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THE DIXMIER-DOUADY CLASS IN THE SIMPLICIAL DE RHAM COMPLEX

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Abstract

On the basis of A. L. Carey, D. Crowley, M. K. Murray's work, we exhibit a cocycle in the simplicial de Rham complex which represents the Dixmier-Douady class.

1. Introduction

In [5, Carey, Crowley, Murray], they proved that when a Lie group G admits a central extension $1 \to U(1) \to \hat{G} \to G \to 1$, there exists a characteristic class of principal G-bundle $\pi : Y \to M$ which belongs to a cohomology group $H^2(M, \underline{U(1)}) \cong H^3(M, \mathbf{Z})$. Here $\underline{U(1)}$ stands for a sheaf of continuous $U(1)$-valued functions on M. This class is called a Dixmier-Douady class associated to the central extension $\hat{G} \to G$.

On the other hand, we have a simplicial manifold $\{NG(*)\}$ for any Lie group G. It is a sequence of manifolds $\{NG(p) = G^p\}_{p=0,1,\dots}$ together with face maps $\varepsilon_i : NG(p) \to NG(p-1)$ for $i = 0, \dots, p$ satisfying relations the $\varepsilon_i \varepsilon_j = \varepsilon_{j-1} \varepsilon_i$ for $i < j$. (The standard definition also involves degeneracy maps but we do not need them here.) Then the n-th cohomology group of classifying space BG is isomorphic to the total cohomology of a double complex $\{\Omega^q(NG(p))\}_{p+q=n}$. See [3] [6] [9] for details.

In this paper we will exhibit a cocycle on $\Omega^*(NG(*))$ which represents the Dixmier-Douady class due to Carey, Crowley, Murray. Such a cocycle is also studied in a general setting by K. Behrend, J.-L. Tu, P. Xu and C. Laurent-Gengoux [1] [2] [13] [14], and G. Ginot, M. Stiénon [7] but our construction of the cocycle is different from theirs, and the proof is more simple. Stevenson [12] also exhibited a cocycle which represents the Dixmier-Douady class in singular cohomology group instead of the de Rham cohomology. As a consequence of our result, we can show that if G is given a discrete topology, the Dixmier-Douady class in $H^3(BG^\delta, \mathbf{R})$ is 0. Furthermore, we can exhibit the "Chern-Simons form" of Dixmier-Douady class on $\Omega^*(N\bar{G}(*))$. Here $N\bar{G}$ is a simplicial manifold which plays the role of universal bundle.

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The outline is as follows. In section 2, we briefly recall the notion of simplicial manifold NG and construct a cocycle in $\Omega^*(NG(*))$. In section 3, we recall the definition of a Dixmier-Douady class and prove the main theorem. In section 4, we give the Chern-Simons form of the Dixmier-Douady class.

2. Cocycle on the double complex

In this section first we recall the relation between the simplicial manifold NG and the classifying space BG, then we construct the cocycle on $\Omega^{*,*}(NG)$.

2.1. The double complex on simplicial manifold

For any Lie group G, we define simplicial manifolds NG, $N\bar{G}$ and a simplicial G-bundle $\gamma : N\bar{G} \to NG$ as follows:

$NG(p) = \overbrace{G \times \cdots \times G}^{p-times} \ni (g_1, \dots, g_p)$:
face operators $\varepsilon_i : NG(p) \to NG(p-1)$

$$\varepsilon_i(g_1, \dots, g_p) = \begin{cases} (g_2, \dots, g_p) & i = 0 \\ (g_1, \dots, g_i g_{i+1}, \dots, g_p) & i = 1, \dots, p-1 \\ (g_1, \dots, g_{p-1}) & i = p \end{cases}$$

$N\bar{G}(p) = \overbrace{G \times \cdots \times G}^{p+1-times} \ni (h_1, \dots, h_{p+1})$:
face operators $\bar{\varepsilon}_i : N\bar{G}(p) \to N\bar{G}(p-1)$

$$\bar{\varepsilon}_i(h_1, \dots, h_{p+1}) = (h_1, \dots, h_i, h_{i+2}, \dots, h_{p+1}) \quad i = 0, 1, \dots, p$$

And we define $\gamma : N\bar{G} \to NG$ as $\gamma(h_1, \dots, h_{p+1}) = (h_1 h_2^{-1}, \dots, h_p h_{p+1}^{-1})$.

