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CERTAIN HOLOMORPHIC SECTIONS RELATING TO 2-POINTED WEIERSTRASS GAP SETS ON A COMPACT RIEMANN SURFACE

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Abstract

For a compact Riemann surface X of genus g, we will construct a holomorphic section of the line bundle $\pi_1^* K_X^{g(g+1)(g+2)/6} \otimes \pi_2^* K_X^{g(g+1)(g+2)/6}$ over $X \times X$ whose zero set consists exactly of the points (P, Q) with the cardinalities of the Weierstrass gap sets $G(P, Q)$ greater than the minimal value $(g^2 + 3g)/2$.

1. Introduction

This is a continuation to the previous paper [G].

Throughout this article, we denote by X a compact Riemann surface of genus g. For a pair of distinct points P and Q in X, the Weierstrass gap set $G(P, Q)$ is defined by

$$G(P, Q) := \{(m, n) \in \mathbf{N}_0 \times \mathbf{N}_0 \mid \nexists f \in \mathcal{M}(X) \text{ such that } (f)_\infty = mP + nQ\}.$$

Here $\mathbf{N}_0$ denotes the set consisting of all non-negative integers, $\mathcal{M}(X)$ the space of meromorphic functions on X, and $(f)_\infty$ the polar divisor of a meromorphic function f. It has then been shown by Kim [K] (see also Homma [H]) that the cardinality of the gap set at (P, Q) satisfies the inequality $(g^2 + 3g)/2 \le \#G(P, Q) \le (3g^2 + g)/2$. The minimal cardinality $(g^2 + 3g)/2$ is attained by generic pair (P, Q) in $X \times X$, while the maximal cardinality $(3g^2 + g)/2$ is attained only when both P and Q are the hyperelliptic Weierstrass points in X.

In the previous paper [G], we have investigated the Wronskian matrices associated to effective divisors on X and constructed a holomorphic section relating to the gap sets, which are summarized as Theorem 1.1 below. Let K_X denotes the canonical line bundle of X, $\Omega(X) = H^0(X, K_X)$ and $\pi_i : X \times X \to X$ the projection onto the i-th component $(i = 1, 2)$.

Theorem 1.1. *For each basis $\omega_1, \dots, \omega_g$ of $\Omega(X)$, there exsists a holomorphic section $\Psi[\omega_1, \dots, \omega_g]$ of the holomorphic line bundle $\pi_1^* K_X^{g(g+1)(g^3+g+4)/12} \otimes \pi_2^* K_X^{g(g+1)(g+2)/6}$ over $X \times X$, for which the following* (1) *and* (2) *are equivalent for each distinct pair* (P, Q).

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(1) *$\#G(P,Q)$ attains the minimal value $(g^2+3g)/2$.*
(2) *There exists a basis $\omega_1,\dots,\omega_g$ of $\Omega(X)$ depending on (P,Q) such that $\Psi[\omega_1,\dots,\omega_g](P,Q)\neq 0$.*

Our motivation of constructing the holomorphic section is that we want to obtain a holomophic section whose zero set consists exactly of the points (P,Q) with the cardinalities of the gap sets greater than the minimal value $(g^2+3g)/2$. Such a holomorphic section is likely to play the role of the Wronskian determinant in the classical Weierstrass point theory. However the holomorphic section $\Psi[\omega_1,\dots,\omega_g]$ does not work well, because of dependence on each point in the assertion (2) above. For Theorem 1.1 gives the following expression only. Putting $Z := \{(P,Q)\in X\times X\backslash\Delta X \mid \#G(P,Q) > (g^2+3g)/2\}$,

$$Z = \bigcap_{\substack{\omega_1,\dots,\omega_g;\\ \text{basis of } \Omega(X)}} \{(P,Q)\in X\times X\backslash\Delta X \mid \Psi[\omega_1,\dots,\omega_g](P,Q)=0\}. \tag{1}$$

Here ΔX denotes the diagonal set in $X\times X$.

In this article, we will construct the other holomorphic sections in order to remove such a weak point. We shall prove the following theorem in section 2.

THEOREM 1.2. *For each basis $\omega_1,\dots,\omega_g$ of $\Omega(X)$, there exsists a holomorphic section $\boldsymbol{S}[\omega_1,\dots,\omega_g]$ of the holomorphic line bundle $\pi_1^*K_X^{g(g+1)(g+2)/6}\otimes \pi_2^*K_X^{g(g+1)(g+2)/6}$ over $X\times X$, for which the following* (1) *and* (2) *are equivalent for each distinct pair (P,Q).*
(1) *$\#G(P,Q)$ attains the minimal value $(g^2+3g)/2$.*
(2) *$\boldsymbol{S}[\omega_1,\dots,\omega_g](P,Q)\neq 0$ for each fixed basis $\omega_1,\dots,\omega_g$ of $\Omega(X)$.*

In contrast with (1), for each fixed basis $\omega_1,\dots,\omega_g$, we have

$$Z = \{(P,Q)\in X\times X\backslash\Delta X \mid \boldsymbol{S}[\omega_1,\dots,\omega_g](P,Q)=0\}.$$

Now we are going to review some known results that will be needed in this article. The Weierstrass gap sequence at P in X is denoted by $\alpha_1^P = 1 < \alpha_2^P < \cdots < \alpha_g^P < 2g$, then the Weierstrass weight at P is defined by $\mathrm{wt}(P) := \sum_{i=1}^g(\alpha_i^P - i)$ as well. After Kim [K], we define the map $\mu : G(P)\to G(Q)$ by $\mu(\alpha) := \min\{\beta \mid (\alpha,\beta)\notin G(P,Q)\}$. One of the key observations made in [G] is the following characterization for $\mu(\alpha)$ in terms of holomorphic 1-forms:

$$\mu(\alpha) = \max\{\beta \mid \exists\omega\in\Omega(X) \text{ such that } \mathrm{ord}_P(\omega)=\alpha-1 \text{ and } \mathrm{ord}_Q(\omega)=\beta-1\}. \tag{2}$$

Let $t(\mu)$ be the number of pairs (i,j) satisfying both $i<j$ and $\mu(\alpha_i^P) > \mu(\alpha_j^P)$, then Homma [H] has proved an expression of $\#G(P,Q)$:

$$\#G(P,Q) = \mathrm{wt}(P) + \mathrm{wt}(Q) - t(\mu) + g(g+1). \tag{3}$$

By virtue of the expression, Homma has found that $\#G(P,Q)$ attains the minimal value $g(g^2+3g)/2$ if and only if both P and Q are non-Weierstrass points and μ is given by $\mu(i)=g+1-i$ $(i=1,\dots,g)$ as well. This fact has been the reason for constructing the holomorphic section $\Psi[\omega_1,\dots,\omega_g]$ in [G].

After constructing the holomorphic section $\boldsymbol{S}[\omega_1,\dots,\omega_g]$ and proving Theorem 1.2 in section 2, we will observe the orders of the holomorphic sections in section 3. In the final section 4, some examples will be examined.

We conclude the introduction with noticing two expansion formulae for determinants, which are used frequently in this article.

