H. YAGISITA
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REMARKS ON SPACE-TIME BEHAVIOR IN THE CAUCHY PROBLEMS OF THE HEAT EQUATION AND THE CURVATURE FLOW EQUATION WITH MILDLY OSCILLATING INITIAL VALUES

Hiroki Yagisita

Abstract

We study two initial value problems of the linear diffusion equation $u_t = u_{xx}$ and the nonlinear diffusion equation $u_t = (1+u_x^2)^{-1}u_{xx}$, when Cauchy data $u(x,0) = u_0(x)$ are bounded and oscillate mildly. The latter nonlinear heat equation is the equation of the curvature flow, when the moving curves are represented by graphs. In the case of $\lim_{|x|\to+\infty}|xu_0'(x)| = 0$, by using an elementary scaling technique, we show

$$\lim_{t\to+\infty}|u(\sqrt{t}x,t) - (F(-x)u_0(-\sqrt{t}) + F(+x)u_0(+\sqrt{t}))| = 0$$

for the linear heat equation $u_t = u_{xx}$, where $x \in \mathbf{R}$ and $F(z) := \dfrac{1}{2\sqrt{\pi}}\displaystyle\int_{-\infty}^{z} e^{-y^2/4}\,dy$. Further, by combining with a theorem of Nara and Taniguchi, we have the same result for the curvature equation $u_t = (1+u_x^2)^{-1}u_{xx}$. In the case of $\lim_{|x|\to+0}|xu_0'(x)| = 0$ and in the case of $\sup_{x\in\mathbf{R}}|xu_0'(x)| < +\infty$, respectively, we also give a similar remark for the linear heat equation $u_t = u_{xx}$.

1. Introduction

In this paper, by using an elementary scaling argument, we study space-time behavior in the Cauchy problem of the heat equation

$$(1.1)\qquad \begin{cases} u_t(x,t) = u_{xx}(x,t), & (x,t) \in \mathbf{R}\times(0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbf{R}, \end{cases}$$

when the initial values $u_0(x)$ are bounded and oscillate mildly. We also study the Cauchy problem of the nonlinear diffusion equation

$$(1.2)\qquad \begin{cases} u_t(x,t) = \dfrac{u_{xx}(x,t)}{1+(u_x(x,t))^2}, & (x,t) \in \mathbf{R}\times(0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbf{R}, \end{cases}$$

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which is the equation of the curvature flow when the moving curves are represented by graphs.

First, we mention criteria for stabilization of the solution $u(x,t)$ to the Cauchy problem of the heat equation $u_t = u_{xx}$. From [3, 11, 4, 2] (e.g.), we see the following:

THEOREM 1. *Let $u_0 \in L^\infty(\mathbf{R})$ and $c \in \mathbf{R}$. Then, the solution $u(x,t)$ to* (1.1) *satisfies $\lim_{t\to+\infty} u(x,t) = c$ if and only if $u_0(x)$ satisfies $\lim_{R\to+\infty} \dfrac{1}{2R}\displaystyle\int_{-R}^{+R} u_0 \cdot (x+y)\,dy = c$. Moreover, $u(x,t)$ satisfies $\lim_{t\to+\infty} \sup_{x\in\mathbf{R}} |u(x,t) - c| = 0$ if and only if $u_0(x)$ satisfies $\lim_{R\to+\infty} \sup_{x\in\mathbf{R}} \left|\dfrac{1}{2R}\displaystyle\int_{-R}^{+R} u_0(x+y)\,dy - c\right| = 0.$*

On the other hand, Collet and Eckmann [1] gave a simple example of a bounded initial value $u_0(x)$ where the solution $u(x,t)$ to (1.1) oscillates forever as $t \to +\infty$:

Example. Let a function $u_0 \in L^\infty(\mathbf{R})$ with $\|u_0\|_{L^\infty(\mathbf{R})} = 1$ satisfy $u_0(\pm x) = (-1)^n$ for all $x \in [n! + 2^n, (n+1)! - 2^{n+1}]$ when $n = 5, 6, 7, \ldots$. Then, the solution $u(x,t)$ to (1.1) satisfies

$$\lim_{n\to\infty} \sup_{(x,t)\in[-L,+L]^2} |u(x, t + (n+1)(n!)^2) - (-1)^n| = 0$$

for all $L > 0$.

