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STABILITY OF F-STATIONARY MAPS OF A CLASS OF FUNCTIONALS RELATED TO CONFORMAL MAPS

Yingbo Han, Shuxiang Feng and Hong Pan

Abstract

In this paper, we study a generalized functional Φ_F related to the conformality of maps between Riemannian manifolds. We derive the first variation formula and the second variation formula of Φ_F, then we study the stability of F-stationary map from or into the standard sphere. We also introduce the F-stress energy tensor associated to Φ_F which is naturally linked to conservation law.

1. Introduction

Let (M^m, g) and (N^n, h) be compact Riemannian manifolds without boundary. A smooth map u from M into N is called a conformal map if there exists a positive function φ on M such that $u^*h = \varphi g$, where u^*h denotes the pullback of the metric h by u, i.e.

$$u^*h(X, Y) = h(du(X), du(Y)).$$

Recently, N. Nakauchi in [8] introduced the following functional,

$$\Phi(u) = \frac{1}{4}\int_M \|T_u\|^2 \, dv_g,$$

(see [6, 9]) where T_u is the symmetric 2-tensor defined by

$$T_u = u^*h - \frac{1}{m}\|du\|^2 g$$

and $\|T_u\|^2$, $\|du\|^2$ as

$$\|T_u\|^2 = \sum_{i,j} T_u(e_i, e_j)^2, \quad \|du\|^2 = \sum_i h(du(e_i), du(e_i)).$$

with respect to a local orthonormal frame $(e_1, \dots, e_m)$ on (M, g). They gave the first variation formula and the second variation formula for this func-

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tional. They also gave a kind of the monotonicity formula and a Bochner type formula.

On the other hand, following Baird and Eells [2], Ara [1] introduced the F-harmonic maps, generalizing harmonic maps. Let $F : [0, \infty) \to [0, \infty)$ be a C^2 function such that $F(0) = 0$ and $F'(t) > 0$ for $t \in [0, \infty)$. A smooth map $u : M \to N$ is said to be an F-harmonic map if it is a critical point of the following F-energy functional E_F given by

$$E_F(u) = \int_M F\left(\frac{\|du\|^2}{2}\right) dv_g,$$

with respect to any compactly supported variation of u. After this, there are many geometers who studied F-harmonic map such as [4, 5, 7].

In this paper, we generalize and unify the concept of critical point of the functional Φ. For this, we define the functional Φ_F by

$$\Phi_F(u) = \int_M F\left(\frac{\|T_u\|^2}{4}\right) dv_g,$$

which is Φ if $F(t) = t$. We call u an F-stationary map for $\Phi_F(u)$, if

$$\frac{d}{dt}\Phi_F(u_t)|_{t=0} = 0$$

for any compactly supported variation $u_t : M \to N$ with $u_0 = u$. We derive the first variation formula and the second variation formula of Φ_F. We also prove that every stable F-stationary map form a compact manifold M into S^n is weakly conformal, provided that

$$\int_{M^m} \|T_u\|^2 \left\{ F''\left(\frac{\|T_u\|^2}{4}\right)\|T_u\|^2 + (4-n)F'\left(\frac{\|T_u\|^2}{4}\right)\right\} dv_g < 0.$$

or every stable F-stationary map from S^m is weakly conformal, provided that

$$\int_{S^m} \|T_u\|^2 \left\{ F''\left(\frac{\|T_u\|^2}{4}\right)\|T_u\|^2 + (4-m)F'\left(\frac{\|T_u\|^2}{4}\right)\right\} dv_g < 0.$$

We also introduce the F-stress energy tensor associated to Φ_F which is naturally linked to conservation law.

The contents of this paper is as follows:

1. Introduction.
2. Preliminaries.
3. The first variation formula for $\Phi_F(u)$.
4. F-stress energy tensor
5. The second variation formula for $\Phi_F(u)$.
6. Stable maps into spheres.
7. Stable maps from spheres.

2. Preliminaries

Let (M^m, g) and (N^n, h) be compact Riemannian manifolds without boundary and let u be a smooth map from M to N. We recall the following notions.