Let $A=(a_{ij})$ be an $m\times n$ matrix. Then for sequences $1\le i_1<\cdots<i_\alpha\le m$ and $1\le j_1<\cdots<j_\beta\le n$, we put

$$A^{i_1,\dots,i_\alpha}=\begin{pmatrix} a_{i_1 1} & a_{i_1 2} & \cdots & a_{i_1 n}\\ a_{i_2 1} & a_{i_2 2} & \cdots & a_{i_2 n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{i_\alpha 1} & a_{i_\alpha 2} & \cdots & a_{i_\alpha n}\end{pmatrix},\quad A_{j_1,\dots,j_\beta}=\begin{pmatrix} a_{1j_1} & a_{1j_2} & \cdots & a_{1j_\beta}\\ a_{2j_1} & a_{2j_2} & \cdots & a_{2j_\beta}\\ \cdots & \cdots & \cdots & \cdots\\ a_{mj_1} & a_{mj_2} & \cdots & a_{mj_\beta}\end{pmatrix}.$$

Then for a $g\times m$ matrix A and a $g\times n$ matrix B with $m+n=g$, the determinant of the square g matrix (A,B) can be expanded as

$$|A,B|=(-1)^{m(m+1)/2}\sum_{\sigma\in\mathfrak{S}_{m,n}}(-1)^{\sigma(1)+\cdots+\sigma(m)}|A^{\sigma(1),\dots,\sigma(m)}|\,|B^{\sigma(m+1),\dots,\sigma(g)}|. \tag{4}$$

Here we denote by $\mathfrak{S}_g$ the symmetric group of degree g, and denote likewise by $\mathfrak{S}_{m,n}$ the set of all (m,n)-shuffles. Namely

$$\mathfrak{S}_{m,n}=\{\sigma\in\mathfrak{S}_{m+n}\mid\sigma(1)<\cdots<\sigma(m),\sigma(m+1)<\cdots<\sigma(m+n)\}.$$

When $m=1$ or $n=1$, (4) is the ordinary expansion of a determinant using cofactors according to the first or the last column respectively. The expansion formula (4) implies another one. Let A, B be square g matices, L a $g\times\ell$ matrix and M a $g\times m$ matrix with $\ell+m=g$. Then

$$|AL,BM|=\sum_{\substack{1\le i_1<\cdots<i_\ell\le g\\ 1\le j_1<\cdots<j_m\le g}}|A_{i_1,\dots,i_\ell},B_{j_1,\dots,j_m}|\,|L^{i_1,\dots,i_\ell}|\,|M^{j_1,\dots,j_m}|. \tag{5}$$

2. Construction of holomorphic sections and the proof of Theorem 1.2

In order to illustrate the reason for constructing the holomorphic section $\boldsymbol{S}[\omega_1,\dots,\omega_g]$ mensioned in the introduction, we shall first review the definition of $\Psi[\omega_1,\dots,\omega_g]$ and write down an explicit expression of it.

Let $\boldsymbol{\omega}=(\omega_1,\dots,\omega_g)$ be a basis of $\Omega(X)$ and z, w be local coordinate functions on open sets U and V respectively on X. Write $\omega_i=f_i\,dz=h_i\,dw$

($i = 1, \dots, g$), where f_i and h_i are holomorphic functions on U and V respectively. We denote by

$$W[\boldsymbol{f}](z) := W[f_1, \dots, f_g](z) := \begin{pmatrix} f_1(z) & f_1'(z) & \cdots & f_1^{(g-1)}(z) \\ f_2(z) & f_2'(z) & \cdots & f_2^{(g-1)}(z) \\ \cdots & \cdots & \cdots & \cdots \\ f_g(z) & f_g'(z) & \cdots & f_g^{(g-1)}(z) \end{pmatrix}$$

the Wronskian matrix for $\boldsymbol{f} = (f_1, \dots, f_g)$, and likewise by $W(\boldsymbol{f})(z)$ its determinant, the Wronskian determinant. A lower triangular matrix $T_{\mathrm{low}}[\boldsymbol{f}]$ is defined by

$$T_{\mathrm{low}}[\boldsymbol{f}]_{ij} = (-1)^{i+j} \frac{W(f_1, \dots, \widehat{f_j}, \dots, f_i)}{W(f_1, \dots, f_{i-1})} \prod_{k=1}^{g-1} W(f_1, \dots, f_k).$$

for $1 \leqq j \leqq i \leqq g$, where the circumflex over a term means that it is to be omitted. Then, in [G], we have defined the function $\psi[\boldsymbol{\omega}](z,w)$ on $U \times V$ by

$$\psi[\boldsymbol{\omega}](z,w) := \prod_{i=1}^{g} \boldsymbol{\delta}^i(W[\boldsymbol{f}](z)) \times \prod_{i=1}^{g} \boldsymbol{\delta}_i((T_{\mathrm{low}}[\boldsymbol{f}](z)) \cdot (W[\boldsymbol{h}](w))).$$

Here for a square g matrix $A = (a_{ij})$ and $1 \le i \le g$, we set

$$\boldsymbol{\delta}^i(A) := \begin{vmatrix} a_{11} & \cdots & a_{1i} \\ \cdots & \cdots & \cdots \\ a_{i1} & \cdots & a_{ii} \end{vmatrix}, \quad \boldsymbol{\delta}_i(A) := \begin{vmatrix} a_{g1} & \cdots & a_{gi} \\ \cdots & \cdots & \cdots \\ a_{g-i+1,1} & \cdots & a_{g-i+1,i} \end{vmatrix}.$$

Those local functions define together a global holomorphic section $\Psi[\omega_1, \dots, \omega_g]$ of the line bundle in Theorem 1.1.

Now, by making a somewhat involved calculation, we have the following explicit expression of $\psi[\boldsymbol{\omega}](z,w)$. Namely,

$$(6) \quad \psi[\boldsymbol{\omega}](z,w) = (-1)^{g(g+1)(g+2)/6} \left(\prod_{i=1}^{g-1} W(f_1, \dots, f_i) \right)^{g(g+1)/2}$$
$$\times \prod_{\nu=0}^{g} \begin{vmatrix} f_1(z) & f_1'(z) & \cdots & f_1^{(g-\nu-1)}(z) & h_1(w) & h_1'(w) & \cdots & h_1^{(\nu-1)}(w) \\ f_2(z) & f_2'(z) & \cdots & f_2^{(g-\nu-1)}(z) & h_2(w) & h_2'(w) & \cdots & h_2^{(\nu-1)}(w) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ f_g(z) & f_g'(z) & \cdots & f_g^{(g-\nu-1)}(z) & h_g(w) & h_g'(w) & \cdots & h_g^{(\nu-1)}(w) \end{vmatrix}.$$

We omit the proof for $\psi[\boldsymbol{\omega}](z,w)$ to have the expression above, because it is not needed in what follows. However, motivated by (6), we define the local functions on $U \times V$ by

$$(7) \qquad S_\nu[\boldsymbol{\omega}](z,w) := |W[\boldsymbol{f}](z)_{1,\dots,g-\nu}, W[\boldsymbol{h}](w)_{1,\dots,\nu}| \quad (\nu = 0,1,\dots,g),$$
$$S[\boldsymbol{\omega}](z,w) := \prod_{\nu=0}^{g} S_\nu[\boldsymbol{\omega}](z,w),$$

hence the expression (6) is rewritten as

$$(8) \qquad \psi[\boldsymbol{\omega}](z,w) = (-1)^{g(g+1)(g+2)/6}\left(\prod_{i=1}^{g-1} W(f_1,\dots,f_i)\right)^{g(g+1)/2} S[\boldsymbol{\omega}](z,w).$$

Note that $S_0[\boldsymbol{\omega}](z,w) = W(\boldsymbol{f})(z)$ and $S_g[\boldsymbol{\omega}](z,w) = W(\boldsymbol{h})(w)$.

In contrast with $\psi[\boldsymbol{\omega}](z,w)$, these functions are subjected to nice transition rules in changing bases of $\Omega(X)$. That is to say, let A be an invertible square g matrix, then

$$(9) \qquad S_\nu[\boldsymbol{\omega}A] = |A| S_\nu[\boldsymbol{\omega}] \quad \text{and} \quad S[\boldsymbol{\omega}A] = |A|^{g+1} S[\boldsymbol{\omega}].$$

Especially the zero sets of $S_\nu[\boldsymbol{\omega}]$ and $S[\boldsymbol{\omega}]$ are determined independently on the choice of the basis $\boldsymbol{\omega}$ of $\Omega(X)$.

