Z. SUN
KODAI MATH. J.
36 (2013), 508–526

PARTIAL GENERALIZATIONS OF SOME CONJECTURES IN LORENTZIAN MANIFOLDS

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Abstract

In this paper, we mainly investigate complete or compact spacelike hypersurfaces with constant mean curvature or constant scalar curvature in Lorentzian manifolds L_1^{n+1}. We give a new estimate of the Laplacian $\triangle S$ of the squared length S of the second fundamental form of such spacelike hypersurfaces. Finally, we give partial generalizations of some Conjectures in Lorentzian manifolds L_1^{n+1}.

1. Introduction

In 1981, S. Stumbles [19] pointed out that spacelike hypersurfaces with constant mean curvature in arbitrary spacetimes are interesting in the relativity. Therefore, complete spacelike hypersurfaces with constant mean curvature in a Lorentz space form $\overline{M}_1^{n+1}(c)$ are studied by many geometers. For example, A. J. Goddard [8] proposed the following Conjecture:

Conjecture 1. *If M^n is a complete spacelike hypersurface of de Sitter space $\mathbf{S}_1^{n+1}(c)$ with constant mean curvature H, then is M^n totally umbilical?*

When $H^2 \leqslant c$ if $n = 2$ or when $n^2H^2 < 4(n-1)c$ if $n \geqslant 3$, K. Akutagawa [1] and J. Ramanathan [18] proved that Goddard's conjecture is true. S. Montiel [13] solved Goddard's problem without restriction over the range of H provided that M^n is compact. For further study in this direction, there are many results such as [10, 14].

In 2004, J. Ok Baek, Q. M. Cheng and Y. Jin Suh [15] studied complete spacelike hypersurfaces with constant mean curvature in a locally symmetric Lorentzian manifold L_1^{n+1} and obtained some rigidity theorems.

On the other hand, concerning the study of spacelike hypersurfaces with constant scalar curvature in a de Sitter space $\mathbf{S}_1^{n+1}(c)$, H. Li proposed an interesting problem:

Key words and phrases. Lorentzian manifolds, Constant mean curvature, Constant scalar curvature, Spacelike hypersurfaces, Second fundamental form.

Received October 16, 2012; revised March 1, 2013.

CONJECTURE 2. *If M^n $(n \geqslant 3)$ is a complete spacelike hypersurface in de Sitter space $\mathbf{S}_1^{n+1}(1)$ with constant normalized scalar curvature R satisfying $\frac{n-2}{n} \leqslant R \leqslant 1$, then is M^n totally umbilical?*

Recently, F. E. C. Camargo et al. [4] proved that Li's question is true if M^n has bounded mean curvature H. For further study in this direction, there are many results such as [3, 5] and [9].

In 2010, J. C. Liu and Z. Y. Sun [12] studied complete or compact spacelike hypersurfaces with constant normalized scalar curvature R in a locally symmetric Lorentzian manifold L_1^{n+1} and obtained some rigidity theorems.

It is natural to study complete or compact spacelike hypersurfaces with constant mean curvature or constant scalar curvature in Lorentzian manifolds L_1^{n+1}.

In Section 3, we give generalizations of [15, Theorem 1 (1)] and [1, Theorem] in Lorentzian manifolds L_1^{n+1}. Thus, we obtain Theorems 3.2 and 3.3.

In Section 4, we give generalizations of [12, Theorems 1.1–1.2(i)] in Lorentzian manifolds L_1^{n+1}. Thus, we get Theorems 4.4 and 4.5.

In order to prove our results, we need some basic facts and notations. First we recall that, for some constants c_1, c_2 and c_3, Jin Ok Baek et al. [15] introduced the class of $(n+1)$-dimensional Lorentz spaces L_1^{n+1} which satisfy the following conditions:

(i) for any spacelike vector u and any timelike vector v

$$K(u,v) = -\frac{c_1}{n}, \tag{1.1}$$

(ii) for any spacelike vectors u and v

$$K(u,v) \geqslant c_2, \tag{1.2}$$

(iii)

$$|\bar{\nabla}\bar{R}| \leqslant \frac{c_3}{n}, \tag{1.3}$$

where $\bar{\nabla}$, K and $\bar{R}$ denote semi-Riemannian connection, sectional curvature and the curvature tensor on L_1^{n+1}, respectively.

When L_1^{n+1} satisfies conditions (1.1) and (1.2), we will say that L_1^{n+1} satisfies condition $(*)$.

When L_1^{n+1} satisfies conditions (1.1) and (1.2) and (1.3), we will say that L_1^{n+1} satisfies condition $(**)$.

Remark 1.1. It can be easily seen that $c_3 = 0$, then the Lorentzian manifold L_1^{n+1} is *locally symmetric*.

Remark 1.2. The Lorentz space form $\overline{M}_1^{n+1}(c) \in L_1^{n+1}$ satisfies the condition $(*)$, where $c = -\frac{c_1}{n} = c_2$.

This class of Lorentzian manifolds L_1^{n+1} contains several examples. For example, semi-Riemannian product manifold $H_1^k(-c_1/n) \times N^{n+1-k}(c_2)$, $c_1 > 0$, and $\mathbf{R}_1^k \times S^{n+1-k}(1)$. Particularly, $\mathbf{R}_1^1 \times S^n(1)$ is the so-called *Einstein Static Universe*. Of course, it is not a Lorentz space form. For more details, we refer the readers to [6, 15] and [20].

2. Preliminaries

In this section, we give a new estimate of the Laplacian $\triangle S$ of the squared length S of the second fundamental form for spacelike hypersurfaces in Lorentzian manifolds L_1^{n+1} satisfying $(**)$. We will use the following convention for the indices throughout this paper: $1 \leqslant A, B, C, \ldots \leqslant n+1; 1 \leqslant i, j, k, \ldots \leqslant n$.