Definition 2.1. (i) A smooth map u is weakly conformal if there exists a non-negative function φ on M such that

$$u^*h = \varphi g, \tag{1}$$

where u^*h denotes the pullback of the metric h by u, i.e.

$$u^*h(X, Y) = h(du(X), du(Y)).$$

(ii) A smooth map u is conformal if there exists a positive function φ on satisfy the equation (1).

The condition (1) is equivalent to

$$u^*h = \frac{1}{m}\|du\|^2 g, \tag{2}$$

Since taking the trace of the both sides of (1) with respect to the metric g, we have $\varphi = \frac{1}{m}\|du\|^2$. Then u is conformal if and only if it satisfies (2) with the assumption $\|du\| \neq 0$. Note that u is weakly conformal if and only if for any point $x \in M$, u is conformal at x, or $du_x = 0$.

In order to state our results, we also need the following Lemmas.

Lemma 2.2 [8]. (a) *T_u is symmetric, i.e. $T_u(X, Y) = T_u(Y, X)$.*

(b) *u is weakly conformal if and only if $T_u = 0$.*

(c) $\|T_u\|^2 = \|u^*h\|^2 - \frac{1}{m}\|du\|^4$.

(d) *T_u is trace-free, i.e.*

$$Trace_g\, T_u = \sum_{i,j} g(e_i, e_j) T_u(e_i, e_j) = 0,$$

where e_i denotes a local orthornormal frame on M.

(e) *The trace of T_u with respect to the pullback u^*h is equal to the norm of T_u, i.e.*

$$Trace_{u^*h}\, T_u = \sum_{i,j} h(du(e_i), du(e_j)) T_u(e_i, e_j) = \|T_u\|^2,$$

We define an $u^{-1}TN$-valued 1-form σ_u on M by

$$\sigma_u(X) = \sum_j T_u(X, e_j)\, du(e_j) = \sum_j h(du(X), du(e_j))\, du(e_j) - \frac{1}{m}\|du\|^2\, du(X)$$

for any vector field X on M.

Lemma 2.3.

$$\sum_i h(du(e_i), \sigma_u(e_i)) = \|T_u\|^2. \tag{3}$$

Proof.

$$\begin{aligned}\sum_i h(du(e_i), \sigma_u(e_i)) &= \sum_{i,j} h(du(e_i), du(e_j))h(du(e_i), du(e_j)) \\ &\quad - \frac{1}{m}\|du\|^2 g(e_i, e_j)h(du(e_i), du(e_j)) \\ &= \|u^*h\|^2 - \frac{1}{m}\|du\|^4 = \|T_u\|^2.\end{aligned}$$

□

3. The first variation formula for $\Phi_F(u)$

Let ∇ and ${}^N\nabla$ always denote the Levi-Civita connections of M and N respectively. Let $\tilde{\nabla}$ be the induced connection on $u^{-1}TN$ defined by $\tilde{\nabla}_X W = {}^N\nabla_{du(X)} W$, where X is a tangent vector of M and W is a section of $u^{-1}TN$. We choose a local orthonormal frame field $\{e_i\}$ on M. We define the F-tension field $\tau_F(u)$ of u by

$$\begin{aligned}\tau_F(u) &= -\delta\left(F'\left(\frac{\|T_u\|^2}{4}\right)\sigma_u\right) \\ &= F'\left(\frac{\|T_u\|^2}{4}\right) div_g(\sigma_u) + \sigma_u\left(grad\left(F'\left(\frac{\|T_u\|^2}{4}\right)\right)\right).\end{aligned} \tag{4}$$

Under the notation above we have the following:

Lemma 3.1 (The first variation formula). *Let $u : M \to N$ be a smooth map. Then*

$$\frac{d}{dt}\Phi_F(u_t)|_{t=0} = -\int_M h(\tau_F(u), V)\, dv_g, \tag{5}$$

where $V = \frac{d}{dt}u_t|_{t=0}$.