See also Krzyżański [5] for another example. So, the large-time behavior of a solution $u(x,t)$ to (1.1) with a bounded initial value $u_0(x)$ may be complex. Indeed, Vázquez and Zuazua [13] showed the general behavior is very complex:

THEOREM 2. (i) *Let $u_0 \in L^\infty(\mathbf{R})$. Then, the set of accumulation points in $L^\infty_{loc}(\mathbf{R})$ of $\{(e^{\triangle t}u_0)(\sqrt{t}\,\cdot)\}_{t>0}$ as $t \to +\infty$ coincides with the set $\{(e^{\triangle}\phi)(\cdot) \mid \phi \in A\}$, where A is the set of accumulation points of $\{u_0(\lambda\,\cdot)\}_{\lambda>0}$ as $\lambda \to +\infty$ in the weak-star topology $\sigma(L^\infty, L^1)$.*

(ii) *Let $c > 0$ and $B_c = \{f \in L^\infty(\mathbf{R}) \mid \|f\|_{L^\infty} \le c\}$. Let $\mathscr{M}_c$ be the set of $f \in B_c$ such that the set of accumulation points of $\{f(\lambda\,\cdot)\}_{\lambda>0}$ as $\lambda \to +\infty$ in the weak-star topology $\sigma(L^\infty, L^1)$ is B_c. Then, $\mathscr{M}_c$ is dense with empty interior in B_c with the weak-star topology $\sigma(L^\infty, L^1)$.*

They also showed the general behavior in a number of evolution equations on $\mathbf{R}^N$ is complex. However, the behavior may be rather simple, if the initial value oscillates mildly. In this paper, we prove the following, which is a remark on the long-time behavior in the Cauchy problem (1.1) when the initial value $u_0(x) \in L^\infty(\mathbf{R}) \cap C^1(\mathbf{R}\backslash\{0\})$ satisfies $\lim_{|x|\to+\infty} |xu_0'(x)| = 0$:

THEOREM 3. *Let* $u_0 \in L^\infty(\mathbf{R}) \cap C^1(\mathbf{R}\setminus\{0\})$ *and* $\lim_{|x|\to+\infty}|xu_0'(x)| = 0$. *Then, the solution* $u(x,t)$ *to* (1.1) *satisfies*

$$\lim_{t\to+\infty} \sup_{x\in[-L,+L]} |u(\sqrt{t}x,t) - (F(-x)u_0(-\sqrt{t}) + F(+x)u_0(+\sqrt{t}))| = 0$$

for all $L > 0$, *where* $F(z) := \dfrac{1}{2\sqrt{\pi}}\displaystyle\int_{-\infty}^{z} e^{-y^2/4}\,dy$.

COROLLARY 4. *Let* $u_0 \in L^\infty(\mathbf{R}) \cap C^1(\mathbf{R}\setminus\{0\})$ *and* $\lim_{|x|\to+\infty}|xu_0'(x)| = 0$. *Then, the set of accumulation points in* $L^\infty_{loc}(\mathbf{R})$ *of* $\{(e^{\triangle t}u_0)(\sqrt{t}\,\cdot)\}_{t>0}$ *as* $t \to +\infty$ *coincides with the set* $\{\alpha F(-\,\cdot\,) + \beta F(+\cdot\,) \mid (\alpha,\beta) \in A\}$, *where* $F(z) := \dfrac{1}{2\sqrt{\pi}}\displaystyle\int_{-\infty}^{z} e^{-y^2/4}\,dy$ *and* A *is the set of accumulation points in* $\mathbf{R}^2$ *of* $\{(u_0(-\lambda), u_0(+\lambda))\}_{\lambda>0}$ *as* $\lambda \to +\infty$.

We also prove the following two:

PROPOSITION 5. *Let* $u_0 \in L^\infty(\mathbf{R}) \cap C^1(\mathbf{R}\setminus\{0\})$ *and* $\lim_{|x|\to+0}|xu_0'(x)| = 0$. *Then, the solution* $u(x,t)$ *to* (1.1) *satisfies*

$$\lim_{t\to+0} \sup_{x\in[-L,+L]} |u(\sqrt{t}x,t) - (F(-x)u_0(-\sqrt{t}) + F(+x)u_0(+\sqrt{t}))| = 0$$

for all $L > 0$, *where* $F(z) := \dfrac{1}{2\sqrt{\pi}}\displaystyle\int_{-\infty}^{z} e^{-y^2/4}\,dy$.