Now we are going to prove the local version of Theorem 1.2.

Theorem 2.1. *Let $\boldsymbol{\omega} = (\omega_1,\dots,\omega_g)$ be a basis of $\Omega(X)$ and (U,z), (V,w) two charts of X. Then the following are equivalent for each distinct pair (z,w) in $U \times V$.*

(1) $\#G(z,w)$ attains the minimal value $(g^2+3g)/2$.

(2) $S[\boldsymbol{\omega}](z,w) \neq 0$.

Proof. First we show that (1) implies (2). When $\#G(z,w) = (g^2+3g)/2$, as mentioned in the introduction (or by Lemma 3.1 in [G] more precisely), there exists a basis $\hat{\boldsymbol{\omega}} = (\hat{\omega}_1,\dots,\hat{\omega}_g)$ of $\Omega(X)$ for which the Wronskian matrix turns to be of the form

$$(W[\hat{\boldsymbol{f}}](z), W[\hat{\boldsymbol{h}}](w)) = \left(\begin{array}{cccc|cccc} A_1 & * & * & * & 0 & \cdots & 0 & B_1 \\ 0 & A_2 & * & * & \vdots & \iddots & B_2 & * \\ \vdots & \ddots & \ddots & * & 0 & \iddots & * & * \\ 0 & \cdots & 0 & A_g & B_g & * & * & * \end{array}\right),$$

where $A_1,\dots,A_g$ and $B_1,\dots,B_g$ are non-zero complex numbers. Here $\hat{\omega}_i = \hat{f}_i\,dz = \hat{h}_i\,dw$ as above. Thus, for each $\nu = 0,1,\dots,g$, we have

$$S_\nu[\hat{\boldsymbol{\omega}}](z,w) = \left|\begin{array}{ccc|ccc} A_1 & \cdots & * & & & \\ & \ddots & \vdots & & 0 & \\ 0 & & A_{g-\nu} & & & \\ \hline & & & 0 & & B_{g-\nu+1} \\ & & & & \iddots & \vdots \\ & 0 & & B_1 & \cdots & * \end{array}\right|$$

$$= (-1)^{\nu(\nu-1)} A_1 \cdots A_{g-\nu} B_{g-\nu+1} \cdots B_g$$

$$\neq 0.$$

The transition rule (9) then shows $\boldsymbol{S}[\boldsymbol{\omega}](z,w) \neq 0$.

Next we suppose $\boldsymbol{S}[\boldsymbol{\omega}](z,w) \neq 0$. Then because $S_0[\boldsymbol{\omega}](z,w) = W(\boldsymbol{f})(z) \neq 0$ and $S_g[\boldsymbol{\omega}](z,w) = W(\boldsymbol{h})(w) \neq 0$ by definition, both P and Q are non-Weierstrass points in X and their gap sequences are the same $\{1,2,\dots,g\}$. Then according to (2) (Theorem 2.6 in [G]), we can take a basis $\hat{\boldsymbol{\omega}}$ of $\Omega(X)$, depending on a point (z,w), with respect to which the orders of $\hat{\omega}_i$ are given by $\mathrm{ord}_z(\hat{\omega}_i) = i-1$ and $\mathrm{ord}_w(\hat{\omega}_i) = \mu(i)-1$ $(i = 1,\dots,g)$. Because $W[\hat{\boldsymbol{f}}](z)$ is an upper triangular matrix, the expansion formula (4) implies

$$S_\nu[\hat{\boldsymbol{\omega}}](z,w) = |W[\hat{\boldsymbol{f}}](z)_{1,\dots,g-\nu}, W[\hat{\boldsymbol{h}}](w)_{1,\dots,\nu}|$$
$$= |W[\hat{\boldsymbol{f}}](z)^{1,\dots,g-\nu}_{1,\dots,g-\nu}|\,|W[\hat{\boldsymbol{h}}](w)^{g-\nu+1,\dots,g}_{1,\dots,\nu}|,$$

which does not vanish by the assumption $\boldsymbol{S}[\boldsymbol{\omega}](z,w) \neq 0$ and the transition rule (9). Therefore $|W[\hat{\boldsymbol{h}}](w)^{g-\nu+1,\dots,g}_{1,\dots,\nu}| \neq 0$ for each $\nu = 1,\dots,g$, which verifies the assertion. For first $|W[\hat{\boldsymbol{h}}](w)^{g}_{1}| \neq 0$ implies $\mu(g) = 1$, second $|W[\hat{\boldsymbol{h}}](w)^{g-1,g}_{1,2}| \neq 0$ implies $\mu(g-1) = 2$. We obtain inductively $\mu(i) = g+1-i$ for each $i = 1,\dots,g$, when $\#G(z,w)$ attains the minimal value $(g^2+3g)/2$ as mentioned in the introduction. ◇

Remark 2.1. If we assume the expression (8), then Theorem 2.1 is obtained as a corollary to Theorem 1.1. However once we have defined $\boldsymbol{S}[\boldsymbol{\omega}]$ directly, $\psi[\boldsymbol{\omega}]$ is not needed to prove Theorem 2.1.

Next we shall proceed to find the transition rules for $\boldsymbol{S}_\nu[\boldsymbol{\omega}]$ and $\boldsymbol{S}[\boldsymbol{\omega}]$ in changing the local coordinate functions. Let z, $\tilde{z}$ and w, $\tilde{w}$ be local coordinate functions on U and V respectively. For a basis $\boldsymbol{\omega} = (\omega_1,\dots,\omega_g)$ of $\Omega(X)$, write $\omega_i = f_i\,dz = \tilde{f}_i\,dz = h_i\,dw = \tilde{h}_i\,dw$ as before. With respect to these local coordinate functions, the functions defined by (7) above are

$$S_\nu[\boldsymbol{\omega}](z,w) := |W[\boldsymbol{f}](z)_{1,\dots,g-\nu}, W[\boldsymbol{h}](w)_{1,\dots,\nu}|,$$
$$\tilde{S}_\nu[\boldsymbol{\omega}](\tilde{z},\tilde{w}) := |W[\tilde{\boldsymbol{f}}](\tilde{z})_{1,\dots,g-\nu}, W[\tilde{\boldsymbol{h}}](\tilde{w})_{1,\dots,\nu}|.$$

It is known that the Wronskian matrices are subjected to the following transition rules in changing the local coordinate functions:

$$W[\boldsymbol{f}](z) = W[\tilde{\boldsymbol{f}}](\tilde{z})L(z) \quad \text{and} \quad W[\boldsymbol{h}](w) = W[\tilde{\boldsymbol{h}}](\tilde{w})M(w).$$

Here $L(z)$ and $M(w)$ are the square g matrices of the forms

$$L(z) := \begin{pmatrix} \frac{d\tilde{z}}{dz} & \cdots\cdots\cdots\cdots\cdots\cdots \\ 0 & \left(\frac{d\tilde{z}}{dz}\right)^2 & \cdots\cdots\cdots\cdots \\ 0 & \cdots & \ddots & \cdots \\ 0 & \cdots\cdots\cdots\cdots & & \left(\frac{d\tilde{z}}{dz}\right)^g \end{pmatrix}, \quad M(w) := \begin{pmatrix} \frac{d\tilde{w}}{dw} & \cdots\cdots\cdots\cdots\cdots\cdots \\ 0 & \left(\frac{d\tilde{w}}{dw}\right)^2 & \cdots\cdots\cdots\cdots \\ 0 & \cdots & \ddots & \cdots \\ 0 & \cdots\cdots\cdots\cdots & & \left(\frac{d\tilde{w}}{dw}\right)^g \end{pmatrix}.$$