We assume that M^n is a spacelike hypersurface in Lorentzian manifolds L_1^{n+1}. Choose a local field of pseudo-Riemannian orthonormal frames $\{e_1, \ldots, e_{n+1}\}$ in L_1^{n+1} such that, restricted to M^n, $\{e_1, \ldots, e_n\}$ are tangent to M^n and e_{n+1} is normal to M^n. That is, $\{e_1, \ldots, e_n\}$ are spacelike vectors and e_{n+1} is a timelike vector. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the fields of dual frames and the connection forms of L_1^{n+1}, respectively. Let $\varepsilon_i = 1$, $\varepsilon_{n+1} = -1$, then the structure equations of L_1^{n+1} are given by

$$d\omega_A = -\sum_B \varepsilon_A \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$d\omega_{AB} = -\sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum_{C,D} \varepsilon_C \varepsilon_D \bar{R}_{ABCD} \omega_C \wedge \omega_D.$$

Here the components $\bar{R}_{CD}$ of the Ricci tensor and the scalar curvature $\bar{R}$ of Lorentzian manifolds L_1^{n+1} are given, respectively, by

$$\bar{R}_{CD} = \sum_B \varepsilon_B \bar{R}_{BCDB}, \quad \bar{R} = \sum_A \varepsilon_A \bar{R}_{AA}.$$

Let $\bar{R}_{CD}$ be the components of the Ricci tensor of L_1^{n+1} satisfying $(*)$, then the scalar curvautre $\bar{R}$ of L_1^{n+1} is given by

$$\bar{R} = \sum_{A=1}^{n+1} \varepsilon_A \bar{R}_{AA} = -2\sum_{i=1}^{n} \bar{R}_{(n+1)ii(n+1)} + \sum_{i,j=1}^{n} \bar{R}_{ijji} = 2c_1 + \sum_{i,j=1}^{n} \bar{R}_{ijji}. \tag{2.1}$$

It is well known that $\bar{R}$ is constant when the Lorentzian manifold L_1^{n+1} is locally symmetric. Together with (2.1), we know that $\sum_{i,j=1}^{n} \bar{R}_{ijji}$ is constant. Hence, when M^n is a spacelike hypersurface in locally symmetric Lorentzian manifolds L_1^{n+1} satisfying $(*)$, we conclude from (2.5) in Section 2 that the normalized scalar curvature R of M^n is constant if and only if P is constant.

The components $\bar{R}_{ABCD;E}$ of the covariant derivative of the Riemannian curvature tensor $\bar{R}$ are defined by

$$\sum_E \varepsilon_E \bar{R}_{ABCD;E} \omega_E = d\bar{R}_{ABCD} - \sum_E \varepsilon_E (\bar{R}_{EBCD}\omega_{EA} + \bar{R}_{AECD}\omega_{EB} + \bar{R}_{ABED}\omega_{EC} + \bar{R}_{ABCE}\omega_{ED}).$$

We restrict these forms to M^n in L_1^{n+1}, then $\omega_{n+1} = 0$. Hence, we have $\sum_i \omega_{(n+1)i} \wedge \omega_i = 0$. Using Cartan's lemma, we know that there are h_{ij} such that $\omega_{(n+1)i} = \sum_j h_{ij}\omega_j$ and $h_{ij} = h_{ji}$, where the h_{ij} are the coefficients of the second fundamental form of M^n. This gives the second fundamental form of M^n, $h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j$.

The Gauss equation, components R_{ij} of the Ricci tensor and the normalized scalar curvature R of M^n are given, respectively, by

$$R_{ijkl} = \bar{R}_{ijkl} - (h_{il}h_{jk} - h_{ik}h_{jl}), \tag{2.2}$$

$$R_{ij} = \sum_k \bar{R}_{kijk} - nHh_{ij} + \sum_k h_{ik}h_{kj}, \tag{2.3}$$

$$n(n-1)R = \sum_{i,j} \bar{R}_{ijji} - n^2H^2 + S, \tag{2.4}$$

where $H = \frac{1}{n}\sum_j h_{jj}$ and $S = \sum_{i,j} h_{ij}^2$ are the mean curvature and the squared length of the second fundamental form of M^n, respectively.

From (2.4), we can define a P such that

$$n(n-1)P = n^2H^2 - S = \sum_{i,j=1}^{n} \bar{R}_{ijji} - n(n-1)R. \tag{2.5}$$

Let h_{ijk}, h_{ijkl} denote the first and the second covariant derivatives of h_{ij}, respectively, so that

$$\sum_k h_{ijk}\omega_k = dh_{ij} - \sum_k h_{ik}\omega_{kj} - \sum_k h_{kj}\omega_{ki},$$

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} - \sum_l h_{ljk}\omega_{li} - \sum_l h_{ilk}\omega_{lj} - \sum_l h_{ijl}\omega_{lk}.$$

Thus, we have the Codazzi equation and the Ricci identity

$$h_{ijk} - h_{ikj} = \bar{R}_{(n+1)ijk}, \tag{2.6}$$

$$h_{ijkl} - h_{ijlk} = -\sum_m h_{im}R_{mjkl} - \sum_m h_{jm}R_{mikl}. \tag{2.7}$$

Let $\bar{R}_{ABCD;E}$ be the covariant derivative of $\bar{R}_{ABCD}$. Thus, restricted on M^n, $\bar{R}_{(n+1)ijk;l}$ is given by

$$\bar{R}_{(n+1)ijk;l} = \bar{R}_{(n+1)ijkl} + \bar{R}_{(n+1)i(n+1)k}h_{jl} + \bar{R}_{(n+1)ij(n+1)}h_{kl} + \sum_m \bar{R}_{mijk}h_{ml}, \tag{2.8}$$

where $\bar{R}_{(n+1)ijk;l}$ denotes the covariant derivative of $\bar{R}_{(n+1)ijk}$ as a tensor on M^n so that

$$\sum_l \bar{R}_{(n+1)ijk;l}\omega_l = d\bar{R}_{(n+1)ijk} - \sum_l \bar{R}_{(n+1)ljk}\omega_{li} - \sum_l \bar{R}_{(n+1)ilk}\omega_{lj} - \sum_l \bar{R}_{(n+1)ijl}\omega_{lk}.$$