Proof. Let $\Psi : (-\varepsilon, \varepsilon) \times M \to N$ be defined by $\Psi(t, x) = u_t(x)$, where $(-\varepsilon, \varepsilon) \times M$ is equipped with the product metric. We extend the vector fields

$\dfrac{\partial}{\partial t}$ on $(-\varepsilon, \varepsilon)$, X on M naturally on $(-\varepsilon, \varepsilon) \times M$, and denote those also by $\dfrac{\partial}{\partial t}$, X. Then

$$V = d\Psi\left(\frac{\partial}{\partial t}\right)\bigg|_{t=0}. \tag{6}$$

We shall use the same notations ∇ and $\tilde{\nabla}$ for the Levi-Civita connection on $(-\varepsilon, \varepsilon) \times M$ and the induced connection on $\Psi^{-1}TN$.

Now we compute

$$\begin{aligned}
(7)\quad & \frac{\partial}{\partial t} F\left(\frac{\|T_{u_t}\|^2}{4}\right) \\
&= F'\left(\frac{\|T_{u_t}\|^2}{4}\right) \frac{1}{4} \frac{\partial}{\partial t} \|T_{u_t}\|^2 \\
&= \frac{1}{2} F'\left(\frac{\|T_{u_t}\|^2}{4}\right) \sum_{i,j=1}^{m} \frac{\partial T_{u_t}(e_i, e_j)}{\partial t} T_{u_t}(e_i, e_j) \\
&= \frac{1}{2} F'\left(\frac{\|T_{u_t}\|^2}{4}\right) \sum_{i,j=1}^{m} \left\{ \frac{\partial}{\partial t} h(du_t(e_i), du_t(e_j)) - \frac{1}{m} \frac{\partial \|du_t\|^2}{\partial t} g(e_i, e_j) \right\} T_{u_t}(e_i, e_j) \\
&= \frac{1}{2} F'\left(\frac{\|T_{u_t}\|^2}{4}\right) \sum_{i,j=1}^{m} \frac{\partial}{\partial t} h(du_t(e_i), du_t(e_j)) T_{u_t}(e_i, e_j) \\
&= \frac{1}{2} F'\left(\frac{\|T_{u_t}\|^2}{4}\right) \sum_{i,j=1}^{m} \frac{\partial}{\partial t} h(d\Psi(e_i), d\Psi(e_j)) T_{u_t}(e_i, e_j) \\
&= F'\left(\frac{\|T_{u_t}\|^2}{4}\right) \sum_{i,j=1}^{m} h(\tilde{\nabla}_{\partial/\partial t}\, d\Psi(e_i), d\Psi(e_j)) T_{u_t}(e_i, e_j) \\
&= F'\left(\frac{\|T_{u_t}\|^2}{4}\right) \sum_{i,j=1}^{m} h\left(\tilde{\nabla}_{e_i}\, d\Psi\left(\frac{\partial}{\partial t}\right), d\Psi(e_j)\right) T_{u_t}(e_i, e_j) \\
&= F'\left(\frac{\|T_{u_t}\|^2}{4}\right) \sum_{i,j=1}^{m} h\left(\tilde{\nabla}_{e_i}\, d\Psi\left(\frac{\partial}{\partial t}\right), du_t(e_j)\right) T_{u_t}(e_i, e_j) \\
&= F'\left(\frac{\|T_{u_t}\|^2}{4}\right) \sum_{i=1}^{m} h\left(\tilde{\nabla}_{e_i}\, d\Psi\left(\frac{\partial}{\partial t}\right), \sigma_{u_t}(e_i)\right) \\
&= F'\left(\frac{\|T_{u_t}\|^2}{4}\right) \sum_{i=1}^{m} \left[e_i h\left(d\Psi\left(\frac{\partial}{\partial t}\right), \sigma_{u_t}(e_i)\right) - h\left(d\Psi\left(\frac{\partial}{\partial t}\right), \tilde{\nabla}_{e_i} \sigma_{u_t}(e_i)\right)\right],
\end{aligned}$$