Making use of the expansion formula (5), we obtain

$$\begin{aligned} (10) \qquad S_v[\boldsymbol{\omega}](z,w) &= |W[\boldsymbol{f}](z)_{1,\dots,g-v}, W[\boldsymbol{h}](w)_{1,\dots,v}| \\ &= |(W[\tilde{\boldsymbol{f}}](\tilde{z}) \cdot L(z))_{1,\dots,g-v}, (W[\tilde{\boldsymbol{h}}](\tilde{w}) \cdot M(w))_{1,\dots,v}| \\ &= |W[\tilde{\boldsymbol{f}}](\tilde{z}) \cdot L(z)_{1,\dots,g-v}, W[\tilde{\boldsymbol{h}}](\tilde{w}) \cdot M(w)_{1,\dots,v}| \\ &= \sum_{\substack{1 \le i_1 < \cdots < i_{g-v} \le g \\ 1 \le j_1 < \cdots < j_v \le g}} |W[\tilde{\boldsymbol{f}}](\tilde{z})_{i_1,\dots,i_{g-v}}, W[\tilde{\boldsymbol{h}}](\tilde{w})_{j_1,\dots,j_v}| \\ &\qquad \times |L(z)^{i_1,\dots,i_{g-v}}_{1,\dots,g-v}|\,|M(w)^{j_1,\dots,j_v}_{1,\dots,v}|. \end{aligned}$$

Substituting

$$|L(z)^{i_1,\dots,i_{g-v}}_{1,\dots,g-v}| = \begin{cases} \left(\frac{d\tilde{z}}{dz}\right)^{(g-v)(g-v+1)/2} & (i_1 = 1, \dots, i_{g-v} = g-v) \\ 0 & (\text{otherwise}), \end{cases}$$

$$|M(w)^{j_1,\dots,j_v}_{1,\dots,v}| = \begin{cases} \left(\frac{d\tilde{w}}{dw}\right)^{v(v+1)/2} & (j_1 = 1, \dots, j_v = v) \\ 0 & (\text{otherwise}), \end{cases}$$

into (10), we have

$$\begin{aligned} (11) \quad S_v[\boldsymbol{\omega}](z,w) &= |W[\tilde{\boldsymbol{f}}](\tilde{z})_{1,\dots,g-v}, W[\tilde{\boldsymbol{h}}](\tilde{w})_{1,\dots,v}| \left(\frac{d\tilde{z}}{dz}\right)^{(g-v)(g-v+1)/2} \left(\frac{d\tilde{w}}{dw}\right)^{v(v+1)/2} \\ &= \tilde{S}_v[\boldsymbol{\omega}](\tilde{z},\tilde{w}) \left(\frac{d\tilde{z}}{dz}\right)^{(g-v)(g-v+1)/2} \left(\frac{d\tilde{w}}{dw}\right)^{v(v+1)/2}. \end{aligned}$$

Taking products on $v = 0, 1, \dots, g$ in (11), we also have

$$S[\boldsymbol{\omega}](z,w) = \tilde{S}[\boldsymbol{\omega}](\tilde{z},\tilde{w})\left(\frac{d\tilde{z}}{dz}\frac{d\tilde{w}}{dw}\right)^{g(g+1)(g+2)/6}. \tag{12}$$

On account of (11) and (12), we put

$$\begin{aligned} \boldsymbol{S}_v[\boldsymbol{\omega}](z,w) &:= S_v[\boldsymbol{\omega}](z,w)(dz)^{\otimes(g-v)(g-v+1)/2} \otimes (dw)^{\otimes v(v+1)/2}, \\ \boldsymbol{S}[\boldsymbol{\omega}](z,w) &:= S[\boldsymbol{\omega}](z,w)(dz)^{\otimes g(g+1)(g+2)/6} \otimes (dw)^{\otimes g(g+1)(g+2)/6} \end{aligned} \tag{13}$$

in order to define holomorphic sections of the line bundles $p_1^*K_X^{(g-v)(g-v+1)/2} \otimes p_2^*K_X^{v(v+1)/2}$ and $p_1^*K_X^{g(g+1)(g+2)/6} \otimes p_2^*K_X^{g(g+1)(g+2)/6}$ over $X \times X$ respectively. Hence Theorem 2.1 implies Theorem 1.2. $\diamondsuit$

3. The orders of $\boldsymbol{S}_v$

In this section, we investigate the orders of the holomorphic section $\boldsymbol{S}_v[\boldsymbol{\omega}]$ ($v = 0, 1, \dots, g$) defined in the previous section. If $\boldsymbol{S}_v[\boldsymbol{\omega}]$ is expressed by $S_v[\boldsymbol{\omega}]$ locally as (13) and local coordinates of the point (P, Q) is (z_0, w_0), the order of $\boldsymbol{S}_v[\boldsymbol{\omega}]$ at a point (P, Q) in $U \times V$ is defined by

$$\begin{aligned} &\operatorname{ord}_{(P,Q)}(\boldsymbol{S}_v[\boldsymbol{\omega}]) \\ &\quad := \min_{N \in \mathbf{N}_0}\left\{\exists m, n \in \mathbf{N}_0 \text{ such that } N = m + n \text{ and } \frac{\partial^N S_v[\boldsymbol{\omega}]}{\partial z^m \partial w^n}(z_0, w_0) \neq 0\right\}. \end{aligned}$$

The definition does not depend on a choice of a local coordinate function.

Let P and Q be distinct points in X and put $\mu_i = \mu(\alpha_i^P)$ for $i = 1, \dots, g$, so that $G(Q) = \{\mu_1, \dots, \mu_g\}$. Take a basis $\omega_1, \dots, \omega_g$ of $\Omega(X)$ with orders $\operatorname{ord}_P(\omega_i) = \alpha_i^P - 1$ and $\operatorname{ord}_Q(\omega_i) = \mu_i - 1$. We also take local coordinate functions z, w centered at P and Q respectively. Write $\omega_i = f_i\,dz = h_i\,dw$, and further

$$f_i(z) = z^{\alpha_i^P - 1}u_i(z) \quad (u_i(0) \neq 0), \qquad h_i(w) = w^{\mu_i - 1}v_i(w) \quad (v_i(0) \neq 0), \tag{14}$$

$u_i(z)$ and $v_i(w)$ being locally defined holomorphic functions around the origin.

Now let a_{ij} denotes the (i, j)-entry of the square g matrix $(W[\boldsymbol{f}](z)_{1,\dots,g-v}, W[\boldsymbol{h}](w)_{1,\dots,v})$. Then by means of (14) and Leibniz formula for higher derivatives of a product of functions, a_{ij} is given by

$$a_{ij} = \begin{cases} \displaystyle\sum_{k=0}^{j-1}\binom{j-1}{k}(z^{\alpha_i^P - 1})^{(k)}u_i^{(j-1-k)}(z) & (1 \leq j \leq g - v), \\ \displaystyle\sum_{k=0}^{j-g+v-1}\binom{j-1}{k}(w^{\mu_i - 1})^{(k)}v_i^{(j-g+v-1-k)}(w) & (g - v + 1 \leq j \leq g). \end{cases}$$