To any simplicial manifold $X = \{X_*\}$, we can associate a topological space $\|X\|$ called the fat realization. Since any G-bundle $\pi : E \to M$ can be realized as the pull-back of the fat realization of γ, $\|\gamma\|$ is an universal bundle $EG \to BG$ [11].

Now we construct a double complex associated to a simplicial manifold.

Definition 2.1. For any simplicial manifold $\{X_*\}$ with face operators $\{\varepsilon_*\}$, we define double complex as follows:

$$\Omega^{p,q}(X) \overset{\text{def}}{=} \Omega^q(X_p)$$

Derivatives are:

$$d' := \sum_{i=0}^{p+1} (-1)^i \varepsilon_i^*, \quad d'' := \text{derivatives on } X_p \times (-1)^p \qquad \square$$

For NG and $N\bar{G}$ the following holds ([3] [6] [9]).

THEOREM 2.1. *There exists a ring isomorphism*

$$H(\Omega^*(NG)) \cong H^*(BG), \quad H(\Omega^*(N\bar{G})) \cong H^*(EG)$$

Here $\Omega^*(NG)$ *and* $\Omega^*(N\bar{G})$ *means the total complexes.* □

For a principal G-bundle $Y \to M$ and an open covering $\{U_\alpha\}$ of M, the transition functions $(g_{\alpha_0\alpha_1}, g_{\alpha_1\alpha_2}, \dots, g_{\alpha_{p-1}\alpha_p}) : U_{\alpha_0\alpha_1\cdots\alpha_p} \to NG(p)$ induce the cohomology map $H^*(NG) \to H^*_{\check{C}ech-deRham}(M)$. The elements in the image are the characteristic class of Y [9].

2.2. Construction of the cocycle

Let $\rho : \hat{G} \to G$ be a central extension of a Lie group G and we recognize it as a $U(1)$-bundle. Using the face operators $\{\varepsilon_i\} : NG(2) \to NG(1) = G$, we can construct the $U(1)$-bundle over $NG(2) = G \times G$ as $\delta\hat{G} := \varepsilon_0^*\hat{G} \otimes (\varepsilon_1^*\hat{G})^{\otimes -1} \otimes \varepsilon_2^*\hat{G}$. Here we define the tensor product $S \otimes T$ of $U(1)$-bundles S and T over M as

$$S \otimes T := \bigcup_{x \in M} (S_x \times T_x/(s,t) \sim (su, tu^{-1}), \quad (u \in U(1))$$

LEMMA 2.1. $\delta\hat{G} \to G \times G$ *is a trivial bundle.*

Proof. We can construct a bundle isomorphism $f : \varepsilon_0^*\hat{G} \otimes \varepsilon_2^*\hat{G} \to \varepsilon_1^*\hat{G}$ as follows. First we define f to be the map sending $[((g_1, g_2), \hat{g}_2), ((g_1, g_2), \hat{g}_1)]$ s.t. $\rho(\hat{g}_2) = g_2$, $\rho(\hat{g}_1) = g_1$ to $((g_1, g_2), \hat{g}_1\hat{g}_2)$. Then we have the inverse f^{-1} that sends $((g_1, g_2), \hat{g})$ s.t. $\rho(\hat{g}) = g_1g_2$ to $[((g_1, g_2), \hat{g}_2), ((g_1, g_2), \hat{g}\hat{g}_2^{-1})]$ s.t. $\rho(\hat{g}_2) = g_2$. □

For any connection θ on $\hat{G}$, there is the induced connection $\delta\theta$ on $\delta\hat{G}$ [4, Brylinski].