where we use that

$$\sum_{i,j=1}^{m} g(e_i, e_j) T_{u_t}(e_i, e_j) = 0$$

for the forth equality, and

$$\tilde{\nabla}_{\partial/\partial t}\, d\Psi(e_i) - \tilde{\nabla}_{e_i}\, d\Psi\left(\frac{\partial}{\partial t}\right) = d\Psi\left[\frac{\partial}{\partial t}, e_i\right] = 0$$

for the seventh equality. Let X_t be a compactly supported vector field on M such that $g(X_t, Y) = h\left(d\Psi\left(\frac{\partial}{\partial t}\right), \sigma_{u_t}(Y)\right)$ for any vector field Y on M. Then

$$\begin{aligned}
(8) \qquad \frac{\partial}{\partial t} F\left(\frac{\|T_{u_t}\|^2}{4}\right) &= F'\left(\frac{\|T_{u_t}\|^2}{4}\right) \sum_{i=1}^{m} e_i g(X_t, e_i) \\
&\quad - F'\left(\frac{\|T_{u_t}\|^2}{4}\right) \sum_{i=1}^{m} \left[h\left(d\Psi\left(\frac{\partial}{\partial t}\right), \tilde{\nabla}_{e_i} \sigma_{u_t}(e_i)\right)\right] \\
&= F'\left(\frac{\|T_{u_t}\|^2}{4}\right) \sum_{i=1}^{m} [g(\nabla_{e_i} X_t, e_i) + g(X_t, \nabla_{e_i} e_i)] \\
&\quad - F'\left(\frac{\|T_{u_t}\|^2}{4}\right) \sum_{i=1}^{m} h\left(d\Psi\left(\frac{\partial}{\partial t}\right), \tilde{\nabla}_{e_i} \sigma_{u_t}(e_i)\right) \\
&= F'\left(\frac{\|T_{u_t}\|^2}{4}\right) div_g(X_t) \\
&\quad - F'\left(\frac{\|T_{u_t}\|^2}{4}\right) \sum_{i=1}^{m} h\left(d\Psi\left(\frac{\partial}{\partial t}\right), \tilde{\nabla}_{e_i} \sigma_{u_t}(e_i) - \sigma_{u_t}(\nabla_{e_i} e_i)\right) \\
&= div\left(F'\left(\frac{\|T_{u_t}\|^2}{4}\right) X_t\right) - g\left(X_t, grad\left(F'\left(\frac{\|T_{u_t}\|^2}{4}\right)\right)\right) \\
&\quad - F'\left(\frac{\|T_{u_t}\|^2}{4}\right) \sum_{i=1}^{m} h\left(d\Psi\left(\frac{\partial}{\partial t}\right), \tilde{\nabla}_{e_i} \sigma_{u_t}(e_i) - \sigma_{u_t}(\nabla_{e_i} e_i)\right) \\
&= div\left(F'\left(\frac{\|T_{u_t}\|^2}{4}\right) X_t\right) - h\left(d\Psi\left(\frac{\partial}{\partial t}\right), F'\left(\frac{\|T_{u_t}\|^2}{4}\right) div_g\, \sigma_{u_t}\right. \\
&\quad \left. + \sigma_{u_t}\left(grad\left(F'\left(\frac{\|T_{u_t}\|^2}{4}\right)\right)\right)\right).
\end{aligned}$$

By (8) and Green's theorem, we get

$$\frac{d}{dt}\Phi_F(u_t)|_{t=0} = \int_M \frac{\partial}{\partial t} F\left(\frac{\|T_{u_t}\|^2}{4}\right)\bigg|_{t=0} dv_g$$

$$= -\int_M h\left(d\Psi\left(\frac{\partial}{\partial t}\right), F'\left(\frac{\|T_{u_t}\|^2}{4}\right) div_g\, \sigma_{u_t}\right.$$

$$\left. + \sigma_{u_t}\left(grad\left(F'\left(\frac{\|T_{u_t}\|^2}{4}\right)\right)\right)\right)\bigg|_{t=0} dv_g$$

$$= -\int_M h(\tau_F(u), V)\, dv_g. \qquad \square$$

The first variation formula allows us to define the notion of F-stationary for the functional Φ_F.