The lowest order term of a_{ij} is

$$b_{ij} := \begin{cases} (z^{\alpha_i^P - 1})^{(j-1)} u_i(z) & (1 \le j \le g - v), \\ (w^{\mu_i - 1})^{(j-g+v-1)} v_i(w) & (g - v + 1 \le j \le g), \end{cases}$$

some of which may vanish. Therefore $S_v[\omega](z,w) = \det(b_{ij}) + (\text{remainder terms})$. We use the expansion formula (4) to compute $\det(b_{ij})$. Namely,

$$(15)\quad \det(b_{ij}) = \begin{vmatrix} z^{\alpha_1^P - 1} u_1(z) \cdots (z^{\alpha_1^P - 1})^{(g-v-1)} u_1(z) \; w^{\mu_1 - 1} v_1(w) \cdots (w^{\mu_1 - 1})^{(v-1)} v_g(w) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ z^{\alpha_g^P - 1} u_g(z) \cdots (z^{\alpha_g^P - 1})^{(g-v-1)} u_g(z) \; w^{\mu_g - 1} v_g(w) \cdots (w^{\mu_g - 1})^{(v-1)} v_g(w) \end{vmatrix}$$

$$= (-1)^{(g-v)(g-v+1)/2} \sum_{\sigma \in \mathfrak{S}_{g-v,\,v}} (-1)^{\sigma(1)+\cdots+\sigma(g-v)}$$

$$\begin{vmatrix} z^{\alpha_{\sigma(1)}^P - 1} u_{\sigma(1)}(z) & \cdots & (z^{\alpha_{\sigma(1)}^P - 1})^{(g-v-1)} u_{\sigma(1)}(z) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ z^{\alpha_{\sigma(g-v)}^P - 1} u_{\sigma(g-v)}(z) & \cdots & (z^{\alpha_{\sigma(g-v)}^P - 1})^{(g-v-1)} u_{\sigma(g-v)}(z) \end{vmatrix}$$

$$\times \begin{vmatrix} w^{\mu_{\sigma(g-v+1)} - 1} v_{\sigma(g-v+1)}(w) & \cdots & (w^{\mu_{\sigma(g-v+1)} - 1})^{(v-1)} v_{\sigma(g-v+1)}(w) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ w^{\mu_{\sigma(g)} - 1} v_{\sigma(g)}(w) & \cdots & (w^{\mu_{\sigma(g)} - 1})^{(v-1)} v_{\sigma(g)}(w) \end{vmatrix}$$

$$= (-1)^{(g-v)(g-v+1)/2} \sum_{\sigma \in \mathfrak{S}_{g-v,\,v}} (-1)^{\sigma(1)+\cdots+\sigma(g-v)}$$

$$\times u_{\sigma(1)}(z) \cdots u_{\sigma(g-v)}(z) W(z^{\alpha_{\sigma(1)}^P - 1}, \ldots, z^{\alpha_{\sigma(g-v)}^P - 1})$$

$$\times v_{\sigma(g-v+1)}(w) \cdots v_{\sigma(g)}(w) W(w^{\mu_{\sigma(g-v+1)} - 1}, \ldots, w^{\mu_{\sigma(g)} - 1}).$$

The Wronskian determinants of monomials are given as follows. See [BD], for instance.

LEMMA 3.1. *For any sequence $\beta_1, \ldots, \beta_m$ of non-negative integers, the Wronskian determinant of monomials $x^{\beta_1}, \ldots, x^{\beta_m}$ is given by*

$$W(x^{\beta_1}, \ldots, x^{\beta_m}) = \prod_{m \ge k > \ell \ge 1} (\beta_k - \beta_\ell) \cdot x^{\beta_1 + \cdots + \beta_m - m(m-1)/2}$$

Now Lemma 3.1 yields

$$W(z^{\alpha_{\sigma(1)}^P - 1}, \ldots, z^{\alpha_{\sigma(g-v)}^P - 1}) = \prod_{g-v \ge k > l \ge 1} (\alpha_{\sigma(k)}^P - \alpha_{\sigma(l)}^P) \cdot z^{\alpha_{\sigma(1)}^P + \cdots + \alpha_{\sigma(g-v)}^P - (g-v)(g-v+1)/2},$$

$$W(w^{\mu_{\sigma(g-v+1)} - 1}, \ldots, w^{\mu_{\sigma(g)} - 1}) = \prod_{g \ge k > l \ge g-v+1} (\mu_{\sigma(k)} - \mu_{\sigma(l)}) \cdot w^{\mu_{\sigma(g-v+1)} + \cdots + \mu_{\sigma(g)} - v(v+1)/2}.$$

Substituting those Wronskian determinants into (15), we obtain

$$\det(b_{ij}) = (-1)^{(g-v)(g-v+1)/2} \sum_{\sigma \in \mathfrak{S}_{g-v,v}} (-1)^{\sigma(1)+\cdots+\sigma(g-v)}$$
$$\times \prod_{g-v \geq k > l \geq 1} (\alpha^P_{\sigma(k)} - \alpha^P_{\sigma(l)}) \cdot \prod_{g \geq k > l \geq g-v+1} (\mu_{\sigma(k)} - \mu_{\sigma(l)})$$
$$\cdot \prod_{k=1}^{g-v} u_{\sigma(k)}(z) \cdot \prod_{k=g-v+1}^{g} v_{\sigma(k)}(w)$$
$$\times z^{\alpha^P_{\sigma(1)}+\cdots+\alpha^P_{\sigma(g-v)}-(g-v)(g-v+1)/2} w^{\mu_{\sigma(g-v+1)}+\cdots+\mu_{\sigma(g)}-v(v+1)/2}.$$

For this reason, for each shuffle $\sigma \in \mathfrak{S}_{g-v,v}$ and $v = 0, 1, \ldots, g$, we set

(16)
$$C_{v,\sigma} := (-1)^{(g-v)(g-v+1)/2}(-1)^{\sigma(1)+\cdots+\sigma(g-v)}$$
$$\times \prod_{g-v \geq k > l \geq 1} (\alpha^P_{\sigma(k)} - \alpha^P_{\sigma(l)}) \cdot \prod_{g \geq k > l \geq g-v+1} (\mu_{\sigma(k)} - \mu_{\sigma(l)})$$
$$\cdot \prod_{k=1}^{g-v} u_{\sigma(k)}(0) \cdot \prod_{k=g-v+1}^{g} v_{\sigma(k)}(0).$$

We set further, for each sequence $\theta = (\theta_1, \ldots, \theta_g)$ and $v = 0, 1, \ldots, g$,

$$\mathrm{wt}_v\{\theta\} := \sum_{i=1}^{v} (\theta_i - i), \quad \mathrm{wt}^v\{\theta\} := \sum_{i=1}^{v} (\theta_{g-v+i} - i), \quad \mathrm{wt}_0\{\theta\} = \mathrm{wt}^0\{\theta\} := 0,$$

and for a point P, $\mathrm{wt}_v(P) := \mathrm{wt}_v\{\alpha^P\}$, $\mathrm{wt}^v(P) := \mathrm{wt}^v\{\alpha^P\}$. Hence the Weierstrass weight at P equals to $\mathrm{wt}_g(P) = \mathrm{wt}^g(P)$ by definition.

We summarize the argument above as follows.

Proposition 3.2. *In accordance with the notation above, $S_v[\boldsymbol{\omega}](z,w)$ is expanded at the origin as follows.*

(17)
$$S_v[\boldsymbol{\omega}](z,w) = \sum_{\sigma \in \mathfrak{S}_{g-v,v}} C_{v,\sigma} z^{\mathrm{wt}_{g-v}\{\alpha^P_\sigma\}} w^{\mathrm{wt}^v\{\mu_\sigma\}} + R(z,w).$$

The remainder term R satisfies

(1) *the order of R along the z-axis is greater than* $\min_{\sigma \in \mathfrak{S}_{g-v,v}} \{\mathrm{wt}_{g-v}\{\alpha^P_\sigma\}\} = \mathrm{wt}_{g-v}(P)$,

(2) *the order of R along the w-axis is greater than* $\min_{\sigma \in \mathfrak{S}_{g-v,v}} \{\mathrm{wt}^v\{\mu_\sigma\}\} = \mathrm{wt}^v(Q)$.

Moreover we have the inequality

(3) $\mathrm{ord}_{(P,Q)}(S_v[\boldsymbol{\omega}]) \geq \min_{\sigma \in \mathfrak{S}_{g-v,v}} \{\mathrm{wt}_{g-v}\{\alpha^P_\sigma\} + \mathrm{wt}^v\{\mu_\sigma\}\}$.