PROPOSITION 2.1. *Let* $c_1(\theta)$ *denote the* 2*-form on* G *which hits* $\left(\frac{-1}{2\pi i}\right) d\theta \in \Omega^2(\hat{G})$ *by* ρ^*, *and* $\hat{s}$ *any global section of* $\delta\hat{G}$. *Then the following equation holds.*

$$(\varepsilon_0^* - \varepsilon_1^* + \varepsilon_2^*)c_1(\theta) = \left(\frac{-1}{2\pi i}\right) d(\hat{s}^*(\delta\theta)) \in \Omega^2(NG(2)).$$

Proof. Choose an open cover $\mathscr{V} = \{V_\lambda\}_{\lambda \in \Lambda}$ of G such that there exist local sections $\eta_\lambda : V_\lambda \to \hat{G}$ of ρ. Then $\{\varepsilon_0^{-1}(V_\lambda) \cap \varepsilon_1^{-1}(V_{\lambda'}) \cap \varepsilon_2^{-1}(V_{\lambda''})\}_{\lambda, \lambda', \lambda'' \in \Lambda}$ is an open cover of $G \times G$ and there are the induced local sections $\varepsilon_0^*\eta_\lambda \otimes (\varepsilon_1^*\eta_{\lambda'})^{\otimes -1} \otimes \varepsilon_2^*\eta_{\lambda''}$ on that covering.

If we pull back $\delta\theta$ by these sections, the induced form on $\varepsilon_0^{-1}(V_\lambda) \cap \varepsilon_1^{-1}(V_{\lambda'}) \cap \varepsilon_2^{-1}(V_{\lambda''})$ is $\varepsilon_0^*(\eta_\lambda^*\theta) - \varepsilon_1^*(\eta_{\lambda'}^*\theta) + \varepsilon_2^*(\eta_{\lambda''}^*\theta)$. We restrict

$(\varepsilon_0^* - \varepsilon_1^* + \varepsilon_2^*)c_1(\theta)$ on $\varepsilon_0^{-1}(V_\lambda) \cap \varepsilon_1^{-1}(V_{\lambda'}) \cap \varepsilon_2^{-1}(V_{\lambda''})$ then it is equal to $\left(\frac{-1}{2\pi i}\right) d(\varepsilon_0^*(\eta_\lambda^*\theta) - \varepsilon_1^*(\eta_{\lambda'}^*\theta) + \varepsilon_2^*(\eta_{\lambda''}^*\theta))$, because $c_1(\theta) = \sum \left(\frac{-1}{2\pi i}\right) d(\eta_\lambda^*\theta)$.

Also $d(\varepsilon_0^*(\eta_\lambda^*\theta) - \varepsilon_1^*(\eta_{\lambda'}^*\theta) + \varepsilon_2^*(\eta_{\lambda''}^*\theta)) = d(\hat{s}^*(\delta\theta))|_{\varepsilon_0^{-1}(V_\lambda) \cap \varepsilon_1^{-1}(V_{\lambda'}) \cap \varepsilon_2^{-1}(V_{\lambda''})}$ since $\delta\theta$ is a connection form. This completes the proof. □

Proposition 2.2. *For the face operators* $\{\varepsilon_i\}_{i=0,1,2,3} : NG(3) \to NG(2)$,

$$(\varepsilon_0^* - \varepsilon_1^* + \varepsilon_2^* - \varepsilon_3^*)(\hat{s}^*(\delta\theta)) = 0.$$

Proof. We consider the $U(1)$-bundle $\delta(\delta\hat{G})$ over $NG(3) = G \times G \times G$ and the induced connection $\delta(\delta\theta)$ on it. Composing $\{\varepsilon_i\} : NG(3) \to NG(2)$ and $\{\varepsilon_i\} : NG(2) \to G$, we define the maps $\{r_i\}_{i=0,1,\dots,5} : NG(3) \to G$ as follows.

$$r_0 = \varepsilon_0 \circ \varepsilon_1 = \varepsilon_0 \circ \varepsilon_0, \quad r_1 = \varepsilon_0 \circ \varepsilon_2 = \varepsilon_1 \circ \varepsilon_0, \quad r_2 = \varepsilon_0 \circ \varepsilon_3 = \varepsilon_2 \circ \varepsilon_0$$

$$r_3 = \varepsilon_1 \circ \varepsilon_2 = \varepsilon_1 \circ \varepsilon_1, \quad r_4 = \varepsilon_1 \circ \varepsilon_3 = \varepsilon_2 \circ \varepsilon_1, \quad r_5 = \varepsilon_2 \circ \varepsilon_3 = \varepsilon_2 \circ \varepsilon_2$$