Next we compute the Laplacian $\triangle h_{ij} = \sum_k h_{ijkk}$. From (2.6) and (2.7), we have

$$\begin{aligned}\triangle h_{ij} &= \sum_k h_{ikjk} + \bar{R}_{(n+1)ijk;k} \\ &= \sum_k \left(h_{kikj} - \sum_l (h_{kl}R_{lijk} + h_{il}R_{lkjk}) + \bar{R}_{(n+1)ijk;k} \right).\end{aligned}$$

From $h_{kikj} = h_{kkij} + \bar{R}_{(n+1)kik;j}$, we get

$$\triangle h_{ij} = (nH)_{ij} + \sum_k (\bar{R}_{(n+1)ijk;k} + \bar{R}_{(n+1)kik;j}) - \sum_{k,l} (h_{kl}R_{lijk} + h_{il}R_{lkjk}). \tag{2.9}$$

From (2.2) and (2.8) and (2.9), we have

$$\begin{aligned}\triangle h_{ij} = (nH)_{ij} &+ \sum_k (\bar{R}_{(n+1)ijk;k} + \bar{R}_{(n+1)kik;j}) \\ &- \sum_k (h_{kk}\bar{R}_{(n+1)ij(n+1)} + h_{ij}\bar{R}_{(n+1)k(n+1)k}) \\ &- \sum_{k,l} (2h_{kl}\bar{R}_{lijk} + h_{jl}\bar{R}_{lkik} + h_{il}\bar{R}_{lkjk}) - nH\sum_l h_{il}h_{lj} + Sh_{ij}.\end{aligned}$$

According to the above equation, the Laplacian of the squared length S of the second fundamental form h_{ij} of M^n is obtained

$$\begin{aligned}\frac{1}{2}\triangle S &= \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij}\triangle h_{ij} \\ &= \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} (nH)_{ij}h_{ij} + \sum_{i,j,k} (\bar{R}_{(n+1)ijk;k} + \bar{R}_{(n+1)kik;j})h_{ij} \\ &\quad - \left(\sum_{i,j} nHh_{ij}\bar{R}_{(n+1)ij(n+1)} + S\sum_k \bar{R}_{(n+1)k(n+1)k} \right) \\ &\quad - 2\sum_{i,j,k,l} (h_{kl}h_{ij}\bar{R}_{lijk} + h_{il}h_{ij}\bar{R}_{lkjk}) - nH\sum_{i,j,l} h_{il}h_{lj}h_{ij} + S^2.\end{aligned} \tag{2.10}$$

Next, we will estimate the right-hand side of (2.10) by using the curvature conditions (**). We choose a local orthonormal frame field $\{e_1, \dots, e_n\}$ such that $h_{ij} = \lambda_i \delta_{ij}$, where λ_i, $1 \leqslant i \leqslant n$, are principal curvatures of M^n. By definition, we see

$$\lambda_i^2 \leqslant S = \sum_i \lambda_i^2$$

and hence we have

$$-\sqrt{S} \leqslant \lambda_i \leqslant \sqrt{S}. \tag{2.11}$$

First of all, we treat with the third term of (2.10). It is seen that we have

$$\begin{aligned} \sum_{i,j,k} (\bar{R}_{(n+1)ijk;k} + \bar{R}_{(n+1)kik;j}) h_{ij} &= \sum_{j,k} \lambda_j (\bar{R}_{(n+1)jjk;k} + \bar{R}_{(n+1)kjk;j}) \\ &\geqslant -\sum_{j,k} |\lambda_j| (|\bar{R}_{(n+1)jjk;k}| + |\bar{R}_{(n+1)kjk;j}|). \end{aligned} \tag{2.12}$$

From (1.3) and (2.12), we have

$$\sum_{i,j,k} (\bar{R}_{(n+1)ijk;k} + \bar{R}_{(n+1)kik;j}) h_{ij} \geqslant -2c_3\sqrt{S}. \tag{2.13}$$

Next, we consider the forth term of (2.10). From (1.1) and $h_{ij} = \lambda_i \delta_{ij}$, we have

$$\begin{aligned} &-\left(nH \sum_{i,j} h_{ij} \bar{R}_{(n+1)ij(n+1)} + S \sum_k \bar{R}_{(n+1)k(n+1)k} \right) \\ &= -\left(nH \sum_k \lambda_k \bar{R}_{(n+1)kk(n+1)} - S \sum_k \bar{R}_{(n+1)kk(n+1)} \right) \\ &= \sum_k (S - nH\lambda_k) \frac{c_1}{n} \\ &= c_1(S - nH^2). \end{aligned} \tag{2.14}$$

Finally, we deal with the fifth term of (2.10). Notice that $S - nH^2 = \frac{1}{2n} \sum_{j,k} (\lambda_j - \lambda_k)^2$ (see Eq. (2.17)). Using (1.2), we also have

$$\begin{aligned} -2 \sum_{i,j,k,l} (h_{kl} h_{ij} \bar{R}_{lijk} + h_{il} h_{ij} \bar{R}_{lkjk}) &= -2 \sum_{j,k} (\lambda_j \lambda_k - \lambda_k^2) \bar{R}_{kjjk} \\ &\geqslant c_2 \sum_{j,k} (\lambda_j - \lambda_k)^2 \\ &= 2c_2(nS - n^2H^2). \end{aligned} \tag{2.15}$$

Now, inserting (2.13), (2.14) and (2.15) into (2.10), we obtain

$$\begin{aligned}\frac{1}{2}\triangle S \geqslant \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} - 2c_3\sqrt{S} \\ + nc(S - nH^2) + \left(S^2 - nH\sum_i \lambda_i^3\right),\end{aligned} \tag{2.16}$$

where $c = 2c_2 + \dfrac{c_1}{n}$ and c_1, c_2, c_3 are given as in $(**)$.

Since we would like to compute $\triangle S$ for spacelike hypersurfaces in Lorentzian manifolds satisfying $(**)$, we need the following algebraic Lemma.

