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RIEMANN ZETA VIA THE CATEGORY OF MONOIDS

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I. Introduction

It is expected by many mathematicians that in order to study the zeta function of $\mathbf{Z}$ it would be nice to be able to treat $\mathbf{Z}$ as a variety over the field F_1 with one element and then follow the Deligne's program on proving the Weil conjectures over finite fields. I do not know who is to be blamed for the origin of this idea. For me its is simple, I have heard about if from C. Soule and he is definitely one of key persons in this subject, compare [S] for example. Of course field F_1 should have characteristic 1 which means that it should carry only one operation. This means that the place where we should look for such a theory is the world of monoids. In the following short note we try to justify the statement that the category of abelian monoids is a good place for calculating the Riemann zeta function of $\mathbf{Z}$.

Of course the idea that $\mathbf{Z}$ treated as a variety over F_1 should live in the category of monoids is well described in the literature, see for example [KOW] or [D]. But most authors instead of working with monoids directly extend their field of scalars from F_1 to $\mathbf{Z}$ (or other rings), assuming that scalar extension from F_1 to $\mathbf{Z}$ should take a monoid A to its monoid ring $\mathbf{Z}[A]$. This agrees well with the expectation that rings should be treated in the category of monoids as monoids with ring multiplication as a monoidal operation. Then the forgetful functor from rings to monoids and the scalar extension as described above give us the nice pair of adjoint functors. But this approach carries one disadvantage. It takes us quickly from formally new approach via monoids to the classical world of rings and modules over them or to other abelian categories. In the present note we show only that the (categorical) zeta function of the category of abelian monoids calculates also the Riemann zeta function of integers.

II. Zeta function for the category of monoids

Let us start from recalling after Kurokawa (compare [K]) the definition of the zeta function of a category with 0. If $\mathscr{C}$ is a category with 0 we say that $X \in Ob(\mathscr{C})$ is simple if for any object Y the set $Hom_{\mathscr{C}}(X, Y)$ consists only of monomorphisms and 0. Let $N(X)$, the norm of X, denote the cardinality of the

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set $End_{\mathscr{C}}(X)$. We say that an object X is finite if $N(X)$ is finite. We denote by $P(\mathscr{C})$ the isomorphism classes of all finite simple objects of $\mathscr{C}$. Then we define the zeta function of $\mathscr{C}$ as

$$\zeta(s, \mathscr{C}) = \prod_{P \in P(\mathscr{C})} (1 - N(P)^{-s})^{-1}$$

In [K] Kurokawa studied the properties of such zeta functions but for us the crucial is the following straightforward observation:

Remark 2.1. Let Ab denote the category of abelian groups and ζ_R stands for the Riemann zeta function of the integers. It is obvious that the finite cyclic groups of prime order form the full set of isomorphism classes of finite simple objects in Ab. So we have

$$\zeta(s, Ab) = \zeta_R$$

Let MAb denote the category of abelian monoids with unit and unital maps. Then

Lemma 2.2. *Assume that M is a finite simple object of MAb. Then M is isomorphic to a cyclic group of prime order or to the monoid B of two elements $\{1, b\}$ where 1 is a unit and $b^2 = b$.*

Proof. First of all observe that in MAb we have well defined notion of a quotient object. If M is a unital monoid and $A \subset M$ is a submonoid then formula

$$m_1 \sim m_2 \Leftrightarrow \exists a_1, a_2 \in A \quad m_1 a_1 = m_2 a_2$$

defines the equivalence relation on M where every element of A is equivalent to the unit. Then the obvious formula $[m_1] \cdot [m_2] = [m_1 m_2]$ defines the monoid structure on the set of equivalence classes. The quotient monoid M/A comes with a monoidal homomorphism $M \to M/A$ given by the formula $q(m) = [m]$. If M has a quotient M/A which contains more than one element then M is usually not simple because then the map q is neither monomorphism nor 0.