Definition 3.2. A smooth map u is called F-stationary map for the functional Φ_F if it is a solution of the Euler-Lagrange equation $\tau_F(u) = 0$.

4. F-stress energy tensor

Following Baird [3], for a smooth map $u : (M, g) \to (N, h)$, we associate a symmetric 2-tensor S_F to the functional Φ_F called the F-stress energy tensor

$$S_F(X, Y) = F\left(\frac{\|T_u\|^2}{4}\right) g(X, Y) - F'\left(\frac{\|T_u\|^2}{4}\right) h(\sigma_u(X), du(Y)), \tag{9}$$

where X, Y are vector fields on M.

Proposition 4.1. *Let $u : (M, g) \to (N, h)$ be a smooth map and S_F be the associated F-stress energy tensor, then for each vector field X on M, we have*

$$(div\ S_F)(X) = -h(\tau_F(u), du(X)). \tag{10}$$

Proof. Let ∇ and ${}^N\nabla$ denote the Levi-Civita connections of M and N, respectively. Let $\tilde{\nabla}$ be the induced connection on $u^{-1}TN$. We choose a local orthonormal frame field $\{e_i\}$ around a point P on M with $\nabla_{e_i} e_j|_P = 0$.

Let X be a vector field on M. At P, we compute

$$(div\ S_F)(X) = \sum_{i=1}^m (\nabla_{e_i} S_F)(e_i, X)$$

$$= \sum_{i=1}^m \{e_i(S_F(e_i, X)) - S_F(\nabla_{e_i} e_i, X) - S_F(e_i, \nabla_{e_i} X)\}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} \Bigg\{ e_i \left(F\left(\frac{\|T_u\|^2}{4}\right) g(e_i, X) \right) - e_i \left(F'\left(\frac{\|T_u\|^2}{4}\right) h(\sigma_u(e_i), du(X)) \right) \\
&\qquad - F\left(\frac{\|T_u\|^2}{4}\right) g(e_i, \nabla_{e_i} X) + F'\left(\frac{\|T_u\|^2}{4}\right) h(\sigma_u(e_i), du(\nabla_{e_i} X)) \Bigg\} \\
&= \sum_{i=1}^{m} \Bigg\{ e_i \left(F\left(\frac{\|T_u\|^2}{4}\right)\right) g(e_i, X) - e_i \left(F'\left(\frac{\|T_u\|^2}{4}\right)\right) h(\sigma_u(e_i), du(X)) \\
&\qquad - F'\left(\frac{\|T_u\|^2}{4}\right) h(\tilde{\nabla}_{e_i} \sigma_u(e_i), du(X)) - F'\left(\frac{\|T_u\|^2}{4}\right) h(\sigma_u(e_i), \tilde{\nabla}_{e_i}\, du(X)) \\
&\qquad + F'\left(\frac{\|T_u\|^2}{4}\right) h(\sigma_u(e_i), du(\nabla_{e_i} X)) \Bigg\} \\
&= X\left(F\left(\frac{\|T_u\|^2}{4}\right)\right) - h\left(\sigma_u\left(grad\ F'\left(\frac{\|T_u\|^2}{4}\right)\right), du(X)\right) \\
&\quad - F'\left(\frac{\|T_u\|^2}{4}\right) h(div\ \sigma_u, du(X)) - \sum_i F'\left(\frac{\|T_u\|^2}{4}\right) h(\sigma_u(e_i), (\nabla_{e_i}\, du)(X)) \\
&= -h(\tau_F(u), du(X)) + F'\left(\frac{\|T_u\|^2}{4}\right) X\left(\frac{\|T_u\|^2}{4}\right) \\
&\quad - F'\left(\frac{\|T_u\|^2}{4}\right) \sum_i h(\sigma_u(e_i), (\nabla_{e_i}\, du)(X)) \\
&= -h(\tau_F(u), du(X)) + F'\left(\frac{\|T_u\|^2}{4}\right) \frac{1}{4} X\left(\|u^*h\|^2 - \frac{1}{m}\|du\|^4 \right) \\
&\quad - F'\left(\frac{\|T_u\|^2}{4}\right) \sum_i h(\sigma_u(e_i), (\nabla_{e_i}\, du)(X)) \\
&= -h(\tau_F(u), du(X)) + F'\left(\frac{\|T_u\|^2}{4}\right) \\
&\quad \times \left(\sum_{i,j} h(\tilde{\nabla}_X\, du(e_i), du(e_j)) h(du(e_i), du(e_j)) - \frac{1}{m}\|du\|^2 \sum_i h(\tilde{\nabla}_X\, du(e_i), du(e_i)) \right) \\
&\quad - F'\left(\frac{\|T_u\|^2}{4}\right) \sum_i h(\sigma_u(e_i), (\nabla_{e_i}\, du)(X))
\end{aligned}
$$