LEMMA 2.1 ([2, 16]). *Let $\mu_1, \dots, \mu_n$ be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = B^2$, where $B \geqslant 0$ is constant. Then*

$$\left|\sum_i \mu_i^3\right| \leqslant \frac{n-2}{\sqrt{n(n-1)}} B^3$$

and equality holds if and only if at least $n-1$ of the μ_i's are equal.

Let $\phi = \sum_{i,j} \phi_{ij}\omega_i \otimes \omega_j$ be a symmetric tensor defined on M^n, where $\phi_{ij} = h_{ij} - H\delta_{ij}$. It is easy to check that ϕ is traceless. Choose a local orthonormal frame field $\{e_1, \dots, e_n\}$ such that $h_{ij} = \lambda_i \delta_{ij}$ and $\phi_{ij} = \mu_i \delta_{ij}$. Let $|\phi|^2 = \sum_i \mu_i^2$. A direct computation gets

$$|\phi|^2 = S - nH^2 = \frac{1}{2n}\sum_{i,j}(\lambda_i - \lambda_j)^2. \tag{2.17}$$

Hence, $|\phi|^2 = 0$ if and only if M^n is totally umbilical. We also get

$$\sum_i \lambda_i^3 = nH^3 + 3H\sum_i \mu_i^2 + \sum_i \mu_i^3.$$

By applying Lemma 2.1 to the real numbers $\mu_1, \dots, \mu_n$, we obtain

$$\begin{aligned}-nH\sum_i \lambda_i^3 &= -n^2H^4 - 3nH^2\sum_i \mu_i^2 - nH\sum_i \mu_i^3 \\ &\geqslant 2n^2H^4 - 3nSH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|(S - nH^2)^{3/2}.\end{aligned} \tag{2.18}$$

Substituting (2.17) and (2.18) into (2.16), we obtain the following lemma.

LEMMA 2.2. *Let M^n be a spacelike hypersurface in an $(n+1)$-dimensional Lorentzian manifold L_1^{n+1} satisfying $(**)$, then*

$$\frac{1}{2}\triangle S \geqslant \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} - 2c_3\sqrt{S} + |\phi|^2 L_{|H|}(|\phi|), \tag{2.19}$$

where $|\phi|^2 = S - nH^2$ *and* $L_{|H|}(|\phi|) = |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\,|\phi| + nc - nH^2.$

In the proof of main Theorems, we need the well known generalized *Maximum Principle* due to H. Omori [17].

LEMMA 2.3 ([17]). *Let M^n be an n-dimensional complete Riemannion manifold whose sectional curvature is bounded from below and $F : M^n \to \mathbf{R}$ be a smooth function which is bounded from above on M^n. Then there exists a sequence of points $\{x_k\} \in M^n$ such that*

$$\lim_{k\to\infty} F(x_k) = \sup F,$$

$$\lim_{k\to\infty} |\nabla F(x_k)| = 0,$$

$$\lim_{k\to\infty} \sup \max\{(\nabla^2 F(x_k))(X,X) : |X| = 1\} \leqslant 0.$$

3. Spacelike hypersurfaces with constant mean curvature in Lorentzian manifolds L_1^{n+1} satisfying $(**)$

In 2004, J. Ok Baek, Q. M. Cheng and Y. Jin Suh [15] studied a complete spacelike hypersurface with constant mean curvature in locally symmetric Lorentzian manifolds and proved the following result.

THEOREM 3.1. *Let M^n $(n \geqslant 3)$ be a complete spacelike hypersurface with constant mean curvature H in an $(n+1)$-dimensional locally symmetric Lorentzian manifold L_1^{n+1} satisfying $(*)$. If $n^2H^2 < 4(n-1)c$, $c = 2c_2 + \frac{c_1}{n}$, where c_1 and c_2 are given as in $(*)$, then $c > 0$ and M^n is totally umbilical.*

In this Section, first we give a generalization of Theorem 3.1 and prove the following result.

THEOREM 3.2. *Let M^n $(n \geqslant 3)$ be a complete spacelike hypersurface with constant mean curvature H in an $(n+1)$-dimensional Lorentzian manifold L_1^{n+1} satisfying $(**)$. If $n^2H^2 < 4(n-1)c$, $c = 2c_2 + \frac{c_1}{n}$ and $c_3 \geqslant 0$, where c_1, c_2 and c_3*

are given as in $(**)$, *then* $c > 0$ *and the squared length* S *of the second fundamental form of* M^n *satisfies*

$$nH^2 \leqslant \sup S \leqslant \left[\frac{4(n-1)c_3 + \sqrt{16(n-1)^2 c_3^2 + n^3 H^2 [4(n-1)c - n^2 H^2]^2}}{n[4(n-1)c - n^2 H^2]}\right]^2.$$

On the other hand, when $n = 2$, we obtain the following theorem.

THEOREM 3.3. *Let* M^2 *be a complete spacelike hypersurface with constant mean curvature* H *in a* 3*-dimensional Lorentzian manifold* L_1^3 *satisfying* $(**)$.

(i) *If* $H^2 < c$, $c = 2c_2 + \frac{c_1}{2}$ *and* $c_3 \geqslant 0$, *where* c_1, c_2 *and* c_3 *are given as in* $(**)$, *then*

$$2H^2 \leqslant \sup S \leqslant \left[\frac{c_3 + \sqrt{c_3^2 + 8H^2(c - H^2)^2}}{2(c - H^2)}\right]^2.$$

(ii) *If* $H^2 = c$, $c = 2c_2 + \frac{c_1}{2}$ *and* $c_3 \geqslant 0$, *where* c_1, c_2 *and* c_3 *are given as in* $(**)$, *then*

$$(\sup S - 2H^2)^2 \leqslant 2c_3\sqrt{\sup S}.$$

Remark 3.4. When $c_3 = 0$ in Theorem 3.2, we know that the Lorentzian manifold L_1^{n+1} is locally symmetric and $\sup S = nH^2$. Together with (2.17), we know that $\sup|\phi|^2 = 0$ which shows M^n is totally umbilical. Hence, Theorem 3.2 is a generalization of Theorem 3.1. Furthermore, if L_1^{n+1} is the de Sitter space $\mathbf{S}_1^{n+1}(c)$ in Theorem 3.2, then $-\frac{c_1}{n} = c_2 = c$ and $c_3 = 0$. Therefore, Theorem 3.2 is also a generalization of the result due to K. Akutagawa [1, Theorem (ii)], saying that a complete spacelike hypersurface M^n $(n \geqslant 3)$ in a de Sitter space $\mathbf{S}_1^{n+1}(c)$ with constant mean curvature H satisfying $n^2H^2 < 4(n-1)c$ must be totally umbilical.

