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KODAI MATH. J.
36 (2013), 491–507

EXPLICIT ESTIMATES ON DISTANCE ESTIMATOR METHOD FOR JULIA SETS OF POLYNOMIALS

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Abstract

The distance estimator method is well-known as an iterative method which estimates the Euclidean distance between a given point and Julia set. Although it brings a remarkable effect in drawing Julia set, it seems to be not known about how accurate this method is. In the present paper, we give explicit estimates on this method.

1. Introduction and results

Let P be a polynomial with degree $d \geq 2$, and J the Julia set of P. Let $A = A(\infty) \subset \hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ be the basin of the super attracting fixed point at ∞ of P, and $K = \hat{\mathbf{C}} \setminus A$ the filled-in Julia set. The domain A is the component of the Fatou set F of P containing ∞.

Throughout the present paper, we assume that

(I) the domain A is simply connected,

(II) P is monic, and K contains the origin.

The condition (I) holds if and only if the forward orbit $\{P^n(z)\}$ of z is bounded for each critical point z $(\neq \infty)$ of P ([1], [3], [6]). Typical example is given by $P(z) = z^d + c$, where $\{P^n(0)\}$ is bounded. The condition (II) is not essential. Indeed, if P satisfies (I), then the conjugate polynomial $Q = T \circ P \circ T^{-1}$ satisfies (II) for some linear transformation $T(z) = az + b$.

To draw the filled-in Julia set K (or $J = \partial K$), we need approximation algorithms for K. The level set method (LSM) is the simplest one. Let $R > 4$ and $N \in \mathbf{N}$. For a given point $z \in \mathbf{C}$, set $z_n = P^n(z)$ $(n = 0, 1, 2, \ldots)$ and $n(z, R) = \min\{n \mid |z_n| \geq R\}$. A point z belongs to K if and only if $n(z, R) = \infty$. Let

$$\begin{aligned} K_{\mathrm{LSM}}(N, R) &= \{z \in \mathbf{C} \mid n(z, R) > N\} \\ &= K \cup \{z \in A_0 \mid n(z, R) > N\}, \end{aligned}$$

2010 *Mathematics Subject Classification*. 30F45, 37F10.

Key words and phrases. Complex dynamics, Julia set, DEM, DEM/J, LSM, Böttcher map, Böttcher domain, Köbe distortion theorem, Hyperbolic metric, Conformal map, Mandelbrot set.

The first author is partially supported by Grant-in-Aid for Scientific Research (C) 22540240.

Received September 18, 2012; revised March 5, 2013.

where $A_0 = A \backslash \{\infty\}$. Then $K_{\mathrm{LSM}} \downarrow K$ as point sets as $N \to \infty$, although unfortunately K_{LSM} is not a good approximation of K for the purpose of drawing K in general (Figure 1). Much better approximation can be obtained by estimating the Euclidean distance between z and K.

The Böttcher map

$$\varphi(z) = \lim_{n\to\infty} \varphi_n(z) \quad (\varphi_n(z) = (P^n(z))^{1/d^n})$$

is a conformal map of A onto $\hat{\mathbf{C}} \backslash \bar{\Delta}$ satisfying

$$\varphi(P(z)) = (\varphi(z))^d,$$

where $\Delta = \{|z| < 1\}$ ([1]). Then, appealing to the Köbe one-quarter theorem (see Proposition 1 below), we obtain

$$\frac{1}{4} \cdot \frac{|dz|}{d(z,J)} \leq \frac{|dw|}{d(w, \partial\Delta)} \leq 4 \cdot \frac{|dz|}{d(z,J)}, \quad z \in A_0,\ w = \varphi(z),$$

where d denotes the Euclidean distance, which means

$$d(z,J) \approx \frac{|\varphi(z)| - 1}{|\varphi'(z)|} \approx \frac{|\varphi_n(z)| - 1}{|\varphi_n'(z)|}, \quad z \in A_0.$$

Moreover, if $z \in A_0$ is near J, then

$$|\varphi_n(z)| - 1 \approx \log|\varphi_n(z)| = \frac{1}{d^n} \log|P^n(z)|,$$

$$|\varphi_n'(z)| = \frac{1}{d^n} |(P^n)'(z)|\, |P^n(z)|^{1/d^n - 1} = \frac{|\varphi_n(z)|\, |(P^n)'(z)|}{d^n |P^n(z)|} \approx \frac{|(P^n)'(z)|}{d^n |P^n(z)|}.$$

It follows that if $z \in A_0$ is near J and n is sufficiently large, then

$$d(z,J) \approx d_n(z,J), \tag{1}$$

where

$$d_n(z,J) = \frac{|z_n| \log|z_n|}{|z_n'|} \quad (z_n = P^n(z), z_n' = (P^n)'(z)).$$

Since z_n and z_n' satisfy

$$z_0 = z, \quad z_0' = 1, \quad z_{n+1} = P(z_n), \quad z_{n+1}' = z_n' P'(z_n),$$

the result (1) provides an algorithm to draw the ε like neighborhood of K, which is known as the *distance estimator method* (DEM) ([7], [9]). Much clear pictures can be obtained by virtue of DEM (Figure 2).

However, since the argument above utilizes the sequence φ_n, it seems difficult to obtain explicit bounds for $d(z,J)/d_n(z,J)$. The main aim of the present paper is to give its explicit bounds by appealing to another argument, which utilizes mainly the invariance of the hyperbolic metric under covering maps.