The pull-back of the point divisor P by the projection $\pi_i : X \times X \to X$ is denoted by $\pi_i^* P$. Then Proposition 3.2(1), (2) are interpreted as

Corollary 3.3. *The orders of* $\boldsymbol{S}_\nu[\boldsymbol{\omega}]$ *along* π_1^*P *and* π_2^*Q *are given by*

$$\mathrm{ord}_{\pi_1^*P}(\boldsymbol{S}_\nu[\boldsymbol{\omega}]) = \mathrm{wt}_{g-\nu}(P) \quad \textit{and} \quad \mathrm{ord}_{\pi_2^*Q}(\boldsymbol{S}_\nu[\boldsymbol{\omega}]) = \mathrm{wt}^\nu(Q).$$

Especially the gap sequences at P and Q can be obtained by the following formula.

$$\begin{cases} \alpha_1^P = \mathrm{ord}_{\pi_1^*P}(\boldsymbol{S}_{g-1}[\boldsymbol{\omega}]) + 1, \\ \alpha_2^P = \mathrm{ord}_{\pi_1^*P}(\boldsymbol{S}_{g-2}[\boldsymbol{\omega}]) - \mathrm{ord}_{\pi_1^*P}(\boldsymbol{S}_{g-1}[\boldsymbol{\omega}]) + 2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_g^P = \mathrm{ord}_{\pi_1^*P}(\boldsymbol{S}_0[\boldsymbol{\omega}]) - \mathrm{ord}_{\pi_1^*P}(\boldsymbol{S}_1[\boldsymbol{\omega}]) + g. \end{cases}$$

$$\begin{cases} \alpha_1^Q = \mathrm{ord}_{\pi_2^*Q}(\boldsymbol{S}_1[\boldsymbol{\omega}]) + 1, \\ \alpha_2^Q = \mathrm{ord}_{\pi_2^*Q}(\boldsymbol{S}_2[\boldsymbol{\omega}]) - \mathrm{ord}_{\pi_2^*Q}(\boldsymbol{S}_1[\boldsymbol{\omega}]) + 2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_g^Q = \mathrm{ord}_{\pi_2^*Q}(\boldsymbol{S}_g[\boldsymbol{\omega}]) - \mathrm{ord}_{\pi_2^*Q}(\boldsymbol{S}_{g-1}[\boldsymbol{\omega}]) + g. \end{cases}$$

Proof. Because $C_{\nu,\sigma} \neq 0$ for any ν, σ by definition (16), those orders are verified from (17) and Proposition 3.2(1) and (2). The expressions of α_i^P and α_i^Q are obtained from the definition of wt_ν and wt^ν. ◇

When $\nu = 0$ or $\nu = g$, these mean that $\mathrm{ord}_P(W(\boldsymbol{f})) = \mathrm{wt}(P)$ or $\mathrm{ord}_Q(W(\boldsymbol{h})) = \mathrm{wt}(Q)$ respectively, which is a classical result in the theory of Weierstrass points. On the diagonal set ΔX, the orders of $\boldsymbol{S}_\nu[\boldsymbol{\omega}]$ are given explicitly.

Corollary 3.4. *For each* $\nu = 1, \dots, g-1$, *we have the following.*
(1) $\boldsymbol{S}_\nu[\boldsymbol{\omega}](P, P) = 0$ *for all* $P \in X$.
(2) $\mathrm{ord}_{(P,P)}(\boldsymbol{S}_\nu[\boldsymbol{\omega}]) = \mathrm{wt}(P) + \nu(g-\nu)$.
(3) $\mathrm{ord}_{\Delta X}(\boldsymbol{S}_\nu[\boldsymbol{\omega}]) = \nu(g-\nu)$.

Proof. We take a local coordinate function (z, w) centered at (P, P) to be $w = z$, then (14) turns to

$$f_i(z) = z^{\alpha_i^P - 1} u_i(z), \quad h_i(w) = w^{\alpha_i^P - 1} u_i(w) \quad (u_i(0) \neq 0).$$

Hence the expansion (17) turns to

$$S_\nu(z, w) = \sum_{\sigma \in \mathfrak{S}_{g-\nu,\nu}} C_{\nu,\sigma} z^{\mathrm{wt}_{g-\nu}\{\alpha_\sigma^P\}} w^{\mathrm{wt}^\nu\{\alpha_\sigma^P\}} + R(z, w). \tag{18}$$

In this case, the degree of $z^{\mathrm{wt}_{g-\nu}\{\alpha_\sigma^P\}} w^{\mathrm{wt}^\nu\{\alpha_\sigma^P\}}$ equals to $\mathrm{wt}(P) + \nu(g-\nu)$ for each σ, which is positive because $0 < \nu < g$ by assumption. Therefore Proposition 3.2(3) implies $\mathrm{ord}_{(P,P)}(S_\nu[\boldsymbol{\omega}]) \geq \mathrm{wt}(P) + \nu(g-\nu) > 0$, which shows (1). Practically we have the equality (2). For in the summation (18), the coefficient of $z^{\mathrm{wt}_{g-\nu}(P)} w^{\mathrm{wt}^\nu(P)}$ is just $C_{\nu,\mathrm{id}}$, id being the identity permutation, that is not zero by definition (16). Thus we obtain (2). See also Remark 3.1 below. Finally, (3) comes from (2), because $\mathrm{wt}(P) = 0$ for all but finitely many P's. ◇

Remark 3.1. It seems likely to be true that the equality in Proposition 3.2(3) holds, although it is not proved at present. We make some remarks when this is the case.

To this end, we put $d_v^{P,Q}(\sigma) := \mathrm{wt}_{g-v}\{\alpha_\sigma^P\} + \mathrm{wt}^v\{\mu_\sigma\}$ for each $\sigma \in \mathfrak{S}_{g-v,v}$ and also $d_v^{P,Q} := \min_{\sigma \in \mathfrak{S}_{g-v,v}}\{d_v^{P,Q}(\sigma)\}$. Thus the problem is whether or not the equality $\mathrm{ord}_{(P,Q)}(S_v[\boldsymbol{\omega}]) = d_v^{P,Q}$ holds. Take $\sigma_0 \in \mathfrak{S}_{g-v,v}$ with $d_v^{P,Q}(\sigma_0) = d_v^{P,Q}$. For a shuffle $\sigma \in \mathfrak{S}_{g-v,v}$, we denote $\sigma \sim \sigma_0$ when

$$\mathrm{wt}_{g-v}\{\alpha_\sigma^P\} = \mathrm{wt}_{g-v}\{\alpha_{\sigma_0}^P\} \quad \text{and} \quad \mathrm{wt}^v\{\mu_\sigma\} = \mathrm{wt}^v\{\mu_{\sigma_0}\}.$$

Then the coefficient of $z^{\mathrm{wt}_{g-v}\{\alpha_{\sigma_0}^P\}} w^{\mathrm{wt}^v\{\mu_{\sigma_0}\}}$ in the summation (17) is $\sum_{\sigma \sim \sigma_0} C_{v,\sigma}$. Therefore the equality in Proposition 3.2(3) holds if and only if there exists such a σ_0 for which $\sum_{\sigma \sim \sigma_0} C_{v,\sigma}$ does not vanish. The following are when this is the case.

(1) There is no σ with $\sigma \sim \sigma_0$ other than σ_0 itself.
(2) σ_0 can be taken as the idendity permutation. This is the case of Corollary 3.4(2).
(3) $v = 1$ or $v = g - 1$.
(4) Because of (3), the equality holds for each v when the genus of X is three.

4. Examples

In this section, we shall apply the argument in the preceding sections to some examples, that is, the hyperelliptic curves and the Fermat curve of genus three.