Then $\{\bigcap r_i^{-1}(V_{\lambda^{(i)}})\}$ is a covering of $NG(3)$. Since each $\bigcap r_i^{-1}(V_{\lambda^{(i)}})$ is equal to

$$\begin{aligned}\varepsilon_0^{-1}(\varepsilon_0^{-1}(V_\lambda) \cap \varepsilon_1^{-1}(V_{\lambda'}) \cap \varepsilon_2^{-1}(V_{\lambda''})) \cap \varepsilon_1^{-1}(\varepsilon_0^{-1}(V_\lambda) \cap \varepsilon_1^{-1}(V_{\lambda^{(3)}}) \cap \varepsilon_2^{-1}(V_{\lambda^{(4)}}))\\ \cap \varepsilon_2^{-1}(\varepsilon_0^{-1}(V_{\lambda'}) \cap \varepsilon_1^{-1}(V_{\lambda^{(3)}}) \cap \varepsilon_2^{-1}(V_{\lambda^{(5)}}))\\ \cap \varepsilon_3^{-1}(\varepsilon_0^{-1}(V_{\lambda''}) \cap \varepsilon_1^{-1}(V_{\lambda^{(4)}}) \cap \varepsilon_2^{-1}(V_{\lambda^{(5)}}))\end{aligned}$$

there are the following induced local sections on that.

$$\begin{aligned}\varepsilon_0^*(\varepsilon_0^*\eta_\lambda \otimes (\varepsilon_1^*\eta_{\lambda'})^{\otimes -1} \otimes \varepsilon_2^*\eta_{\lambda''}) \otimes \varepsilon_1^*(\varepsilon_0^*\eta_\lambda \otimes (\varepsilon_1^*\eta_{\lambda^{(3)}})^{\otimes -1} \otimes \varepsilon_2^*\eta_{\lambda^{(4)}})^{\otimes -1}\\ \otimes \varepsilon_2^*(\varepsilon_0^*\eta_{\lambda'} \otimes (\varepsilon_1^*\eta_{\lambda^{(3)}})^{\otimes -1} \otimes \varepsilon_2^*\eta_{\lambda^{(5)}}) \otimes \varepsilon_3^*(\varepsilon_0^*\eta_{\lambda''} \otimes (\varepsilon_1^*\eta_{\lambda^{(4)}})^{\otimes -1} \otimes \varepsilon_2^*\eta_{\lambda^{(5)}})^{\otimes -1}.\end{aligned}$$

From direct computations we can check that the pull-back of $\delta(\delta\theta)$ by this section is equal to 0. This means $\delta(\delta\theta)$ is the Maurer-Cartan connection. Hence if we pull back $\delta(\delta\theta)$ by the induced section $\varepsilon_0^*\hat{s} \otimes (\varepsilon_1^*\hat{s})^{\otimes -1} \otimes \varepsilon_2^*\hat{s} \otimes (\varepsilon_3^*\hat{s})^{\otimes -1}$, it is also equal to 0 and this pull-back is nothing but $(\varepsilon_0^* - \varepsilon_1^* + \varepsilon_2^* - \varepsilon_3^*)(\hat{s}^*(\delta\theta))$. □

The propositions above give the cocycle $c_1(\theta) - \left(\frac{-1}{2\pi i}\right)\hat{s}^*(\delta\theta) \in \Omega^3(NG)$ below.

$$\begin{array}{ccccc} 0 & & & & \\ \uparrow d & & & & \\ c_1(\theta) \in \Omega^2(G) & \xrightarrow{\varepsilon_0^* - \varepsilon_1^* + \varepsilon_2^*} & \Omega^2(G \times G) & & \\ & & \uparrow -d & & \\ & & -\left(\frac{-1}{2\pi i}\right)\hat{s}^*(\delta\theta) \in \Omega^1(G \times G) & \xrightarrow{\varepsilon_0^* - \varepsilon_1^* + \varepsilon_2^* - \varepsilon_3^*} & 0 \end{array}$$

PROPOSITION 2.3. *The cohomology class* $\left[c_1(\theta) - \left(\frac{-1}{2\pi i}\right)\hat{s}^*(\delta\theta)\right] \in H^3(\Omega(NG))$ *does not depend on* θ.