$$= -h(\tau_F(u), du(X)) + F'\left(\frac{\|T_u\|^2}{4}\right)\sum_i h$$

$$\times\left(\tilde{\nabla}_X\, du(e_i), \left[\sum_j h(du(e_i), du(e_j))\, du(e_j) - \frac{1}{m}\|du\|^2\, du(e_i)\right]\right)$$

$$- F'\left(\frac{\|T_u\|^2}{4}\right)\sum_i h(\sigma_u(e_i), (\nabla_{e_i}\, du)(X))$$

$$= -h(\tau_F(u), du(X)) + F'\left(\frac{\|T_u\|^2}{4}\right)\sum_i h((\nabla_X\, du)(e_i), \sigma_u(e_i))$$

$$- F'\left(\frac{\|T_u\|^2}{4}\right)\sum_i h(\sigma_u(e_i), (\nabla_{e_i}\, du)(X)).$$

Since $(\nabla_X\, du)(e_i) = (\nabla_{e_i}\, du)(X)$, we obtain

$$(div\ S_F)(X) = -h(\tau_F(u), du(X)). \qquad \square$$

From the above Proposition, we know that if $u : M \to N$ is an F-stationary map, we have

$$div\ S_F = 0, \tag{11}$$

that is, u satisfies the Φ_F-conservation law.

Recall that for two 2-tensors $T_1, T_2 \in \Gamma(T^*M \otimes T^*M)$, their inner product is defined as follows;

$$\langle T_1, T_2\rangle = \sum_{ij} T(e_i, e_j)T_2(e_i, e_j), \tag{12}$$

where $\{e_i\}$ is an orthonormal basis of with respect to g. For a vector field $X \in \Gamma(TM)$, we denote by θ_X is dual one form i.e. $\theta_X(Y) = g(X, Y)$. The covariant derivative of θ_X gives a 2-tensor field $\nabla\theta_X$:

$$(\nabla\theta_X)(Y, Z) = (\nabla_Z\theta_X)(Y) = g(\nabla_Z X, Y). \tag{13}$$

If $X = \nabla\varphi$ is the gradient of some function φ on M, then $\theta_X = d\varphi$ and $\nabla\theta_X = Hess\ \varphi$.

LEMMA 4.2 (CF. [3, 4]). *Let T be a symmetric $(0,2)$-type tensor field and let X be a vector field, then*

$$div(i_X T) = (div\ T)(X) + \langle T, \nabla\theta_X\rangle = (div\ T)(X) + \frac{1}{2}\langle T, L_X g\rangle. \tag{14}$$

Let D be any bounded domain of M with C^1 boundary. By using the stokes' theorem, we immediately have the following integral formula:

$$\int_{\partial D} T(X, \nu)\, ds_g = \int_D \left[\left\langle T, \frac{1}{2} L_X g\right\rangle + (div\ T)(X)\right] dv_g, \tag{15}$$

where ν is the unit outward normal vector field along ∂D. By (11) and (15), we have

$$\int_{\partial D} S_F(X, \nu)\, ds_g = \int_D \left\langle S_F, \frac{1}{2} L_X g\right\rangle dv_g. \tag{16}$$

5. The second variation formula for $\Phi_F(u)$

In this section, we calculate the second variation of the functional $\Phi_F(u)$.