On the other hand, S. Montiel [13] exhibited examples of complete spacelike hypersurfaces in $\mathbf{S}_1^{n+1}(1)$ with constant mean curvature H satisfying $n^2H^2 \geqslant 4(n-1)$ and being non-totally umbilical, the so-called hyperbolic cylinders (cf. [1] and [10]), which are isometric to the Riemannian product $\mathbf{H}^1(\sinh r) \times \mathbf{S}^{n-1}(\cosh r)$ of a hyperbolic line and an $(n-1)$-dimensional sphere of constant sectional curvatures $1 - \coth^2 r$ and $1 - \tanh^2 r$, respectively. Hence, when $n \geqslant 3$, the assumption $n^2H^2 < 4(n-1)c$ in Theorem 3.2 is essential.

Remark 3.5. When $c_3 = 0$ in Theorem 3.3, we know that the Lorentzian manifold L_1^3 is locally symmetric and $\sup S = 2H^2$. Together with (2.17), we know that $\sup|\phi|^2 = 0$ which shows M^n is totally umbilical. Furthermore, if L_1^3 is the de Sitter space $\mathbf{S}_1^3(c)$ in Theorem 3.3, then $-\frac{c_1}{2} = c_2 = c$ and $c_3 = 0$. Therefore, theorem 3.3 is a generalization of [1, Theorem (i)].

On the other hand, K. Akutagawa [1] constructed, for any constant H satisfying $H^2 > c$, complete, noncompact embedded spacelike surfaces in $\mathbf{S}_1^3(c)$, which have constant mean curvature H and which are not totally umbilical. Hence, when $n = 2$, the assumption $H^2 \leqslant c$ in Theorem 3.3 is essential.

Proof of Theorem 3.2. Since H is constant, it follows from (2.19) that

$$\frac{1}{2}\triangle|\phi|^2 = \frac{1}{2}\triangle S \geqslant -2c_3\sqrt{S} + |\phi|^2\left(|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\,|\phi| + nc - nH^2\right). \tag{3.1}$$

Choose a local orthonormal frame field $\{e_1, \ldots, e_n\}$ such that $h_{ij} = \lambda_i\delta_{ij}$. By a similar reasoning as in the proof of [15, Theorem 1], we obtain that there is a sequence of points $\{x_k\} \in M^n$ such that

$$\begin{aligned} &\lim_{k\to\infty} |\phi|^2(x_k) = \sup|\phi|^2, \\ &\lim_{k\to\infty} S(x_k) = \sup S, \\ &\lim_{k\to\infty} \sup(\triangle|\phi|^2)(x_k) \leqslant 0. \end{aligned} \tag{3.2}$$

Evaluating (3.1) at the points x_k of the sequence, taking the limit and using (2.17) and (3.2), we obtain that

$$\begin{aligned} 0 &\geqslant \lim_{k\to\infty} \sup\left(\frac{1}{2}\triangle|\phi|^2\right)(x_k) \\ &\geqslant -2c_3\sqrt{\sup S} + (\sup S - nH^2)L_{|H|}(\sup S), \end{aligned} \tag{3.3}$$

where $L_{|H|}(\sup S) = \sup S - 2nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\sqrt{\sup S - nH^2} + nc$.

Since

$$\left(\sqrt{\frac{n(n-2)}{2\sqrt{n(n-1)}}}|H| - \sqrt{\frac{2\sqrt{n(n-1)}}{n(n-2)}}\sqrt{S - nH^2}\right)^2 \geqslant 0,$$

we have

$$\begin{aligned} -|H|\sqrt{S - nH^2} &= -\sqrt{\frac{n(n-2)}{2\sqrt{n(n-1)}}}|H| \cdot \sqrt{\frac{2\sqrt{n(n-1)}}{n(n-2)}}\sqrt{S - nH^2} \\ &\geqslant -\frac{\frac{n(n-2)}{2\sqrt{n(n-1)}}H^2 + \frac{2\sqrt{n(n-1)}}{n(n-2)}(S - nH^2)}{2}. \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we have

$$0 \geqslant \frac{n[4(n-1)c - n^2H^2]}{4(n-1)}\sup S - 2c_3\sqrt{\sup S} - \frac{n^2H^2[4(n-1)c - n^2H^2]}{4(n-1)}. \tag{3.5}$$

Since $n^2H^2 < 4(n-1)c$, it follows from (3.5) and (2.17) that

$$nH^2 \leqslant \sup S \leqslant \left[\frac{4(n-1)c_3 + \sqrt{16(n-1)^2c_3^2 + n^3H^2[4(n-1)c - n^2H^2]^2}}{n[4(n-1)c - n^2H^2]}\right]^2$$

and $c > 0$. This completes the proof of Theorem 3.2. □

Proof of Theorem 3.3. (i) When $n = 2$, it follows from (3.3) that

$$(3.6) \qquad \begin{aligned} 0 &\geqslant -2c_3\sqrt{\sup S} + (\sup S - 2H^2)(\sup S - 2H^2 + 2c - 2H^2) \\ &\geqslant -2c_3\sqrt{\sup S} + 2(\sup S - 2H^2)(c - H^2) \\ &= 2(c - H^2)\sup S - 2c_3\sqrt{\sup S} - 4H^2(c - H^2). \end{aligned}$$

Since $H^2 < c$, it follows from (3.6) and (2.17) that

$$2H^2 \leqslant \sup S \leqslant \left[\frac{c_3 + \sqrt{c_3^2 + 8H^2(c - H^2)^2}}{2(c - H^2)}\right]^2$$

and $c > 0$.