PROPOSITION 6. *Let* $u_0 \in C^1(\mathbf{R}\setminus\{0\})$ *and* $\sup_{x\in\mathbf{R}\setminus\{0\}}|xu_0'(x)| < +\infty$. *Then, the solution* $u(x,t)$ *to* (1.1) *satisfies*

$$\begin{aligned} &|u(\sqrt{t}x,t) - (F(-x)u_0(-\sqrt{t}) + F(+x)u_0(+\sqrt{t}))| \\ &\quad \le G(-x)\left(\sup_{y<0}|yu_0'(y)|\right) + G(+x)\left(\sup_{y>0}|yu_0'(y)|\right) \end{aligned}$$

for all $(x,t) \in \mathbf{R}\times(0,+\infty)$, *where* $F(z) := \dfrac{1}{2\sqrt{\pi}}\displaystyle\int_{-\infty}^{z} e^{-y^2/4}\,dy$ *and* $G(z) := \dfrac{1}{2\sqrt{\pi}}\displaystyle\int_{0}^{+\infty} e^{-(z-y)^2/4}|\log y|\,dy$.

Remark 1. (i) Let $(a,b) \in \mathbf{R}^2$ and

$$u_0(x) = \begin{cases} a & (x < 0), \\ b & (x > 0). \end{cases}$$

Then, the solution $u(x,t)$ to (1.1) satisfies

$$u(\sqrt{t}x,t) = aF(-x) + bF(+x)$$

for all $(x,t) \in \mathbf{R}\times(0,+\infty)$, where $F(z) := \dfrac{1}{2\sqrt{\pi}}\displaystyle\int_{-\infty}^{z} e^{-y^2/4}\,dy$.

(ii) Let $u_1(x) = \phi_1(\log(-x))$ and $u_2(x) = \phi_2(\log(+x))$. Then, $xu_1'(x) = \phi_1'(\log(-x))$ and $xu_2'(x) = \phi_2'(\log(+x))$.

(iii) Let $u(x,t)$ be the solution to (1.1). Then, the function

$$v(x,t) := u(e^{t/2}x, e^t)$$

is the solution to

$$\begin{cases} v_t(x,t) = v_{xx}(x,t) + \dfrac{x}{2} v_x(x,t), & (x,t) \in \mathbf{R}^2, \\ v(x,0) = (e^{\triangle} u_0)(x), & x \in \mathbf{R}. \end{cases}$$

(iv) Because of (ii), (iii), Theorem 3 and Proposition 5, if two functions $a \in L^\infty(\mathbf{R}) \cap C^1(\mathbf{R})$ and $b \in L^\infty(\mathbf{R}) \cap C^1(\mathbf{R})$ satisfy

$$\lim_{t\to\pm\infty} |a'(t)| = \lim_{t\to\pm\infty} |b'(t)| = 0,$$

then the solution $v(x,t)$ to the equation

$$v_t(x,t) = v_{xx}(x,t) + \frac{x}{2} v_x(x,t)$$

with the initial data

$$v(x,0) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{0} e^{-(x-y)^2/4} a(\log(y^2))\,dy + \frac{1}{2\sqrt{\pi}} \int_{0}^{+\infty} e^{-(x-y)^2/4} b(\log(y^2))\,dy$$

satisfies

$$\lim_{t\to\pm\infty} \sup_{x\in[-L,+L]} |v(x,t) - (a(t)F(-x) + b(t)F(+x))| = 0$$

for all $L > 0$, where $F(z) := \dfrac{1}{2\sqrt{\pi}} \displaystyle\int_{-\infty}^{z} e^{-y^2/4}\,dy$.

Nara and Taniguchi [9] showed that the difference between the solution to the heat equation (1.1) and that to the curvature flow equation (1.2) with the same initial value is of order $O(t^{-1/2})$ as $t \to +\infty$. Precisely, they given the following theorem:

Theorem 7. *Let* $\varepsilon > 0$. *Suppose* $u_0 \in C^2(\mathbf{R})$ *satisfies* $\sup_{x\in\mathbf{R}}(|u_0(x)| + |u_0'(x)| + |u_0''(x)|) < +\infty$ *and* $\sup_{x_1,x_2\in\mathbf{R},x_1\neq x_2} \dfrac{|u_0''(x_1) - u_0''(x_2)|}{|x_1 - x_2|^\varepsilon} < +\infty$. *Then, the maximum interval of existence of the classical solution* $u(x,t)$ *to* (1.2) *is* $[0,+\infty)$ *and the solution* $u(x,t)$ *satisfies*

$$\sup_{t>0,x\in\mathbf{R}} t^{1/2} \left| u(x,t) - \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4t} u_0(y)\,dy \right| < +\infty.$$

Therefore, by combining it with Theorem 3, we have the following remark on the long-time behavior in the Cauchy problem (1.2):

