1}{n + 1 - d(M^1)}.$$

If $T_p \subsetneq M^1$, then $d(T_p) < d(M^1)$ and

$$\frac{2N - n + 1 - \#T_p}{n + 1 - d(T_p)} = \frac{\#M^1 - \#T_p}{d(M^1) - d(T_p)}.$$

As $M^1 \in \mathcal{O}$, this means that X is of type II from Definition 4.2, which is a contradiction. We have that $T_p = M^1$. From Lemma 4.1(I) we obtain that $d(M^1) = d(T_p) < (n + 1)/2$. \square

THEOREM 4.5. *Suppose that $d(\mathcal{M}^1) \geq 1$, $N > n \geq 2$ and that $X^n(2)$ is in N -subgeneral position. Then,*

$$\sum_{\mathbf{a} \in X^n(2)} \mu_n(\mathbf{a}, f) < 2N - n + 1.$$

Proof. Suppose to the contrary that

$$\sum_{\mathbf{a} \in X^n(2)} \mu_n(\mathbf{a}, f) = 2N - n + 1.$$

Then, from Lemma 4.8, we have that

$$h = (2N - n + 1 - \#\mathcal{M}^1)/(n + 1 - d(\mathcal{M}^1)).$$

From Lemma 4.1(II)(c),

$$(2N - n + 1 - \#\mathcal{M}^1)/(n + 1 - d(\mathcal{M}^1)) \leq (2N - n + 1)/(n + 1),$$

which reduces to the inequality

$$d(\mathcal{M}^1) < \frac{2N - n + 1}{n + 1} d(\mathcal{M}^1) \leq \#\mathcal{M}^1$$

and we have that $\#\mathcal{M}^1 \geq d(\mathcal{M}^1) + 1$.

1) When $d(\mathcal{M}^1) = 1$. It is trivial that there are two vectors \mathbf{a} and \mathbf{b} in \mathcal{M}^1 satisfying $\mathbf{a} = c\mathbf{b}$ ($c \neq 0$, constant). This is absurd since $\mathbf{a}, \mathbf{b} \in X^n(2)$.

2) When $d(\mathcal{M}^1) = d \geq 2$. We suppose that $\mathbf{b}_1, \dots, \mathbf{b}_d$ are linearly independent vectors belonging to \mathcal{M}^1 . Then, any vector $\mathbf{a} \in \mathcal{M}^1 \setminus \{\mathbf{b}_1, \dots, \mathbf{b}_d\}$ can be represented by $\mathbf{b}_1, \dots, \mathbf{b}_d$ as a linear combination over \mathbf{C} :

$$\mathbf{a} = c_1 \mathbf{b}_1 + \dots + c_d \mathbf{b}_d.$$

From this relation

$$(4.18) \quad (\mathbf{a}, f) = \sum_{v=1}^d c_v (\mathbf{b}_v, f).$$

As $\mathbf{a}, \mathbf{b}_1, \dots, \mathbf{b}_d$ are in \mathcal{M}^1 , from (4.18) we obtain that there is an integer k ($1 \leq k \leq d$) such that

$$(\mathbf{a}, f) = c_k (\mathbf{b}_k, f) \quad (c_k \neq 0)$$

due to a Borel's theorem (see [1, 1^o, p. 19]). This relation reduces to $\mathbf{a} = c_k \mathbf{b}_k$ since f is linearly non-degenerate over \mathbf{C} . This is absurd since $\mathbf{a}, \mathbf{b}_k \in X^n(2)$.

From 1) and 2) our theorem must hold. \square

THEOREM 4.6. *Suppose that $N > 2m$ ($m \in \mathbf{N}$) and $X^n(2)$ is in N -subgeneral position. Then*

$$\sum_{\mathbf{a} \in X^n(2)} \mu_n(\mathbf{a}, f) < 2N - n + 1.$$

Proof. We suppose to the contrary that there exist $f, X^n(2)$ such that

$$(4.19) \quad \sum_{\mathbf{a} \in X^n(2)} \mu_n(\mathbf{a}, f) = 2N - n + 1.$$

Then, from Lemma 4.9, $\lambda < (n+1)/(2N-n+1)$, so that from Lemma 4.10 \mathcal{M}^1 satisfies

$$\#\mathcal{M}^1 = d(\mathcal{M}^1) + N - n; \quad d(\mathcal{M}^1) < (n+1)/2.$$

This means that $d(\mathcal{M}^1) \geq 1$. From Theorem 4.5, we have that (4.19) does not hold. We have our theorem. \square

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