(ii) When $n = 2$ and $H^2 = c$, it follows from (3.3) that

$$(\sup S - 2H^2)^2 \leqslant 2c_3\sqrt{\sup S}.$$

This completes the proof of Theorem 3.3. □

4. Spacelike hypersurfaces with constant scalar curvature in Lorentzian manifolds L_1^{n+1} satisfying (∗∗)

According to Cheng and Yau's definition in [7], we introduce the self-adjoint operator □ acting on any C^2-function f by

$$(4.1) \qquad \square(f) = \sum_{i,j}(nH\delta_{ij} - h_{ij})f_{ij}.$$

In this Section, in order to prove Theorems 4.4–4.5, first we give a new estimate of $\square(nH)$ for spacelike hypersurfaces in Lorentzian manifolds L_1^{n+1}.

Proposition 4.1. *Let M^n $(n \geqslant 3)$ be a spacelike hypersurface with constant P defined by* (2.5) *in an $(n+1)$-dimensional Lorentzian manifold L_1^{n+1} satisfying* (∗∗). *Suppose that the constant $P \geqslant 0$. Then*

$$(4.2) \qquad \square(nH) \geqslant -2c_3\sqrt{S} + |\phi|^2 L_{|H|}(|\phi|),$$

where $|\phi|^2 = S - nH^2$ and $L_{|H|}(|\phi|) = |\phi|^2 - \dfrac{n(n-2)}{\sqrt{n(n-1)}}|H|\,|\phi| + nc - nH^2.$

Proof. Choose a local orthonormal frame field $\{e_1, \dots, e_n\}$ such that $h_{ij} = \lambda_i \delta_{ij}$. In view of (4.1), $\square(nH)$ is given by

$$\square(nH) = nH\triangle(nH) - \sum_{i,j} h_{ij}(nH)_{ij}. \tag{4.3}$$

Notice that

$$nH\triangle(nH) = \frac{1}{2}\triangle(nH)^2 - n^2|\nabla H|^2. \tag{4.4}$$

Thus, it follows from (4.3) and (4.4) that

$$\square(nH) = \frac{1}{2}\triangle(nH)^2 - n^2|\nabla H|^2 - \sum_{i,j} h_{ij}(nH)_{ij}. \tag{4.5}$$

Moreover, as P is constant, by (2.5), we have $\triangle S = \triangle(nH)^2$. Therefore, it follows from (2.19) and (4.5) that

$$\square(nH) \geqslant \sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 - 2c_3\sqrt{S} + |\phi|^2 L_{|H|}(|\phi|). \tag{4.6}$$

Next, we claim that

$$\sum_{i,j,k} h_{ijk}^2 \geqslant n^2|\nabla H|^2. \tag{4.7}$$

Indeed, since P is a constant, differentiating formula (2.5) exteriorly yields $n^2HH_k = \sum_{i,j} h_{ij}h_{ijk}$, then by using Cauchy-Schwarz inequality we have

$$\sum_k n^4H^2(H_k)^2 = \sum_k \left(\sum_{i,j} h_{ij}h_{ijk}\right)^2 \leqslant \left(\sum_{i,j} h_{ij}^2\right)\left(\sum_{i,j,k} h_{ijk}^2\right).$$

That is, $n^4H^2|\nabla H|^2 \leqslant S\sum_{i,j,k} h_{ijk}^2$. Together with the fact $n^2H^2 - S \geqslant 0$ since $P \geqslant 0$, we obtain that $\sum_{i,j,k} h_{ijk}^2 \geqslant n^2|\nabla H|^2$ which proves our claim. Consequently, (4.2) follows from (4.6) and (4.7). Finally, Proposition 4.1 is proved. □

In 2010, J. C. Liu and Z. Y. Sun [12] studied a complete or compact spacelike hypersurface with constant normalized scalar curvature R in a locally symmetric Lorentzian manifold L_1^{n+1} and proved Theorems 4.2–4.3.

THEOREM 4.2. *Let M^n $(n \geqslant 3)$ be a complete spacelike hypersurface with constant normalized scalar curvature R in an $(n+1)$-dimensional locally symmetric Lorentzian manifold L_1^{n+1} satisfying $(*)$. Suppose that M^n has bounded mean curvature H. If $0 \leqslant P \leqslant \dfrac{2c}{n}$ and $c > 0$, where the constant P defined by* (2.5), *$c = 2c_2 + \dfrac{c_1}{n}$ and c_1, c_2 are given as in $(*)$, then M^n is totally umbilical.*

THEOREM 4.3. *Let M^n $(n \geqslant 3)$ be a compact spacelike hypersurface with constant normalized scalar curvature R in an $(n+1)$-dimensional locally symmetric Lorentzian manifold L_1^{n+1} satisfying $(*)$. If $0 \leqslant P \leqslant \frac{2c}{n}$ and $c > 0$, where the constant P defined by (2.5), $c = 2c_2 + \frac{c_1}{n}$ and c_1, c_2 are given as in $(*)$, then M^n is totally umbilical.*

In this Section, we give generalizations of Theorems 4.2–4.3 and obtain the following results.