4.1. Hyperelliptic curves

Let Y be a hyperelliptic curve of genus g. This is a compactification of an affine plane curve $y^2 = a(x - b_1)(x - b_2)\cdots(x - b_{2g+2})$ with two points at infinity. Here $b_1, b_2, \ldots, b_{2g+2}$ are distinct complex numbers and a a non-zero complex number. The Weierstrass points on Y are exactly $(b_1, 0), \ldots, (b_{2g+2}, 0)$. We denote by σ the hyperelliptic involution of Y. That is defined by $\sigma(x, y) = (x, -y)$.

Now put $\omega_\alpha = x^\alpha\, dx/y$ $(\alpha = 0, \ldots, g-1)$ on Y. Then those 1-forms become a basis of $\Omega(Y)$. Let $((x, y), (\xi, \eta))$ be a point on $Y \times Y$. When $y \neq 0$ and $\eta \neq 0$, we can take (x, ξ) as a local coordinate function around there. We put $f_\alpha := x^\alpha/y$ and $h_\alpha := \xi^\alpha/\eta$ away from $y = 0$ and $\eta = 0$ respectively. Using the expansion formula (4) to obtain

$$\begin{aligned} S_v[\boldsymbol{\omega}]((x,y),(\xi,\eta)) &= |W[\boldsymbol{f}]_{1,\ldots,g-v}, W[\boldsymbol{h}]_{1,\ldots,g-v}| \\ &= (-1)^{(g-v)(g-v+1)} \sum_{\sigma \in \mathfrak{S}_{g-v,v}} (-1)^{\sigma(1)+\cdots+\sigma(g-v)} \\ &\quad \times |W[\boldsymbol{f}]_{1,\ldots,g-v}^{\sigma(1),\ldots,\sigma(g-v)}|\,|W[\boldsymbol{h}]_{1,\ldots,g-v}^{\sigma(g-v+1),\ldots,\sigma(g)}|, \end{aligned} \tag{19}$$

and substituting

$$|W[\boldsymbol{f}]_{1,\dots,g-\nu}^{\sigma(1),\dots,\sigma(g-\nu)}| = W(x^{\sigma(1)-1}/y,\dots,x^{\sigma(g-\nu)-1}/y)$$
$$= W(x^{\sigma(1)-1},\dots,x^{\sigma(g-\nu)-1})/y^{g-\nu},$$
$$|W[\boldsymbol{h}]_{1,\dots,g-\nu}^{\sigma(g-\nu+1),\dots,\sigma(g)}| = W(\xi^{\sigma(g-\nu+1)-1}/\eta,\dots,\xi^{\sigma(g)-1}/\eta)$$
$$= W(\xi^{\sigma(g-\nu+1)-1},\dots,\xi^{\sigma(g)-1})/\eta^{\nu}$$

into (19) and using (4) again, we have

$$(20)\quad S_\nu[\boldsymbol{\omega}]((x,y),(\xi,\eta)) = \frac{1}{y^{g-\nu}\eta^{\nu}}\underbrace{\begin{vmatrix} 1 & 0 & \cdots \\ x & 1 & \cdots \\ x^2 & 2x & \cdots \\ \cdots & \cdots & \cdots \\ x^{g-1} & (g-1)x^{g-2} & \cdots \end{vmatrix}}_{g-\nu \text{ columns}}\!\!\underbrace{\begin{vmatrix} 1 & 0 & \cdots \\ \xi & 1 & \cdots \\ \xi^2 & 2\xi & \cdots \\ \cdots & \cdots & \cdots \\ \xi^{g-1} & (g-1)\xi^{g-2} & \cdots \end{vmatrix}}_{\nu \text{ columns}}$$

$$= C\frac{(\xi - x)^{\nu(g-\nu)}}{y^{g-\nu}\eta^{\nu}},$$

where $C = (\prod_{k=0}^{g-\nu-1} k!)(\prod_{k=0}^{\nu-1} k!)$.

At a point, say, where $y \neq 0$ and $\eta = 0$, taking (x,η) as a local coordinate function, (20) and the transition low (11) imply

$$S_\nu[\boldsymbol{\omega}]((x,y),(\xi,\eta)) = C\left(\frac{\xi'}{\eta}\right)^{\nu}\frac{(\xi - x)^{\nu(g-\nu)}(\xi')^{\nu(\nu-1)/2}}{y^{g-\nu}}.$$

Here $\xi' = d\xi/d\eta$. Note that $\mathrm{ord}_{\eta=0}(\xi') = 1$ and hence ξ'/η does not vanish at $\eta = 0$.

Let Σ be the curve in $Y \times Y$ defined by the equation $\xi - x = 0$, which contains precisely the diagonal set ΔY and the graph of the hyperelliptic involution σ as the irreducible components. Then because of (20), we find that the orders of $\boldsymbol{S}_\nu[\boldsymbol{\omega}]$ are given by the table below.

P, Q		$\mathrm{ord}_{(P,Q)}\, \boldsymbol{S}_\nu[\boldsymbol{\omega}]$
Both P and Q are Weierstrass points	$(P,Q) \in \Sigma$	$g(g-1)/2 + \nu(\nu-1)$
	$(P,Q) \notin \Sigma$	$\{(g-\nu)(g-\nu-1) + \nu(\nu-1)\}/2$
Q is a Weierstrass point and P is not		$\nu(\nu-1)/2$
P is a Weierstrass point and Q is not		$(g-\nu)(g-\nu-1)/2$
Both P and Q are not Weierstrass points	$(P,Q) \in \Sigma$	$\nu(g-\nu)$
	$(P,Q) \notin \Sigma$	0

The gap sets for pairs of points in a hyperelliptic curve have been well investigated in [K] or [H]. Compare also the table with one in [G, p. 334]. As a consequence, we know that $t(\mu)$ takes the extremal values only, that is, $t(\mu) = 0$ (minimum) or $g(g-1)/2$ (maximum) at each pair of points in $Y \times Y$. This fact verifies that the equality in Proposition 3.2(3) holds for each (P, Q). This is the case of Remark 3.1.(2).

4.2. Fermat curve of genus three

Let C be the Fermar curve of genus three over the field of complex numbers. Namely C is a smooth projective plane curve of degree four defined by the equation $x_0^4 + x_1^4 + x_2^4 = 0$, where x_0, x_1, x_2 are the homogeneous coordinates of $\mathbf{P}^2$. For $U = \{[x_0 : x_1 : x_2] \in \mathbf{P}^2 \mid x_2 \neq 0\}$, $C \cap U$ is isomorphic to the affine plane curve A defined by $f(z_0, z_1) = z_0^4 + z_1^4 + 1 = 0$, where $z_0 = x_0/x_2$, $z_1 = x_1/x_2$ are the inhomogeneous coordinates of $\mathbf{C}^2$. We define three 1-forms on A by

$$\omega_1 := dz_0/z_1^3, \quad \omega_2 := dz_0/z_1^2, \quad \omega_3 := z_0 dz_0/z_1^3.$$

Then each of these 1-forms becomes a holomorphic 1-form on C, moreover ω_1, ω_2, ω_3 form a basis of $\Omega(C)$. See Miranda [M, p. 112], for construction of these 1-forms. Let ζ_j $(j = 1, 2, 3, 4)$ be the 4-th roots of -1, then the divisors associated to those 1-forms are given by $(\omega_1) = \sum_{j=1}^4 [\zeta_j : 1 : 0]$, $(\omega_2) = \sum_{j=1}^4 [\zeta_j : 0 : 1]$, $(\omega_3) = \sum_{j=1}^4 [0 : \zeta_j : 1]$.