THEOREM 1. *Let* $z \in A_0$, $R > 4$, *and* $|z_n| \geq R$. *Then*

$$\frac{1}{2}\left(1 - \frac{|z|}{12}\right)\left(1 - \frac{4}{R}\right) \leq \frac{d(z, J)}{d_n(z, J)} \leq 2\left(1 + \frac{4}{R}\right).$$

The left-hand side estimate is not good, unless z is near the origin, even if z is near J. We can improve this point as follows.

THEOREM 2. *Let* $z \in A_0$, $R \geq 9$, *and* $|z_n| \geq R$. *Then*

$$\frac{1}{2}\left(1 - \frac{9}{R}\right)\left(1 - \frac{d_n(z, J)}{24}\right) d_n(z, J) \leq d(z, J) \leq 2\left(1 + \frac{4}{R}\right) d_n(z, J).$$

Note that our bounds depend neither on the number n of iteration nor on the degree d of the polynomial P.

Let

$$K_{\mathrm{DEM}}(R, \varepsilon) = K \cup \{z \in A_0 \mid d_{n(z,R)}(z, J) < \varepsilon\}.$$

Our K_{DEM} is different from the usual, computable version

$$\begin{aligned}\tilde{K}_{\mathrm{DEM}}(N, R, \varepsilon) &= K_{\mathrm{LSM}}(N, R) \cup K_{\mathrm{DEM}}(R, \varepsilon) \\ &= K_{\mathrm{LSM}}(N, R) \cup \{z \in A_0 \mid n(z, R) \leq N, d_{n(z,R)}(z, J) < \varepsilon\}.\end{aligned}$$

However, if N is sufficiently large, then $K_{\mathrm{LSM}}(N, R) \subset K_{\mathrm{DEM}}(R, \varepsilon)$, and so these two types of DEM approximation coincide with each other.

Figures 1 and 2 correspond to $K_{\mathrm{LSM}}(N, R)$ and $\tilde{K}_{\mathrm{DEM}}(N, R, \varepsilon)$ respectively, where

$$P(z) = z^2 + c, \quad c = -0.770826391 + 0.115528513i,$$

$N = 200$, $R = 10$, and $\varepsilon = 0.0058$. Calculation was performed to 1000×1000 lattice points. In this case the set K is connected, but it is too thin to be captured by LSM.

FIGURE 1. LSM.

FIGURE 2. DEM.

Let $K_\varepsilon = \{z \mid d(z, K) < \varepsilon\}$.

Corollary 1. *Let* $0 < \varepsilon < 1$ *and* $R \geq 9$. *Then*

$$K_{\varepsilon_1} \subset K_{\mathrm{DEM}}(R, \varepsilon) \subset K_{\varepsilon_2},$$

where

$$\varepsilon_1 = \frac{1}{2}\left(1 - \frac{9}{R}\right)\left(1 - \frac{\varepsilon}{24}\right)\varepsilon, \quad \varepsilon_2 = 2\left(1 + \frac{4}{R}\right)\varepsilon.$$

Note $\varepsilon_2/\varepsilon$, $\varepsilon/\varepsilon_1 \to 2$ as $R \to \infty$ and $\varepsilon \to 0$. For instance, if $\varepsilon \leq 1/100$ and $R \geq 1000$, then

$$K_{0.495\varepsilon} \subset K_{\mathrm{DEM}}(R, \varepsilon) \subset K_{2.008\varepsilon}.$$

We establish Theorems 1 and 2 by showing the fact that DEM is just the algorithm computing the hyperbolic metric of the domain A_0. Let $D \subset \hat{\mathbf{C}}$ be a domain with holomorphic universal covering $\pi : \Delta \to D$. The hyperbolic metric $\rho_D(z)|dz|$ of D is defined by the equation

$$\rho_D(z) = \frac{2}{|\pi'(\zeta)|(1 - |\zeta|^2)},$$

where $z = \pi(\zeta)$. We obtain a sharp estimate for $\rho_{A_0}(z) d_n(z, J)$.

Theorem 3. *Let* $z \in A_0$, $R > 4$, *and* $|z_n| \geq R$. *Then*

$$p_1(R) \leq \rho_{A_0}(z) d_n(z, J) \leq p_2(R).$$

As to the functions p_1, p_2, see §3. In particular

$$\rho_{A_0}(z) = \lim_{n \to \infty} \frac{1}{d_n(z, J)}, \quad z \in A_0.$$

Also we give a remark on the conjecture of Milnor ([7]) concerning the Euclidean distance between a given point and the Mandelbrot set (Remark 6).

We would like to express our thanks to the referee for helpful advices about the paper.

2. Sharp estimate for ρ_{A_0}

Our argument depends on the distortion estimates for conformal maps.

Proposition 1 (cf. [4], [10]). *Let* f *be a conformal map of* Δ *into* $\mathbf{C}$ *satisfying* $f(0) = 0$, $f'(0) = 1$.

(a) *Let* $z \in \Delta$. *Then*

$$\frac{|z|}{(1 + |z|)^2} \leq |f(z)| \leq \frac{|z|}{(1 - |z|)^2},$$

$$\frac{1 - |z|}{1 + |z|} \leq \left| z \frac{f'(z)}{f(z)} \right| \leq \frac{1 + |z|}{1 - |z|}.$$

(b) $f(\Delta)$ *contains the open disk with center the origin and radius* $1/4$.

The first inequality in (a) is known as the Köbe distortion theorem and the result (b) the Köbe one-quarter thereom.

Let $z = \phi(w)$ be the inverse map of the Böttcher map $w = \varphi(z)$, and $f(z) = 1/\phi(1/z)$. Then, from the condition (II), f satisfies the condition of the proposition above, and so we obtain the following two lemmas.