THEOREM 4.4. *Let M^n $(n \geqslant 3)$ be a complete spacelike hypersurface with constant P defined by (2.5) in an $(n+1)$-dimensional Lorentzian manifold L_1^{n+1} satisfying $(**)$. Suppose that M^n has bounded mean curvature H.*

(i) *If $0 \leqslant P < \frac{2c}{n}$, $c = 2c_2 + \frac{c_1}{n}$ and $c_3 \geqslant 0$, where c_1, c_2 and c_3 are given as in $(**)$, then $c > 0$ and the squared length S of the second fundamental form of M^n satisfies*

$$nP \leqslant \sup S \leqslant \left[\frac{c_3 + \sqrt{c_3^2 + nP(n-1)^2\left(c - \frac{n}{2}P\right)^2}}{(n-1)\left(c - \frac{n}{2}P\right)}\right]^2.$$

(ii) *If $P = \frac{2c}{n}$, $c = 2c_2 + \frac{c_1}{n} > 0$ and $c_3 \geqslant 0$, where c_1, c_2 and c_3 are given as in $(**)$, then*

$$\frac{n-1}{n}(\sup S - 2c)L(\sup S) \leqslant 2c_3\sqrt{\sup S} \quad \textit{and} \quad L(\sup S) > 0,$$

where

$$L(\sup S) = \frac{n-2}{n}\left[(n-2)c + \sup S - \sqrt{(2(n-1)c + \sup S)(\sup S - 2c)}\right].$$

THEOREM 4.5. *Let M^n $(n \geqslant 3)$ be a compact spacelike hypersurface with constant P defined by (2.5) in an $(n+1)$-dimensional Lorentzian manifold L_1^{n+1} satisfying $(**)$.*

(i) *Suppose that*

$$S \geqslant \left[\frac{c_3 + \sqrt{c_3^2 + nP(n-1)^2\left(c - \frac{n}{2}P\right)^2}}{(n-1)\left(c - \frac{n}{2}P\right)}\right]^2.$$

If $0 \leqslant P < \frac{2c}{n}$, $c = 2c_2 + \frac{c_1}{n}$ *and* $c_3 \geqslant 0$, *where* c_1, c_2 *and* c_3 *are given as in* $(**)$, *then* $c > 0$ *and*

$$S = \left[\frac{c_3 + \sqrt{c_3^2 + nP(n-1)^2\left(c - \frac{n}{2}P\right)^2}}{(n-1)\left(c - \frac{n}{2}P\right)} \right]^2.$$

(ii) *Suppose that*

$$\frac{n-1}{n}(S - 2c)L(S) \geqslant 2c_3\sqrt{S}.$$

If $P = \frac{2c}{n}$, $c = 2c_2 + \frac{c_1}{n} > 0$ *and* $c_3 \geqslant 0$, *where* c_1, c_2 *and* c_3 *are given as in* $(**)$, *then*

$$\frac{n-1}{n}(S - 2c)L(S) = 2c_3\sqrt{S} \quad \textit{and} \quad L(S) > 0,$$

where

$$L(S) = \frac{n-2}{n}\left[(n-2)c + S - \sqrt{(2(n-1)c + S)(S - 2c)}\right].$$

Remark 4.6. When $c_3 = 0$ in Theorem 4.4, we know that the Lorentzian manifold L_1^{n+1} is locally symmetric and $\sup S = nP$. Together with (4.9), we know that $\sup|\phi|^2 = 0$ which shows M^n is totally umbilical provided that M^n has bounded mean curvature H. Hence, Theorem 4.4 is a generalization of Theorem 4.1. Furthermore, if L_1^{n+1} is the de Sitter space $\mathbf{S}_1^{n+1}(c)$ in Theorem 4.4, then $-\frac{c_1}{n} = c_2 = c$, $c_3 = 0$ and $P = c - R$ following from (2.5). At the same time, the assumption $0 \leqslant P \leqslant \frac{2c}{n}$ in Theorem 4.4 becomes $\frac{n-2}{n}c \leqslant R \leqslant c$. Hence, Theorem 4.4 is also a generalization of the result due to F. E. C. Camargo et al. in [4], saying that a complete spacelike hypersurface M^n $(n \geqslant 3)$ in the de Sitter space $\mathbf{S}_1^{n+1}(c)$ with constant normalized scalar curvature R satisfying $\frac{n-2}{n}c \leqslant R \leqslant c$ must be totally umbilical provided that M^n has bounded mean curvature H.

On the other hand, consider the spacelike hypersurface immersed into $\mathbf{S}_1^{n+1}(1)$ defined by $T_{k,r} = \{x \in \mathbf{S}_1^{n+1}(1) \mid -x_0^2 + x_1^2 + \cdots + x_k^2 = -\sinh^2 r\}$, where r is a positive real number and $1 \leqslant k \leqslant n-1$. $T_{k,r}$ is complete and isometric to the Riemannian product $\mathbf{H}^k(1 - \coth^2 r) \times \mathbf{S}^{n-k}(1 - \tanh^2 r)$ of a k-dimensional hyperbolic space and an $(n-k)$-dimensional sphere of constant sectional curvatures $1 - \coth^2 r$ and $1 - \tanh^2 r$, respectively. It follows from [9] that if $k = 1$, then R satisfies $0 < R = \frac{n-2}{n}(1 - \tanh^2 r) < \frac{n-2}{n}$; similarly, if $k = n-1 \geqslant 2$, we see that $R = \frac{n-2}{n}(1 - \coth^2 r) < 0$. Thus, for any R satisfying $0 < R < \frac{n-2}{n}$ and for any $R < 0$, we can choose r such that the hypersurfaces $T_{1,r}$ and $T_{n-1,r}$, respectively, are complete, non-totally umbilical and have constant normalized scalar curvature R. Hence, the assumption $0 \leqslant P \leqslant \frac{2c}{n}$ in Theorem 4.4 is essential.

Remark 4.7. When $c_3 = 0$ in Theorem 4.5, we know that the Lorentzian manifold L_1^{n+1} is locally symmetric and $S = nP$. Together with (4.9), we know that $|\phi|^2 = 0$ which shows M^n is totally umbilical. Hence, Theorem 4.5 is a generalization of Theorem 4.2. Theorem 4.5 reduces to Li's result [11, Theorem 4.3].