COROLLARY 8. *Let* $\varepsilon > 0$. *Suppose* $u_0 \in C^2(\mathbf{R})$ *satisfies* $\sup_{x\in\mathbf{R}}(|u_0(x)| + |u_0'(x)| + |u_0''(x)|) < +\infty$, $\sup_{x_1, x_2 \in \mathbf{R}, x_1 \neq x_2} \dfrac{|u_0''(x_1) - u_0''(x_2)|}{|x_1 - x_2|^\varepsilon} < +\infty$ *and* $\lim_{|x|\to+\infty} |xu_0'(x)| = 0$. *Then, the solution* $u(x,t)$ *to* (1.2) *satisfies*

$$\lim_{t\to+\infty} \sup_{x\in[-L,+L]} |u(\sqrt{t}x, t) - (F(-x)u_0(-\sqrt{t}) + F(+x)u_0(+\sqrt{t}))| = 0$$

for all $L > 0$, *where* $F(z) := \dfrac{1}{2\sqrt{\pi}} \displaystyle\int_{-\infty}^{z} e^{-y^2/4}\, dy$.

Nara [8] showed that the difference between the solution to the heat equation on $\mathbf{R}^N$ and that to the mean curvature flow equation on $\mathbf{R}^N$ with the same initial value is of order $O(t^{-1/2})$ as $t \to +\infty$, when the initial value is radially symmetric. See [12, 6] for the difference between the behavior of a disturbed planar front in a bistable reaction-diffusion equation and that of a mean curvature flow with a drift term. See [10, 14, 12, 7] for other nontrivial large-time behaviors in nonlinear diffusion equations.

2. Proof

LEMMA 9. *The solution* $u(x,t)$ *to* (1.1) *satisfies*

$$\sup_{x\in[-L,+L]} |u(\sqrt{t}x, t) - (aF(-x) + bF(+x))| \le \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} \rho_L(z)(|u_0(-\sqrt{t}z) - a| + |u_0(+\sqrt{t}z) - b|)\, dz$$

for all $(L,t) \in (0,+\infty)^2$ *and* $(a,b) \in \mathbf{R}^2$, *where* $\rho_L(z) := \sup_{z_0\in[-L,+L]} e^{-(z-z_0)^2/4}$.

Proof. From

$$\begin{aligned} u(\sqrt{t}x, t) &= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4} u_0(\sqrt{t}y)\, dy \\ &= \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} e^{-(x+z)^2/4} u_0(-\sqrt{t}z)\, dz + \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} e^{-(x-z)^2/4} u_0(+\sqrt{t}z)\, dz, \end{aligned}$$

we see

$$\begin{aligned} u(\sqrt{t}x, t) - (aF(-x) + bF(+x)) = &\frac{1}{2\sqrt{\pi}} \int_0^{+\infty} e^{-(z-(-x))^2/4}(u_0(-\sqrt{t}z) - a)\, dz \\ &+ \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} e^{-(z-(+x))^2/4}(u_0(+\sqrt{t}z) - b)\, dz. \end{aligned}$$

So, we have the conclusion. ∎

LEMMA 10. *Let* $u_0 \in C^1(\mathbf{R}\backslash\{0\})$ *and* $\alpha > 0$. *Then*, $\lim_{|x|\to+\infty}|xu_0'(x)| = 0$ *implies* $\lim_{|s|\to+\infty}|u_0(s\alpha) - u_0(s)| = 0$. *Also*, $\lim_{|x|\to+0}|xu_0'(x)| = 0$ *implies* $\lim_{|s|\to+0}|u_0(s\alpha) - u_0(s)| = 0$.

Proof. We see

$$\begin{aligned}|u_0(s\alpha) - u_0(s)| = \left|\int_1^\alpha su_0'(sz)\,dz\right| &\leq \left(\alpha + \frac{1}{\alpha}\right) \sup_{\min\{\alpha,1/\alpha\}\leq|z|\leq\max\{\alpha,1/\alpha\}} |su_0'(sz)| \\ &\leq \left(\alpha + \frac{1}{\alpha}\right)^2 \sup_{\min\{\alpha,1/\alpha\}\leq|z|\leq\max\{\alpha,1/\alpha\}} |szu_0'(sz)| \\ &= \left(\alpha + \frac{1}{\alpha}\right)^2 \sup_{\min\{\alpha,1/\alpha\}|s|\leq|x|\leq\max\{\alpha,1/\alpha\}|s|} |xu_0'(x)|.\end{aligned}$$

So, we have the conclusion. ∎

Proof of Theorem 3 *and Proposition* 5. We see

$$|u_0(-\sqrt{t}z) - u_0(-\sqrt{t})| + |u_0(+\sqrt{t}z) - u_0(+\sqrt{t})| \leq 4\|u_0\|_{L^\infty(\mathbf{R})}$$

for all $t > 0$ and $z > 0$. Hence, because of $\rho_L \in L^1((0,+\infty))$, we have the conclusions by Lemmas 9 and 10. ∎