Proof. Suppose θ_0 and θ_1 are two connections on $\hat{G}$. Consider the $U(1)$-bundle $\hat{G} \times [0,1] \to G \times [0,1]$ and the connection form $t\theta_0 + (1-t)\theta_1$ on it. Then we obtain the cocycle $c_1(t\theta_0 + (1-t)\theta_1) - \left(\frac{-1}{2\pi i}\right)\hat{s}^*(\delta(t\theta_0 + (1-t)\theta_1))$ on $\Omega^3(NG \times [0,1])$. Let $i_0 : NG \times \{0\} \to NG \times [0,1]$ and $i_1 : NG \times \{1\} \to NG \times [0,1]$ be the natural inclusion map. When we identify $NG \times \{0\}$ with $NG \times \{1\}$, $(i_0^*)^{-1} i_1^* : H(\Omega^*(NG \times \{0\})) \to H(\Omega^*(NG \times \{1\}))$ is the identity map. Hence $\left[c_1(\theta_0) - \left(\frac{-1}{2\pi i}\right)\hat{s}^*(\delta\theta_0)\right] = \left[c_1(\theta_1) - \left(\frac{-1}{2\pi i}\right)\hat{s}^*(\delta\theta_1)\right]$. □

3. Dixmier-Douady class on the double complex

First, we recall the definition of Dixmier-Douady classes, following [5]. Let $\pi : Y \to M$ be a principal G-bundle and $\{U_\alpha\}$ a Leray covering of M. When G has a central extension $\rho : \hat{G} \to G$, the transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$ lift to $\hat{G}$. i.e. there exist continuous maps $\hat{g}_{\alpha\beta} : U_{\alpha\beta} \to \hat{G}$ such that $\rho \circ \hat{g}_{\alpha\beta} = g_{\alpha\beta}$. This is because each $U_{\alpha\beta}$ is contractible so the pull-back of ρ by $g_{\alpha\beta}$ has a global section. Now the $U(1)$-valued functions $c_{\alpha\beta\gamma}$ on $U_{\alpha\beta\gamma}$ are defined as $c_{\alpha\beta\gamma} := \hat{g}_{\beta\gamma}\hat{g}_{\alpha\gamma}^{-1}\hat{g}_{\alpha\beta}$. Note that here they identify $g_{\beta\gamma}^*\hat{G} \otimes (g_{\alpha\gamma}^*\hat{G})^{\otimes -1} \otimes g_{\alpha\beta}^*\hat{G}$ with $U_{\alpha\beta\gamma} \times U(1)$. Then it is easily seen that $\{c_{\alpha\beta\gamma}\}$ is a $U(1)$-valued Čech-cocycle on M and hence define a cohomology class in $H^2(M, \underline{U(1)}) \cong H^3(M, \mathbf{Z})$. This class is called the Dixmier-Douady class of Y.

Here G can be infinite dimensional, but we require G to have a partition of unity so that we can consider a connection form on the $U(1)$-bundle over G. A good example which satisfies such a condition is the loop group of a finite dimensional Lie group [4] [10].

Secondly, we fix any trivialization $\delta\hat{G} \cong \hat{G} \times U(1)$. Then since $g_{\beta\gamma}^*\hat{G} \otimes (g_{\alpha\gamma}^*\hat{G})^{\otimes -1} \otimes g_{\alpha\beta}^*\hat{G}$ is the pull-back of $\delta\hat{G}$ by $(g_{\alpha\beta}, g_{\beta\gamma}) : U_{\alpha\beta\gamma} \to G \times G$, there is the induced trivialization $g_{\beta\gamma}^*\hat{G} \otimes (g_{\alpha\gamma}^*\hat{G})^{\otimes -1} \otimes g_{\alpha\beta}^*\hat{G} \cong U_{\alpha\beta\gamma} \times U(1)$. So we have the Dixmier-Douady cocycle by using this identification.