Lemma 1. *Let $w \in \mathbf{C}\backslash\bar{\Delta}$ and $z = \phi(w)$. Then*

$$\frac{(|w|-1)^2}{|w|} \le |z| \le \frac{(|w|+1)^2}{|w|},$$

$$\frac{|w|-1}{|w|+1} \le \frac{|w|}{|z|}|\phi'(w)| \le \frac{|w|+1}{|w|-1}.$$

Corollary 2. *Let $z \in A_0$ and $w = \varphi(z)$. Then*

$$\left(\frac{\sqrt{|z|}+\sqrt{|z|-4}}{2}\right)^2 \le |w| \le \left(\frac{\sqrt{|z|}+\sqrt{|z|+4}}{2}\right)^2, \tag{2}$$

$$\frac{|w|-1}{|w|+1} \le \frac{|z|}{|w|}|\varphi'(z)| \le \frac{|w|+1}{|w|-1}.$$

However, the left-hand side inequality of (2) *holds only for $z \in A_0$ satisfying $|z| \ge 4$.*

Lemma 2. $K \subset \{|z| \le 4\}$.

Remark 1. All estimates in Lemma 1 and Corollary 2 are sharp. Let $P(z) = (z-2)^2$. Then $J = [0, 4]$, $z = \phi(w) = w + w^{-1} + 2$, and the equalities are attained on $(4, +\infty)$ or on $(-\infty, 0)$. This example shows the constant 4 in Lemma 2 is also sharp.

The hyperbolic metric of the domain $\mathbf{C}\backslash\bar{\Delta}$ is given by

$$\rho_{\mathbf{C}\backslash\bar{\Delta}}(w)|dw| = \frac{|dw|}{|w| \log|w|}.$$

Since $\rho_{A_0}(z) = \rho_{\mathbf{C}\backslash\bar{\Delta}}(w)|\varphi'(z)|$, where $w = \varphi(z)$, Corollary 2 implies

$$\frac{1}{|z|}\cdot\frac{|w|-1}{(|w|+1)\log|w|} \le \rho_{A_0}(z) \le \frac{1}{|z|}\cdot\frac{|w|+1}{(|w|-1)\log|w|}. \tag{3}$$

Set

$$q_1(x) = \left\{2\sqrt{x^2+4x}\cdot\log\left(\frac{\sqrt{x}+\sqrt{x+4}}{2}\right)\right\}^{-1},$$

$$q_2(x) = q_1(x-4) = \left\{2\sqrt{x^2-4x}\cdot\log\left(\frac{\sqrt{x}+\sqrt{x-4}}{2}\right)\right\}^{-1}.$$

Theorem 4.
(a) *Let* $z \in A_0$. *Then*

$$\rho_{A_0}(z) \geq q_1(|z|).$$

(b) *Let* $|z| > 4$. *Then*

$$\rho_{A_0}(z) \leq q_2(|z|).$$

Remark 2. These estimates are sharp. Let $P(z) = (z-2)^2$ be the polynomial in Remark 1. The equalities hold in the first inequality for $z < 0$ and in the second one for $z > 4$.

Proof of Theorem 4. First, let $z \in A_0$. Set $t = \frac{1}{4}(\sqrt{|z|} + \sqrt{|z|+4})^2$. Then $1 < |w| \leq t$ by Corollary 2. Let $g_1(x) = \dfrac{x-1}{(x+1)\log x}$, $x > 1$. Since g_1 is non-increasing, it follows from (3) that

$$\rho_{A_0}(z) \geq \frac{g_1(|w|)}{|z|} \geq \frac{g_1(t)}{|z|} = q_1(|z|).$$

Next, let $|z| > 4$. In this case set $s = \frac{1}{4}(\sqrt{|z|} + \sqrt{|z|-4})^2$. Then $|w| \geq s > 1$ by Corollary 2 again. Since $g_2(x) = \dfrac{x+1}{(x-1)\log x}$, $x > 1$, is non-increasing, we obtain

$$\rho_{A_0}(z) \leq \frac{g_2(|w|)}{|z|} \leq \frac{g_2(s)}{|z|} = q_2(|z|). \qquad \square$$

Remark 3. Our argument actually solves the following extremal problem for the hyperbolic metric. Let $\mathscr{K}$ be the family of all full compact sets $K \subset \mathbf{C}$ satisfying $0 \in K$ and $\mathrm{cap}(K) = 1$, where $\mathrm{cap}(K)$ denotes the logarithmic capacity of K. Then

$$\inf \rho_{\mathbf{C}\backslash K}(z) = \rho_{\mathbf{C}\backslash[-4,0]}(|z|) \quad (= q_1(|z|)), \quad z \neq 0,$$

where the infimum is taken over all $K \in \mathscr{K}$ satisfying $z \notin K$, and

$$\sup \rho_{\mathbf{C}\backslash K}(z) = \rho_{\mathbf{C}\backslash[0,4]}(|z|) \quad (= q_2(|z|)), \quad |z| > 4,$$

where the supremum is taken over all $K \in \mathscr{K}$.