Remark 2. (i) Let $u_1(x) = \phi_1(\log(-x))$, $u_2(x) = \phi_2(\log(+x))$ and $\alpha > 0$. Then, $\lim_{|z|\to+\infty}|\phi_1(z + \log\alpha) - \phi_1(z)| = 0$ implies $\lim_{s\to-\infty}|u_1(s\alpha) - u_1(s)| = 0$ and $\lim_{s\to-0}|u_1(s\alpha) - u_1(s)| = 0$. Also, $\lim_{|z|\to+\infty}|\phi_2(z + \log\alpha) - \phi_2(z)| = 0$ implies $\lim_{s\to+\infty}|u_2(s\alpha) - u_2(s)| = 0$ and $\lim_{s\to+0}|u_2(s\alpha) - u_2(s)| = 0$.

(ii) Because of (i) and Remark 1 (iii), if two functions $a \in L^\infty(\mathbf{R})$ and $b \in L^\infty(\mathbf{R})$ satisfy

$$\lim_{t\to\pm\infty} |a(t+\beta) - a(t)| = \lim_{t\to\pm\infty} |b(t+\beta) - b(t)| = 0$$

for all $\beta \in \mathbf{R}$, then the solution $v(x,t)$ to the equation

$$v_t(x,t) = v_{xx}(x,t) + \frac{x}{2}v_x(x,t)$$

with the initial data

$$v(x,0) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^0 e^{-(x-y)^2/4}a(\log(y^2))\,dy + \frac{1}{2\sqrt{\pi}}\int_0^{+\infty} e^{-(x-y)^2/4}b(\log(y^2))\,dy$$

satisfies

$$\lim_{t\to\pm\infty} \sup_{x\in[-L,+L]} |v(x,t) - (a(t)F(-x) + b(t)F(+x))| = 0$$

for all $L > 0$, where $F(z) := \dfrac{1}{2\sqrt{\pi}}\displaystyle\int_{-\infty}^z e^{-y^2/4}\,dy$.

Proof of Proposition 6. From

$$
\begin{aligned}
& u(\sqrt{t}x, t) - (F(-x)u_0(-\sqrt{t}) + F(+x)u_0(+\sqrt{t})) \\
&\quad = \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} e^{-(x+z)^2/4} (u_0(-\sqrt{t}z) - u_0(-\sqrt{t}))\, dz \\
&\qquad + \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} e^{-(x-z)^2/4} (u_0(+\sqrt{t}z) - u_0(+\sqrt{t}))\, dz \\
&\quad = \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} e^{-(x+z)^2/4} \left(\int_1^z \frac{(-\sqrt{t}y)u_0'(-\sqrt{t}y)}{y}\, dy \right) dz \\
&\qquad + \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} e^{-(x-z)^2/4} \left(\int_1^z \frac{(+\sqrt{t}y)u_0'(+\sqrt{t}y)}{y}\, dy \right) dz,
\end{aligned}
$$

we see

$$
\begin{aligned}
& |u(\sqrt{t}x, t) - (F(-x)u_0(-\sqrt{t}) + F(+x)u_0(+\sqrt{t}))| \\
&\quad \le \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} e^{-(x+z)^2/4} \left| \int_1^z \frac{|(-\sqrt{t}y)u_0'(-\sqrt{t}y)|}{y}\, dy \right| dz \\
&\qquad + \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} e^{-(x-z)^2/4} \left| \int_1^z \frac{|(+\sqrt{t}y)u_0'(+\sqrt{t}y)|}{y}\, dy \right| dz \\
&\quad \le \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} e^{-(x+z)^2/4} \left| \int_1^z \frac{\sup_{s<0} |su_0'(s)|}{y}\, dy \right| dz \\
&\qquad + \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} e^{-(x-z)^2/4} \left| \int_1^z \frac{\sup_{s>0} |su_0'(s)|}{y}\, dy \right| dz \\
&\quad = \frac{\sup_{s<0} |su_0'(s)|}{2\sqrt{\pi}} \int_0^{+\infty} e^{-((-x)-z)^2/4} |\log z|\, dz \\
&\qquad + \frac{\sup_{s>0} |su_0'(s)|}{2\sqrt{\pi}} \int_0^{+\infty} e^{-((+x)-z)^2/4} |\log z|\, dz.
\end{aligned}
$$

So, we have the conclusion. ■

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Hiroki Yagisita
Department of Mathematics
Faculty of Science
Kyoto Sangyo University
Japan
E-mail: hrk0ygst@cc.kyoto-su.ac.jp