Proof of Theorem 4.4. (i) By using Lemma 2.3 and taking the similar method as in the proof of [12, Lemma 2.4 (ii)], we obtain that there is a sequence of points $\{x_k\} \in M^n$ such that

$$\begin{aligned} &\lim_{k\to\infty} nH(x_k) = \sup(nH), \\ &\lim_{k\to\infty} |\nabla(nH)(x_k)| = 0, \\ &\lim_{k\to\infty} \sup(\Box(nH)(x_k)) \leqslant 0. \end{aligned} \tag{4.8}$$

From (2.5) and (2.17), we have

$$|\phi|^2 = n(n-1)(H^2 - P) = \frac{n-1}{n}(S - nP). \tag{4.9}$$

Since $\lim_{k\to\infty}(nH)(x_k) = \sup(nH)$ and P is a constant, it follows from (4.9) that

$$\lim_{k\to\infty} |\phi|^2(x_k) = \sup|\phi|^2, \quad \lim_{k\to\infty} S(x_k) = \sup S. \tag{4.10}$$

Evaluating (4.2) at the points x_k of the sequence, taking the limit and using (4.8) and (4.9) and (4.10), we obtain that

$$\begin{aligned} 0 &\geqslant \lim_{k\to\infty} \sup(\Box(nH)(x_k)) \\ &\geqslant -2c_3\sqrt{\sup S} \\ &\quad + \sup|\phi|^2\left(\sup|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}\sup|H|\sup|\phi| + nc - n\sup H^2\right) \\ &= -2c_3\sqrt{\sup S} + \frac{n-1}{n}(\sup S - nP)L_P(\sup S), \end{aligned} \tag{4.11}$$

where

$$L_P(\sup S) = nc - 2(n-1)P + \frac{n-2}{n}\sup S - \frac{n-2}{n}\sqrt{(n(n-1)P + \sup S)(\sup S - nP)}.$$

Since

$$(\sqrt{n(n-1)P + S} - \sqrt{S - nP})^2 \geqslant 0,$$

we have

$$-\sqrt{(n(n-1)P + S)(S - nP)} \geqslant -\frac{n(n-1)P + S + S - nP}{2}. \tag{4.12}$$

From (4.11) and (4.12), we have

$$0 \geqslant (n-1)\left(c - \frac{n}{2}P\right)\sup S - 2c_3\sqrt{\sup S} - nP(n-1)\left(c - \frac{n}{2}P\right). \tag{4.13}$$

Since $0 \leqslant P < \frac{2c}{n}$, it follows from (4.13) and (4.9) that

$$nP \leqslant \sup S \leqslant \left[\frac{c_3 + \sqrt{c_3^2 + nP(n-1)^2\left(c - \frac{n}{2}P\right)^2}}{(n-1)\left(c - \frac{n}{2}P\right)}\right]^2$$

and $c > 0$.

(ii) Since $P = \frac{2c}{n}$, it follows from (4.11) that

$$\frac{n-1}{n}(\sup S - 2c)L(\sup S) \leqslant 2c_3\sqrt{\sup S},$$

where

$$L(\sup S) = \frac{n-2}{n}[(n-2)c + \sup S - \sqrt{(2(n-1)c + \sup S)(\sup S - 2c)}].$$

As $c > 0$, a direct computation gets

$$L(\sup S) > 0.$$

This completes the proof of Theorem 4.4. □

Proof of theorem 4.5. (i) From (4.2) and (4.9), we have

$$\Box(nH) \geqslant -2c_3\sqrt{S} + \frac{n-1}{n}(S - nP)L_P(S), \tag{4.14}$$

where

$$L_P(S) = nc - 2(n-1)P + \frac{n-2}{n}S - \frac{n-2}{n}\sqrt{(n(n-1)P + S)(S - nP)}.$$

Since

$$(\sqrt{n(n-1)P + S} - \sqrt{S - nP})^2 \geqslant 0,$$

we have

$$-\sqrt{(n(n-1)P + S)(S - nP)} \geqslant -\frac{n(n-1)P + S + S - nP}{2}. \tag{4.15}$$

From (4.14) and (4.15), we have

$$\Box(nH) \geqslant (n-1)\left(c - \frac{n}{2}P\right)S - 2c_3\sqrt{S} - nP(n-1)\left(c - \frac{n}{2}P\right). \tag{4.16}$$

Since M^n is compact and $\Box$ is self-adjoint operator, we have

$$\int_{M^n} \Box(nH)\, dv_{M^n} = 0. \tag{4.17}$$

Thus, it follows from (4.16) and (4.17) that

$$0 \geqslant \int_{M^n} \left((n-1)\left(c - \frac{n}{2}P\right)S - 2c_3\sqrt{S} - nP(n-1)\left(c - \frac{n}{2}P\right)\right) dv_{M^n}. \tag{4.18}$$

Since $S \geqslant \left[\dfrac{c_3 + \sqrt{c_3^2 + nP(n-1)^2\left(c - \frac{n}{2}P\right)^2}}{(n-1)\left(c - \frac{n}{2}P\right)}\right]^2$ and $0 \leqslant P < \dfrac{2c}{n}$, we obtain that $c > 0$ and

$$(n-1)\left(c - \frac{n}{2}P\right)S - 2c_3\sqrt{S} - nP(n-1)\left(c - \frac{n}{2}P\right) \geqslant 0. \tag{4.19}$$

Hence, it follows from (4.18) and (4.19) that

$$S = \left[\frac{c_3 + \sqrt{c_3^2 + nP(n-1)^2\left(c - \frac{n}{2}P\right)^2}}{(n-1)\left(c - \frac{n}{2}P\right)}\right]^2.$$

(ii) Since $P = \dfrac{2c}{n}$, it follows from (4.14) and (4.17) that

$$0 \geqslant \int_{M^n} \left(-2c_3\sqrt{S} + \frac{n-1}{n}(S - 2c)L(S)\right) dv_{M^n}, \tag{4.20}$$

where

$$L(S) = \frac{n-2}{n}[(n-2)c + S - \sqrt{(2(n-1)c+S)(S-2c)}].$$

Since $\frac{n-1}{n}(S-2c)L(S) \geqslant 2c_3\sqrt{S}$, it follows from (4.20) that

$$\frac{n-1}{n}(S-2c)L(S) = 2c_3\sqrt{S}.$$

As $c > 0$, a direct computation gets $L(S) > 0$. This completes the proof of Theorem 4.5. □

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