3. Proof of Theorem 3

Let $h(w) = w^d$. Then we obtain the following commutative diagram which consists of covering maps.

$$\begin{array}{ccc} A_0 & \xrightarrow{P^n} & A_0 \\ \varphi\big\downarrow & & \big\downarrow\varphi \\ \mathbf{C}\backslash\bar{\Delta} & \xrightarrow{h^n} & \mathbf{C}\backslash\bar{\Delta} \end{array}$$

Since the hyperbolic metric is invariant under covering maps, we obtain

$$\begin{array}{ccc} \rho_{A_0}(z)|dz| & = & \rho_{A_0}(Z)|dZ| \\ \| & & \| \\ \rho_{\mathbf{C}\backslash\bar{\Delta}}(w)|dw| & = & \rho_{\mathbf{C}\backslash\bar{\Delta}}(W)|dW| \end{array} \qquad \left(\begin{array}{ccc} z & \xrightarrow{P^n} & Z \\ \varphi\big\downarrow & & \big\downarrow\varphi \\ w & \xrightarrow{h^n} & W \end{array}\right).$$

Let $z \in A_0$ and $Z = z_n = P^n(z)$. Assume $|Z| > 1$. Then

$$\frac{|dz|}{d_n(z,J)} = \rho_{\mathbf{C}\backslash\bar{\Delta}}(Z)|dZ|. \tag{4}$$

For $z \in A_0 \cap (\mathbf{C}\backslash\bar{\Delta})$, set

$$H(z) = \frac{\rho_{A_0}(z)}{\rho_{\mathbf{C}\backslash\bar{\Delta}}(z)},$$

equivalently

$$H(z) = |z| \log|z| \rho_{A_0}(z) = \frac{|z| \log|z|}{|w| \log|w|} |\varphi'(z)|, \quad w = \varphi(z). \tag{5}$$

The functions p_1, p_2 which appeared in Theorem 3 are given by

$$p_1(x) = (x \log x) q_1(x) = \left\{ 2\sqrt{1 + \frac{4}{x}} \cdot \log\left(\frac{\sqrt{x} + \sqrt{x+4}}{2}\right) \right\}^{-1} \log x,$$

$$p_2(x) = (x \log x) q_2(x) = \left\{ 2\sqrt{1 - \frac{4}{x}} \cdot \log\left(\frac{\sqrt{x} + \sqrt{x-4}}{2}\right) \right\}^{-1} \log x.$$

Theorem 4 gives a sharp estimate

$$p_1(|z|) \le H(z) \le p_2(|z|), \quad |z| > 4. \tag{6}$$

Since $p_1(x) \ge (1 + 4/x)^{-1}$, $p_2(x) \le (1 - 4/x)^{-1}$, $x > 4$, we obtain

$$\left(1 + \frac{4}{|z|}\right)^{-1} \le H(z) \le \left(1 - \frac{4}{|z|}\right)^{-1}, \quad |z| > 4. \tag{7}$$

Now Theorem 3 is a consequence of the following lemma.

LEMMA 3. *Let $z \in A_0$ and $z_n \in \mathbf{C}\backslash\bar{\Delta}$. Then*

$$H(z_n) = \rho_{A_0}(z) d_n(z, J).$$

Proof. Let $Z = z_n = P^n(z)$. Because of (4), we have

$$H(z_n) = \frac{\rho_{A_0}(Z)|dZ|}{\rho_{\mathbf{C}\backslash\bar{\Delta}}(Z)|dZ|} = \frac{\rho_{A_0}(z)|dz|}{\rho_{\mathbf{C}\backslash\bar{\Delta}}(Z)|dZ|} = \rho_{A_0}(z) d_n(z, J). \qquad \square$$

Remark 4. Theorem 3 provides a sharp estimate. Let $P(z) = z^2 + 4z$, $z > 0$, and $R = z_n > 4$. Then $J = [-4, 0]$ and the equality holds in the left-hand side inequality. Also, let $P(z) = (z-2)^2$, the polynomial in Remark 1, $z > 4$, and $R = z_n$. Then the equality holds in the other inequality.

COROLLARY 3. *Let* $z \in A_0$, $R > 4$, *and* $|z_n| \geq R$. *Then*

$$\left(1+\frac{4}{R}\right)^{-1} \leq \rho_{A_0}(z) d_n(z, J) \leq \left(1-\frac{4}{R}\right)^{-1}.$$

4. Proof of Theorems 1 and 2

Let B_z be the open disc with center $z \in A_0$ and radius $d(z, J)$, and $\Omega_w = \mathbf{C} \backslash (\bar{\Delta} \cup l_w)$, where $|w| > 1$ and $l_w = \{tw/|w| \mid t \leq -1\}$.

LEMMA 4. *Let* $z \in A_0$ *and* $w = \varphi(z)$. *Then*

$$\rho_{\Omega_w}(w)|\varphi'(z)| \leq \rho_{B_z}(z) \leq 4\rho_{\Omega_w}(w)|\varphi'(z)|.$$

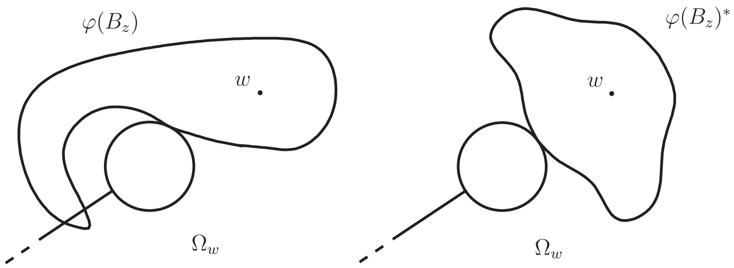
Proof. Let $\psi : \Delta \to \varphi^{-1}(\Omega_w)$ be a conformal map satisfying $\psi(0) = z$. Applying the Köbe one-quarter theorem, we have

$$\frac{|\psi'(0)|}{4} \leq d(z, \partial\varphi^{-1}(\Omega_w)) \leq d(z, J).$$

Since $\rho_{\varphi^{-1}(\Omega_w)}(z) = 2/|\psi'(0)|$, we obtain

$$\rho_{\Omega_w}(w)|\varphi'(z)| = \rho_{\varphi^{-1}(\Omega_w)}(z) \geq \frac{1}{2d(z, J)} = \frac{1}{4}\rho_{B_z}(z).$$

Next, let $\varphi(B_z)^*$ be the circular symmetrization of the domain $\varphi(B_z)$ with respect to the half line $\{-tw \mid t \geq 0\}$.