Let now $1 \neq x \in M$ and let $C = \langle x \rangle$ be a multiplicative set in M generated by x. It means it consists of the powers of x. We have to consider three possibilities:

1. There is $x \in M$ such that $1 \in C$.

Then C is a finite cyclic group C_k of order k, where k is the smallest integer such that $x^k = 1$. It is easy to see that assumption that M is simple implies k is a prime. If not then let $p|k$. Then M contains a submonoid C_p and $M \to M/C_p$ is neither monomorphism nor 0. Observe that cyclic groups of prime order are simple as monoids: If $f : \langle x \rangle = C_p \to X$ is a nontrivial map of monoids then $f^{-1}(1)$ is a submonoid of C_p and hence consists only of the unit. The image of x has to be an element of order p in X so f is an injection of sets. It means here that f is a categorical monomorphism. Hence the proof of our lema in this case will be finished if we show that in our case $M = C$.

Assume that $y \in M \backslash C$. Then for every $c \in C$ the classes $[y]$ and $[c]$ are different in M/C because if $[c] = [y]$ then there exist $s, t \in C$ such that $sc = ty$. But then $y = t^{-1}sc \in C$. So if $M \neq C$ then the order of M/C is bigger than one and M is not simple. To justify the last statement consider two monoid maps $f, g : C_p \to M$ from the cyclic group of order p to M. The first one is trivial, $f(C_p) = 1$, and the second is is an isomorphism between C and C_p. If $q : M \to M/C$ is a quotient map then q is not 0 and $q \circ f = q \circ g$ but $f \neq g$. Hence q is not a monomorphism.

2. There exists $x \in M$ such that for every n we have $x^n \neq 1$ and C is infinite as a set.

It is easy to see that in this case C is isomorphic as a monoid (not unital) to the monoid of natural numbers with addition. The point is that the assumption on the cardinality of C implies that for $k < m$, $x^k \neq x^m$. Otherwise, if for $k < m$, $x^k = x^m$ then C is equal to the set $\{x, x^2, \ldots, x^k, \ldots, x^{m-1}\}$ which is finite. So C is infinite and let A be a submonoid of M consisting of 1 and even powers of x. Then the quotient map $q : M \to M/A$ is neither 0 nor a monomorphism. Hence M is not simple.

3. For every $x \in M$ and n we have $x^n \neq 1$ and C is finite as a set.

We want to show first that in such a case the monoid B (isomorphic to) is contained in M. Because C is finite there exist natural numbers k and l such that $x^l x^k = x^k$. Assume that we have such k and l and we choose minimal k with this property and for it we choose minimal l. If $l = k$ we have $x^k x^k = x^k$ so the set $\{1, x^k\}$ is a submonoid of M isomorphic to B, as we wanted. By minimality of k and l we know that $C = \{x, x^2, \ldots, x^k, \ldots, x^{k+l-1}\}$ and the multiplication by x acts as a bijection on the set $\{x^k, \ldots, x^{k+l-1}\}$. Assume that $l > k$. From the previous observation it follows immediately that $x^l x^l = x^l$ and again we have B in M. Assume now that $l < k$. Then there exists s such that $k \leq sl < k + l$. Again we check directly that $x^{sl} x^{sl} = x^{sl}$.

So we know that if condition **3** is satisfied then M contains a submonoid isomorphic to B. Denote it $M' = \{1, m\}$. Of course we want to underline that $m \neq 1$. Then m is not invertible in M because if $1 = mx$ then $m = mmx = mx = 1$ and we have a contradiction. Let M^* denote the set of invertible elements of M. M^* is not empty and also $m \in (M \backslash M^*)$. We have a homomorphism of monoids $f : M \to B$ which takes elements of M^* to the unit and $f(M \backslash M^*) = b$. If $M \neq M'$ then this map is neither 0 nor a monomorphism. To see this let $f' : M \to M$ be a homomorphism of monoids which takes M^* to 1 and $M \backslash M^*$ to $m \in M'$. Then $f \circ f' = f \circ id$ but $f' \neq id$. This finishes the proof of our lemma.

THEOREM 2.3.

$$\zeta_R \cdot (1 - 2^{-s})^{-1} = \zeta(s, MAb)$$

Proof. This is obvious after the previous lemma. The isomorphism classes of finite simple objects in MAb are the same as in Ab plus additional class of B. Of course the cardinality of $End_{MAb}(B)$ equals to 2 and hence we have our formula.

Nothing special follows from the observation above. But we can view it as a strong evidence supporting the idea that in order to study the Riemann zeta function of integers we can work in the category of abelian monoids.

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