Now we compute the functions $S_v[\boldsymbol{\omega}]$ $(v = 0, 1, 2, 3)$ on $A \times A$. When $z_1 \neq 0$ and $w_1 \neq 0$, we can take (z_0, w_0) as a local coordinate function of $A \times A$. Then elementary calculations yield the following.

$$\begin{aligned} S_0[\boldsymbol{\omega}]((z_0, z_1), (w_0, w_1)) &= -3z_0^2/z_1^{16}, \\ S_1[\boldsymbol{\omega}]((z_0, z_1), (w_0, w_1)) &= -(z_0^3 w_0 + z_1^3 w_1 + 1)/(z_1^9 w_1^3), \\ S_2[\boldsymbol{\omega}]((z_0, z_1), (w_0, w_1)) &= -(z_0 w_0^3 + z_1 w_1^3 + 1)/(z_1^3 w_1^9), \\ S_3[\boldsymbol{\omega}]((z_0, z_1), (w_0, w_1)) &= -3w_0^2/w_1^{16}. \end{aligned} \tag{21}$$

With respect to the other local coordinate functions, say, at the point where $z_0 \neq 0$ and $w_0 \neq 0$, we have the expression $S_1[\boldsymbol{\omega}]((z_0, z_1), (w_0, w_1)) = -(z_0^3 w_0 + z_1^3 w_1 + 1)/(z_0^9 w_0^3)$ for instance. $S_0[\boldsymbol{\omega}]((z_0, z_1), (w_0, w_1))$ is nothing but the Wronskian determinant $W(\boldsymbol{\omega})(z_0, z_1)$, which is a holomorphic section of $K_C^{\otimes 6}$. Therefore we find from (21) that the divisor associated to $W(\boldsymbol{\omega})$ is given by

$$(W(\boldsymbol{\omega})) = \sum_{j=1}^4 2[\zeta_j : 1 : 0] + \sum_{j=1}^4 2[\zeta_j : 0 : 1] + \sum_{j=1}^4 2[0 : \zeta_j : 1].$$

This means that there are exactly twelve Weierstrass points $[\zeta_j : 1 : 0]$, $[\zeta_j : 0 : 1]$ and $[0 : \zeta_j : 1]$ $(j = 1, 2, 3, 4)$, all of whose weight are two. According to Kuribayashi and Komiya [KK], there are exactly two compact Riemann surfaces of genus three that have twelve Weierstrass points.

Let $P = (z_0, z_1)$ be a point in A with, say, $z_0 \neq 0$. Then by means of (21), the order of $\boldsymbol{S}_v[\boldsymbol{\omega}]$ along $\pi_1^* P$ is given as follow.

- When $z_1 \neq 0$, $\mathrm{ord}_{\pi_1^* P}(S_\nu[\omega]) = 0$ for all $\nu = 0, 1, 2$.
- When $z_1 = 0$, $\mathrm{ord}_{\pi_1^* P}(S_0[\omega]) = 2$ and $\mathrm{ord}_{\pi_1^* P}(S_\nu[\omega]) = 0$ for $\nu = 1, 2$.

In the former case, by virture of Corollary 3.3, the gap sequence at P is $\alpha^P = (1, 2, 3)$, namely P is a non-Weierstrass point. In the latter case, the gap sequence is likewise $\alpha^P = (1, 2, 5)$, namely P is a Weierstrass point. Therefore, as mentioned above, the weight of every Weierstrass point equals to two.

Next let (P, Q) be a point in $C \times C$ with $P \neq Q$. We shall explain how the sequence $\mu_j = \mu(\alpha_j^P)$ $(j = 1, 2, 3)$ can be obtained from the orders of $S_\nu[\omega]$ $(\nu = 0, 1, 2, 3)$ at the point (P, Q). The key fact is the equality

$$\mathrm{ord}_{(P,Q)}(S_\nu[\omega]) = \min_{\sigma \in \mathfrak{S}_{3-\nu,\nu}} \{\mathrm{wt}_{g-\nu}\{\alpha_\sigma^P\} + \mathrm{wt}^\nu\{\mu_\sigma\}\}, \tag{22}$$

which holds in the present case (genus three) as mentioned in Remark 3.1.(4). As before, we assume the point (P, Q) to be contained in $A \times A$. We also assume that both P and Q are non-Weierstrass points. Then $\mathrm{ord}_{(P,Q)}(S_0[\omega]) = \mathrm{ord}_{(P,Q)}(S_3[\omega]) = 0$. Because of (21), the orders of $S_1[\omega]$ and $S_2[\omega]$ at (P, Q) equal to 0 or 1. When $\mathrm{ord}_{(P,Q)}(S_1[\omega]) = 0$, (22) implies $(\alpha_{\sigma(1)}^P - 1) + (\alpha_{\sigma(2)}^P - 2) + (\mu_{\sigma(3)} - 1) = 0$ for some $\sigma \in \mathfrak{S}_{2,1}$. Then $\sigma = (1, 2, 3)$ and $\mu_3 = 1$. When $\mathrm{ord}_{(P,Q)}(S_2[\omega]) = 1$, (22) implies $(\alpha_{\sigma(1)}^P - 1) + (\alpha_{\sigma(2)}^P - 2) + (\mu_{\sigma(3)} - 1) = 1$ for some $\sigma \in \mathfrak{S}_{1,2}$. Then $\sigma = (1, 2, 3)$ and $(\mu_2 - 1) + (\mu_3 - 2) = 1$ or $\sigma = (2, 1, 3)$ and $(\mu_1 - 1) + (\mu_3 - 2) = 0$. Therefore if $\mathrm{ord}_{(P,Q)} S_1 = 0$ and $\mathrm{ord}_{(P,Q)} S_2 = 1$, we obtain $\mu = (2, 3, 1)$, and then $\#G(P, Q) = \mathrm{wt}(P) + \mathrm{wt}(Q) - t(\mu) + 12 = 0 + 0 - 2 + 12 = 10$ by means of (3). In all other cases, we can likewise obtain the sequence μ_j at each point.

We denote by Σ_ν $(\nu = 1, 2)$ the zero sets of $\boldsymbol{S}_\nu[\omega]$, which are defined by the following equations with respect to the homogeneous coordinates in $\mathbf{P}^2 \times \mathbf{P}^2$.

$$\Sigma_1 : x_0^3 y_0 + x_1^3 y_1 + x_2^3 y_2 = 0, \quad \Sigma_2 : x_0 y_0^3 + x_1 y_1^3 + x_2 y_2^3 = 0.$$

Therefore the argument above is summarized as the table below.

P, Q $(P \neq Q)$		μ_1, μ_2, μ_3	$t(\mu)$	$\#G(P, Q)$
Both P and Q are Weierstrass points		5, 2, 1	3	13
P is not a Weierstrass point and Q a Weierstrass point	$(P, Q) \in \Sigma_1 \backslash \Sigma_2$	5, 1, 2	2	12
	$(P, Q) \notin \Sigma_1 \cup \Sigma_2$	5, 2, 1	3	11
Both P and Q are not Weierstrass point	$(P, Q) \in \Sigma_1 \cap \Sigma_2$	1, 3, 2 2, 1, 3	1	11
	$(P, Q) \in \Sigma_1 \backslash \Sigma_2$	3, 1, 2	2	10
	$(P, Q) \in \Sigma_2 \backslash \Sigma_1$	2, 3, 1	2	10
	$(P, Q) \notin \Sigma_1 \cup \Sigma_2$	3, 2, 1	3	9

The final comment: The examples above suggest that the holomorphic sections $\boldsymbol{S}_\nu$ $(\nu = 0, 1, \ldots, g)$ are closely related to the structure of the gap sets. For example, it seems to be reasonable to conjecture that $\#G(P, Q)$ remains constant on each Σ_ν, the zero set of $\boldsymbol{S}_\nu$, except at the intersections with the other Σ_μ's.

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