Then $\varphi(B_z)^* \subset \Omega_w$ and $\rho_{\varphi(B_z)^*}(w) \leq \rho_{\varphi(B_z)}(w)$ (cf. [5] Theorem 4.9), hence

$$\rho_{B_z}(z)|\varphi'(z)|^{-1} = \rho_{\varphi(B_z)}(w) \geq \rho_{\varphi(B_z)^*}(w) \geq \rho_{\Omega_w}(w). \qquad \square$$

LEMMA 5.

$$\rho_{\Omega_w}(w) = \frac{|w| + 1}{2|w|(|w| - 1)}.$$

Proof. We may assume $w > 1$. Then the conformal map $\psi : \Omega_w \to \Delta$ is given by

$$\psi = h_5 \circ h_4 \circ h_3 \circ h_2 \circ h_1,$$

where

$$h_1(z) = \frac{1}{z}, \quad h_2(z) = \sqrt{z}, \quad h_3(z) = \frac{i - z}{i + z}, \quad h_4(z) = z^2, \quad h_5(z) = \frac{z - i}{z + i},$$

and $|\arg \sqrt{z}| < \frac{\pi}{2}$. Hence, a simple calculation shows

$$\rho_{\Omega_w}(w) = \frac{2|\psi'(w)|}{1 - |\psi(w)|^2} = \frac{|w| + 1}{2|w|(|w| - 1)}. \qquad \square$$

Then Lemmas 4 and 5 mean

COROLLARY 4. *Let $z \in A_0$ and $w = \varphi(z)$. Then*

$$\frac{|w|(|w| - 1)}{(|w| + 1)|\varphi'(z)|} \le d(z, J) \le 4 \cdot \frac{|w|(|w| - 1)}{(|w| + 1)|\varphi'(z)|}.$$

Let G be the Green function of A with pole at ∞. Then $G = \log|\varphi|$, hence

COROLLARY 5. *Let $z \in A_0$. Then*

$$\frac{\tanh \frac{G(z)}{2}}{|\nabla G(z)|} \le d(z, J) \le 4 \cdot \frac{\tanh \frac{G(z)}{2}}{|\nabla G(z)|}.$$

If P is a quadratic polynomial, then an improved version can be obtained (§6).

Remark 5. Let $P(z) = z^2 - 2$. Then $z = \phi(w) = w + w^{-1}$ and $J = [-2, 2]$. Hence, if $z > 2$, then $w > 1$ and

$$\rho_{\Omega_w}(w)|\varphi'(z)| = \rho_{\mathbf{C} \setminus (-\infty, 2]}(z) = \frac{1}{2(z - 2)}, \quad \rho_{B_z}(z) = \frac{2}{z - 2}.$$

Hence the right-hand side inequality of Lemma 4 (and so the left-hand side inequalities of Corollaries 4 and 5) is sharp.

Remark 6. Let $P_c(z) = z^2 + c$. The Mandelbrot set M is the set of parameter $c \in \mathbf{C}$ such that the sequence $\{P_c^n(0)\}$ is bounded. Then

$$\varphi_0(c) = \lim_{n\to\infty} (P_c^n(c))^{1/2^n}$$

is a conformal map of $\hat{\mathbf{C}} \setminus M$ onto $\hat{\mathbf{C}} \setminus \bar{\Delta}$ ([1]). Hence we can show the corresponding result for the Mandelbrot set in the same way;

$$\frac{\tanh \dfrac{G_0(z)}{2}}{|\nabla G_0(z)|} \le d(z, M) \le 4 \cdot \frac{\tanh \dfrac{G_0(z)}{2}}{|\nabla G_0(z)|}, \quad z \in \mathbf{C} \setminus M, \tag{8}$$

where $G_0(z) = \log|\varphi_0(z)|$ is the Green function of $\hat{\mathbf{C}} \setminus M$ with pole at ∞. Our estimate (8) improves the known result ([7], [9])

$$\frac{\sinh G_0(z)}{2e^{G_0(z)}|\nabla G_0(z)|} \le d(z, M) \le \frac{2 \sinh G_0(z)}{|\nabla G_0(z)|}.$$

Milnor [7] conjectured that the function $p(z) = \dfrac{2 \sinh G_0(z)}{|\nabla G_0(z)|}$ on the right-hand side gives the right order for $d(z, M)$ in the sense that $p(z)/4 \le d(z, M) \le p(z)$ holds near M. Later, he stated in [6] (p. 273) that from the Köbe one-quarter theorem this inequality holds globally on $\mathbf{C} \setminus M$. However, this claim does not hold, since $d(z, M)|\nabla G_0(z)| \to 1$ as $z \to \infty$. We can not apply the Köbe one-quarter theorem to $\hat{\mathbf{C}} \setminus M$ which is not a plane domain.

Our result shows that $q(z) = \dfrac{4 \tanh(G_0(z)/2)}{|\nabla G_0(z)|}$ is just the function which provides the right order even in global sense.