Now we are ready to state the main theorem.

DEFINITION 3.1. For the global section $\hat{s} : G \times G \to 1$, we call the sum of $c_1(\theta) \in \Omega^2(NG(1))$ and $-\left(\frac{-1}{2\pi i}\right)\hat{s}^*(\delta\theta) \in \Omega^1(NG(2))$ the simplicial Dixmier-Douady cocycle associated to θ and the trivialization $\delta\hat{G} \cong \hat{G} \times U(1)$.

THEOREM 3.1. *The simplicial Dixmier-Douady cocycle represents the universal Dixmier-Douady class associated to* ρ.

Proof. We show that the $[C_{2,1}+C_{1,2}]$ below is equal to $\left[\left\{\left(\frac{-1}{2\pi i}\right)d\log c_{\alpha\beta\gamma}\right\}\right]$ as a Čech-de Rham cohomology class of $M=\bigcup U_\alpha$.

$$\begin{array}{ccc} C_{2,1}\in\prod\Omega^2(U_{\alpha\beta}) & & \\ \uparrow{\scriptstyle -d} & & \\ \prod\Omega^1(U_{\alpha\beta}) & \xrightarrow{\check{\delta}} & C_{1,2}\in\prod\Omega^1(U_{\alpha\beta\gamma}) \end{array}$$

$$C_{2,1}=\{(g^*_{\alpha\beta}c_1(\theta))\},\quad C_{1,2}=\left\{-\left(\frac{-1}{2\pi i}\right)(g_{\alpha\beta},g_{\beta\gamma})^*\hat{s}^*(\delta\theta)\right\}$$

Since $g^*_{\alpha\beta}c_1(\theta)=\hat{g}^*_{\alpha\beta}\rho^*(c_1(\theta))=d\left(\frac{-1}{2\pi i}\right)\hat{g}^*_{\alpha\beta}\theta$, we can see $[C_{2,1}+C_{1,2}]=\left[\check{\delta}\left\{\left(\frac{-1}{2\pi i}\right)\hat{g}^*_{\alpha\beta}\theta\right\}+C_{1,2}\right]$. By definition $(\hat{s}\circ(g_{\alpha\beta},g_{\beta\gamma}))(p)\cdot c_{\alpha\beta\gamma}(p)=(\hat{g}_{\beta\gamma}\otimes\hat{g}^{\otimes-1}_{\alpha\gamma}\otimes\hat{g}_{\alpha\beta})(p)$ for any $p\in U_{\alpha\beta\gamma}$. Hence $(g_{\alpha\beta},g_{\beta\gamma})^*\hat{s}^*(\delta\theta)+d\log c_{\alpha\beta\gamma}=\check{\delta}\{\hat{g}^*_{\alpha\beta}\theta\}$. □

Corollary 3.1. *If the principal G-bundle over M is flat, then its Dixmier-Douady class is* 0 *in* $H^3(M,\mathbf{R})$.

Proof. This is because the cocycle in Theorem 3.1 vanishes when G is given a discrete topology. □

Corollary 3.2. *If the first Chern class of* $\rho:\hat{G}\to G$ *is not* 0 *in* $H^2(G,\mathbf{R})$, *then the corresponding Dixmier-Douady class of the universal G-bundle is not* 0.

Proof. In that situation, any differential form $x\in\Omega^1(NG(1))$ does not hit $c_1(\theta)\in\Omega^2(NG(1))$ by $d:\Omega^1(NG(1))\to\Omega^2(NG(1))$. □

4. Chern-Simons form

As mentioned in section 2.1, $N\overline{G}$ plays the role of the universal G-bundle and NG, the classifying space BG. Then, the pull-back of the cocycle in Definition 3.1 to $\Omega^*(N\overline{G})$ by $\gamma:N\overline{G}\to NG$ should be a coboundary of a cochain on $N\overline{G}$. In this section we shall exhibit an explicit form of the cochain, which can be called Chern-Simons form for the Dixmier-Douady class.