Let

$$g(x) = \frac{x - 1}{(x + 1) \log x}, \quad x > 1.$$

Lemma 6. *Let $z \in A_0$ and $w = \varphi(z)$. Assume $|z_n| > 4$. Then*

$$\frac{g(|w|)}{H(z_n)} \le \frac{d(z, J)}{d_n(z, J)} \le 4 \cdot \frac{g(|w|)}{H(z_n)}. \tag{9}$$

Proof. Since $\rho_{A_0}(z)|dz| = \rho_{\mathbf{C} \setminus \bar{\Delta}}(w)|dw|$, the assertion is a consequence of Lemma 3 and Corollary 4. □

The factor $g(|w|)$ corresponds to the initial error and $H(z_n)$ the terminal error. It is to be noted that there is no factor corresponding to the process of the iteration, which is a consequence of the invariance of the hyperbolic metric.

Proof of Theorem 1. Let $z \in A_0$, $w = \varphi(z)$, and $|z_n| > 4$. Set $t = \frac{1}{4}(\sqrt{|z|} + \sqrt{|z| + 4})^2$. Then $|w| \le t$ by Corollary 2. Since g is non-increasing function satisfying $g(1+0) = 1/2$, we have

$$r(|z|) = g(t) \le g(|w|) \le \frac{1}{2}, \tag{10}$$

where

$$r(x) = \sqrt{\frac{x}{x+4}}\left(2\log\left(\frac{\sqrt{x} + \sqrt{x+4}}{2}\right)\right)^{-1}, \quad x > 0.$$

Thus, from (7) and Lemma 6, we obtain

$$r(|z|)\left(1 - \frac{4}{|z_n|}\right) \le \frac{d(z,J)}{d_n(z,J)} \le 2\left(1 + \frac{4}{|z_n|}\right). \tag{11}$$

Hence, Theorem 1 follows from the estimate

$$r(x) \ge \frac{1}{2}\left(1 - \frac{x}{12}\right), \quad x > 0. \qquad \square$$

Remark 7. The left-hand side inequalities of (9) and (10) are sharp. The equalities hold for $P(z) = (z-2)^2$ and $z > 4$. Hence the left-hand side inequality of (11) is also sharp in the sense that we can not replace the function $r(|z|)$ with larger one.

Proof of Theorem 2. Let $z \in A_0$, $R \ge 9$, and $|z_n| \ge R$. Set $\tilde{P}(\zeta) = P(\zeta + \alpha) - \alpha$, where α is the point on J satisfying $d(z,J) = |z - \alpha|$. Then the Julia set $\tilde{J}$ of $\tilde{P}$ is given by $\tilde{J} = J - \alpha$. Let $\tilde{z} = z - \alpha$. Then $\tilde{z}_n = z_n - \alpha$, $\tilde{z}_n' = z_n'$. Applying Theorem 1 to $\tilde{P}$ and $\tilde{z}$, we obtain

$$\frac{1}{2}\left(1 - \frac{|\tilde{z}|}{12}\right)\left(1 - \frac{4}{|\tilde{z}_n|}\right) \le \frac{d(\tilde{z},\tilde{J})}{d_n(\tilde{z},\tilde{J})},$$

which implies

$$\frac{1}{2}\left(1 - \frac{d(z,J)}{12}\right)\left(1 - \frac{4}{|z_n - \alpha|}\right)\frac{|z_n - \alpha|\log|z_n - \alpha|}{|z_n|\log|z_n|} \le \frac{d(z,J)}{d_n(z,J)}.$$

Let $u(x) = (x-8)\log(x-4) - (x-9)\log x$, $x \ge 9$. Then u' is increasing on $[9, 28 + 8\sqrt{10}]$, decreasing on $[28 + 8\sqrt{10}, \infty)$, $u'(x) \to +0$ $(x \to \infty)$, $u(27) > 0.2$, and $-0.1 \le u'(27) < 0 < u'(28) < 0.1$. Hence

$$u(x) \ge u(x_0) \ge u(27) - 0.1 > 0,$$

where x_0 $(27 < x_0 < 28)$ is the unique solution of $u'(x) = 0$. Since $|z_n - \alpha| \geq |z_n| - 4$, we have

$$\left(1 - \frac{4}{|z_n - \alpha|}\right)\frac{|z_n - \alpha| \log|z_n - \alpha|}{|z_n| \log|z_n|} \geq \frac{|z_n| - 8}{|z_n|} \cdot \frac{\log(|z_n| - 4)}{\log|z_n|}$$

$$\geq \frac{R - 8}{R} \cdot \frac{\log(R - 4)}{\log R}$$

$$\geq 1 - \frac{9}{R}.$$

Hence

$$\frac{1}{2}\left(1 - \frac{d(z,J)}{12}\right)\left(1 - \frac{9}{R}\right) \leq \frac{d(z,J)}{d_n(z,J)}.$$

Therefore

$$d(z,J) \geq \frac{1}{2}\left(1 - \frac{9}{R}\right)\left(\left(1 - \frac{9}{R}\right)\frac{d_n(z,J)}{24} + 1\right)^{-1} d_n(z,J)$$

$$\geq \frac{1}{2}\left(1 - \frac{9}{R}\right)\left(1 - \frac{d_n(z,J)}{24}\right)d_n(z,J). \qquad \square$$

Proof of Corollary 1. The right-hand side inclusion follows from Theorem 2. Next, let $0 < \varepsilon < 1$ and $z \in K_{\varepsilon_1}$. Then, from Theorem 2 again, we have $u(d_n(z,J)) < u(\varepsilon)$, where $u(x) = (1 - x/24)x$. Moreover, $|z| \leq \varepsilon_1 + 4 \leq 5$. Thus, from Theorem 1, $d_n(z,J) < 12$ holds. Since u is non-decreasing on $[0,12]$, we obtain $d_n(z,J) < \varepsilon$. $\square$

5. Sharpness of Theorems 1 and 2

Let $P(z) = z^2 + 4z$. Then $J = [-4, 0]$, $z = \phi(w) = w + w^{-1} - 2$, and

$$z_n = w^{2^n} + w^{-2^n} - 2, \quad z_n' = \frac{2^n(w^{2^n} - w^{-2^n})}{w - w^{-1}}.$$

Let $z > 0$. Then $w = \left(\frac{1}{2}(\sqrt{z} + \sqrt{z+4})\right)^2 > 1$, $d(z,J) = z$, and

$$d_n(z,J) = \frac{1}{2^n} \cdot \frac{w^2 - 1}{w} \cdot \frac{(w^{2^n} - 1)\log(w^{2^n} + w^{-2^n} - 2)}{w^{2^n} + 1} \tag{12}$$

$$\to \frac{(w^2 - 1)\log w}{w} \quad (n \to \infty)$$

$$= 2z\left(1 + \frac{z}{12}\right) + O(z^3) \quad (z \to +0).$$

Hence

$$\frac{d(z,J)}{d_n(z,J)} \to \frac{zw}{(w^2-1)\log w} \quad (n \to \infty)$$
$$= \frac{1}{2}\left(1 - \frac{z}{12}\right) + O(z^2) \quad (z \to +0).$$

Hence the left-hand side inequality in Theorem 1 is sharp in the sense that we can not replace the constant $1/12$ with smaller one. Similarly, from (12), we can show that the left-hand side inequality in Theorem 2 is sharp in the sense that we can not replace the constant $1/24$ with smaller one. In particular, if R is sufficiently large and z is sufficiently near J, then

$$\frac{1}{2}(1-\varepsilon) \le \frac{d(z,J)}{d_n(z,J)} \le 2(1+\varepsilon), \tag{13}$$

by Theorem 2. The constant factor $\frac{1}{2}$ in (13) is sharp.

Next, we consider about the sharpness of the right-hand side inequality in Theorem 1. Assume that we can take sequences of polynomial P_k and point $z^{(k)} \in A_0^{(k)}$, $d(z^{(k)}, J_k) \to 0$, so that there exists linear transformation $T_k(z) = a_k z + b_k$ satisfying $T_k(z^{(k)}) = 0$ and $T_k(A_0^{(k)}) \to \Delta$ in the sense of kernel convergence with respect to the origin, where $A_0^{(k)}$ and J_k denote A_0 and J respectively with respect to P_k ([4], [10]). Then $\rho_{A_0^{(k)}}(z^{(k)}) d(z^{(k)}, J_k) \to 2$, which implies the constant factor 1 on the left-hand side inequality in Lemma 4 is sharp. Hence, the constant factor 2 in Theorem 1 (and so in Theorem 2) is also sharp.

We expect that such a situation may happen.

Conjecture 1. *The constant factor* 2 *in the right-hand side inequality of Theorem* 1 (*and so of Theorem* 2) *is also sharp.*

Example 1. Let

$$P(z) = z^2 + c, \quad c = 0.250057091821313095 - 0.000000680499928995i,$$

$R = 10000000$, and $z = 0.1272 + 0.2672i$ (Figure 3). Then $d(z,J) \approx 0.00402$, $d_n(z,J) \approx 0.00228$, and so

$$\frac{d(z,J)}{d_n(z,J)} \approx 1.76.$$

A_0 is a disk-like domain with center z in the sense of kernel convergence.

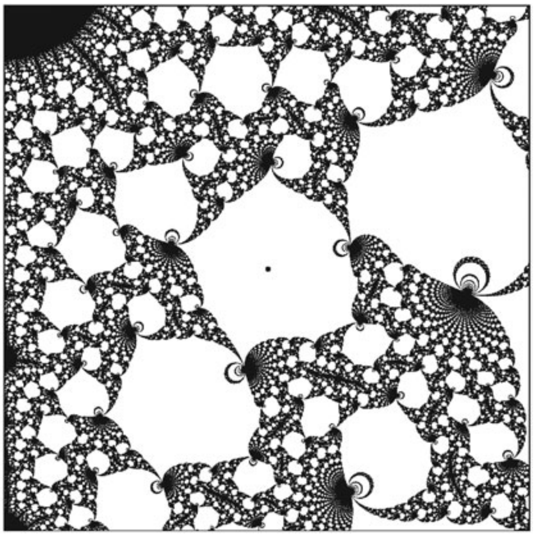


FIGURE 3. $\dfrac{d(z,J)}{d_n(z,J)} \approx 1.76$. z is the center point.

6. Julia sets with rotational symmetricity

In the present section, we assume that J is invariant under rotation of order k, $k \geq 1$, with respect to the origin. Such a polynomial P is characterized as a polynomial of the form

$$P(z) = z^m Q(z^k),$$

where m is non-negative integer and Q is a polynomial ([1], [2]). Then its square root transformation of order k

$$\hat{P}(z) = (P(\sqrt[k]{z}))^k = z^m (Q(z))^k$$

is also a polynomial, and the Böttcher map $\hat{\varphi}$ of $\hat{P}$ is also given by the square root transformation of the Böttcher map φ of P;

$$\hat{\varphi}(z) = (\varphi(\sqrt[k]{z}))^k.$$

The Julia set $\hat{J}$ of $\hat{P}$ is given by

$$\hat{J} = J^k = \{z^k \mid z \in J\}.$$

Hence, applying Lemma 2 to $\hat{P}$, we have

LEMMA 7. $K \subset \{|z| \leq \sqrt[k]{4}\}$.