Recall $N\overline{G}(1)=G\times G$ and $\gamma:N\overline{G}(1)\to NG$ is defined as $\gamma(h_1,h_2)=h_1h_2^{-1}$. Then we consider the $U(1)$-bundle $\bar{\delta}_\gamma\hat{G}:=\bar{\varepsilon}_0^*\hat{G}\otimes\gamma^*\hat{G}\otimes(\bar{\varepsilon}_1^*\hat{G})^{\otimes-1}$ over $G\times G$ and the induced connection $\bar{\delta}_\gamma\theta$ on it. We can check $\bar{\delta}_\gamma\hat{G}$ is trivial using the same argument as that in Lemma 2.1, so there is a global section $\bar{s}_\gamma:G\times G\to\bar{\delta}_\gamma\hat{G}$.

THEOREM 4.1. *If we take* $\bar{s}_\gamma = 1$, *the cochain* $c_1(\theta) - \left(\frac{-1}{2\pi i}\right)\bar{s}_\gamma^*(\bar{\delta}_\gamma\theta) \in \Omega^2(N\bar{G})$ *is a Chern-Simons form of* $c_1(\theta) - \left(\frac{-1}{2\pi i}\right)\hat{s}^*(\delta\theta) \in \Omega^3(NG)$.

$$\begin{array}{ccccc} 0 & & & & \\ \uparrow{\scriptstyle d} & & & & \\ c_1(\theta) \in \Omega^2(G) & \xrightarrow{\bar{\varepsilon}_0^* - \bar{\varepsilon}_1^*} & \Omega^2(N\bar{G}(1)) & & \\ & & \uparrow{\scriptstyle -d} & & \\ & & -\left(\frac{-1}{2\pi i}\right)\bar{s}_\gamma^*(\bar{\delta}_\gamma\theta) \in \Omega^1(N\bar{G}(1)) & \xrightarrow{\bar{\varepsilon}_0^* - \bar{\varepsilon}_1^* + \bar{\varepsilon}_2^*} & \Omega^1(N\bar{G}(2)) \end{array}$$

Proof. Repeating the same argument as that in Proposition 2.1, we can see $(\bar{\varepsilon}_0^* + \gamma^* - \bar{\varepsilon}_1^*)((c_1(\theta)) = \left(\frac{-1}{2\pi i}\right)d(\bar{s}_\gamma^*(\bar{\delta}_\gamma\theta)) \in \Omega^2(N\bar{G}(1))$. Because $(\varepsilon_0, \varepsilon_1, \varepsilon_2) \circ \gamma = (\gamma \circ \bar{\varepsilon}_0, \gamma \circ \bar{\varepsilon}_1, \gamma \circ \bar{\varepsilon}_2)$, $(\bar{\varepsilon}_0^*\bar{\delta}_\gamma\hat{G}) \otimes (\bar{\varepsilon}_1^*\bar{\delta}_\gamma\hat{G})^{\otimes -1} \otimes (\bar{\varepsilon}_2^*\bar{\delta}_\gamma\hat{G})$ is $\gamma^*(\delta\hat{G})$. Hence $(\bar{\varepsilon}_0^* - \bar{\varepsilon}_1^* + \bar{\varepsilon}_2^*)\bar{s}_\gamma^*(\bar{\delta}_\gamma\theta) = \gamma^*(\hat{s}^*(\delta\theta))$. □

By restricting the Chern-Simons form on $\Omega^*(N\bar{G})$ to the edge $\Omega^*(N\bar{G}(0))$, we obtain the cocycle on $\Omega^*(G)$. So there is the induced map of the cohomology class $H^*(BG) \cong H(\Omega^*(NG)) \to H^{*-1}(G)$. This map coincides with the transgression map for the universal bundle $EG \to BG$ in the sense of J. L. Heitsch and H. B. Lawson in [8]. Hence as a corollary of theorem 4.1, we obtain an alternative proof of the following theorem from [5] [12].

THEOREM 4.2. *The transgression map of the universal bundle* $EG \to BG$ *maps the Dixmier-Douady class to the first Chern class of* $\rho : \hat{G} \to G$.

Remark 4.1. Here the meaning of the terminology "transgression map" is different from those in [5] [12], but the statement is essentially same.

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