Remark 8. Let $P(z) = z^2 - 2$. Then $J = [-2, 2]$ is invariant under rotation of order 2. In this case $\hat{P}(z) = (z-2)^2$, which is just the polynomial in Remark

1, and $\hat{J} = [0,4] = J^2$. This example shows that if $k = 2$, the lemma above is sharp. We do not know about the sharpness for general k.

Let $\hat{A}_0$ be the basin of the super attracting fixed point at ∞ of $\hat{P}$, and

$$\hat{H}(\hat{z}) = \frac{\rho_{\hat{A}_0}(\hat{z})}{\rho_{\mathbf{C}\setminus\bar{\Delta}}(\hat{z})}, \quad \hat{z} \in \hat{A}_0 \cap (\mathbf{C}\setminus\bar{\Delta}).$$

Then, $\hat{A}_0 = (A_0)^k$, and $z \mapsto \hat{z} = z^k$ induces covering maps of $\mathbf{C}\setminus\bar{\Delta}$ onto $\mathbf{C}\setminus\bar{\Delta}$ and of A_0 onto $\hat{A}_0$. Hence

$$\hat{H}(\hat{z}) = \frac{\rho_{\hat{A}_0}(\hat{z})|d\hat{z}|}{\rho_{\mathbf{C}\setminus\bar{\Delta}}(\hat{z})|d\hat{z}|} = \frac{\rho_{A_0}(z)|dz|}{\rho_{\mathbf{C}\setminus\bar{\Delta}}(z)|dz|} = H(z), \quad \hat{z} = z^k.$$

Thus, applying (6) and (7) to $\hat{H}$, we have

$$p_1(|z|^k) \le H(z) \le p_2(|z|^k), \quad |z| \ge \sqrt[k]{4}$$

$$\left(1 + \frac{4}{|z|^k}\right)^{-1} \le H(z) \le \left(1 - \frac{4}{|z|^k}\right)^{-1}, \quad |z| \ge \sqrt[k]{4}. \tag{14}$$

Thus, we can generalize Theorem 3 as follows.

Theorem 5. *Let $z \in A_0$, $R \ge \sqrt[k]{4}$, and $|z_n| \ge R$. Then*

$$p_1(|z|^k) \le \rho_{A_0}(z) d_n(z,J) \le p_2(|z|^k).$$

Also Theorem 1 can be generalized as follows.

Theorem 6. *Let $z \in A_0$, $R \ge \sqrt[k]{4}$, and $|z_n| \ge R$. Then*

$$\frac{1}{2}\left(1 - \frac{|z|^k}{12k^2}\right)\left(1 - \frac{4}{R^k}\right) \le \frac{d(z,J)}{d_n(z,J)} \le 2\left(1 + \frac{4}{R^k}\right).$$

Proof. Let $\hat{\phi}(w) = (\phi(\sqrt[k]{w}))^k$, where ϕ is the inverse function of φ. Applying Lemma 1 to $\hat{\phi}$, we have

$$\frac{(|w|^k - 1)^2}{|w|^k} \le |z|^k, \quad z \in A_0.$$

Since

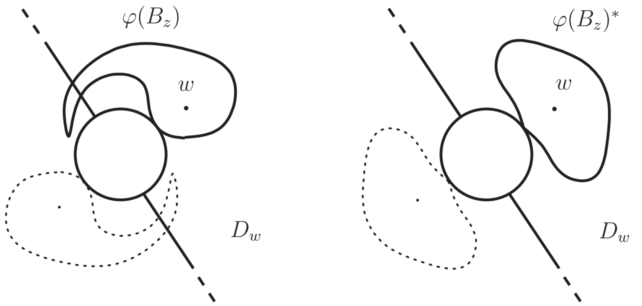
$$g(t) \ge \frac{1}{2} - \frac{(t-1)^2}{24t} \ge \frac{1}{2} - \frac{(t^k - 1)^2}{24k^2 t^k}, \quad t > 1,$$

we obtain

$$g(|w|) \ge \frac{1}{2} - \frac{(|w|^k - 1)^2}{24k^2|w|^k} \ge \frac{1}{2} - \frac{|z|^k}{24k^2}.$$

Thus Lemma 6 and (14) establish the desired result. □

Finally, we give a remark on Corollaries 4 and 5. Let $P(z) = z^2 + c$. Then J has rotational symmetricity of order 2 with respect to the origin. Hence we may replace the inclusion $\varphi(B_z)^* \subset \Omega_w$ in the proof of Lemma 4 with $\varphi(B_z)^* \subset D_w$, where $D_w = \{z \in \mathbf{C} \backslash \overline{\Delta} \mid \mathrm{Re}(z/w) > 0\}$, and so we have $\rho_{D_w}(w)|\varphi'(z)| \le \rho_{B_z}(z)$.



Since

$$\rho_{D_w}(w) = \frac{|w|^2 + 1}{|w|(|w|^2 - 1)},$$

we can improve the right-hand side inequalities of Corollaries 4 and 5 as follows.

Proposition 2. *Let $z \in A_0$ and $w = \varphi(z)$. Then*

$$d(z, J) \le \frac{2|w|(|w|^2 - 1)}{(|w|^2 + 1)|\varphi'(z)|} = \frac{2 \tanh G(z)}{|\nabla G(z)|}.$$

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