THEOREM 5.1 (The second variation formula). *Let $u : (M, g) \to (N, h)$ be an F-stationary map. Let $u_{s,t} : M \to N$ $(-\varepsilon < s, t < \varepsilon)$ be a compactly supported two-parameter variation such that $u_{0,0} = u$ and set $V = \frac{\partial}{\partial t} u_{s,t}|_{s,t=0}$, $W = \frac{\partial}{\partial s} u_{s,t}|_{s,t=0}$. Then*

$$\begin{aligned}
\frac{\partial^2}{\partial s \partial t} \Phi_F(u_{s,t})|_{s,t=0} &= \int_M F''\left(\frac{\|T_u\|^2}{4}\right) \langle \tilde\nabla V, \sigma_u\rangle \langle \tilde\nabla W, \sigma_u\rangle\, dv_g \\
&\quad + \int_M F'\left(\frac{\|T_u\|^2}{4}\right) \sum_{i,j=1}^m h(\tilde\nabla_{e_i} V, \tilde\nabla_{e_j} W) T_u(e_i, e_j)\, dv_g \\
&\quad + \int_M F'\left(\frac{\|T_u\|^2}{4}\right) \sum_{i,j=1}^m h(\tilde\nabla_{e_i} V, du(e_j)) h(\tilde\nabla_{e_i} W, du(e_j))\, dv_g \\
&\quad + \int_M F'\left(\frac{\|T_u\|^2}{4}\right) \sum_{i,j=1}^m h(\tilde\nabla_{e_i} V, du(e_j)) h(du(e_i), \tilde\nabla_{e_j} W)\, dv_g \\
&\quad - \frac{2}{m} \int_M F'\left(\frac{\|T_u\|^2}{4}\right) \sum_{i,j=1}^m h(\tilde\nabla_{e_i} V, du(e_i)) h(du(e_j), \tilde\nabla_{e_j} W)\, dv_g \\
&\quad + \int_M F'\left(\frac{\|T_u\|^2}{4}\right) \sum_{i,j=1}^m h(R^N(V, du(e_i)) W, du(e_j)) T_u(e_i, e_j)\, dv_g.
\end{aligned}$$

where $\langle\, ,\rangle$ is the inner product on $T^*M \otimes u^{-1}TN$ and R^N is the curvature tensor of N.

We put

$$I(V, W) = \frac{\partial^2}{\partial s \partial t} \Phi_F(u_{s,t})|_{s,t=0}. \tag{17}$$

An F-stationary map u is called stable if $I(V,V) \geq 0$ for any compactly supported vector field V along u.

Proof. Let $\Psi : (-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\times M \to N$ be defined by $\Phi(s,t,x) = u_{s,t}(x)$, where $(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\times M$ is equipped with the product metric. We extend the vector fields $\partial/\partial t$ on $(-\varepsilon,\varepsilon)$, $\partial/\partial s$ on $(-\varepsilon,\varepsilon)$, X on M naturally on $(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\times M$, and denote those also by $\partial/\partial t$, $\partial/\partial s$ and X. Then

$$V = d\Psi\left(\frac{\partial}{\partial t}\right)\bigg|_{s,t=0}, \quad W = d\Psi\left(\frac{\partial}{\partial s}\right)\bigg|_{s,t=0}. \tag{18}$$

We shall use the same notations ∇ and $\tilde{\nabla}$ for the Levi-Civita connection on $(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\times M$ and the induced connection on $\Psi^{-1}TN$. We choose a local orthonormal frame $\{e_i\}_{i=1}^m$ around a point P on M with $\nabla_{e_i}e_j|_P = 0$.

Using (5) we have

$$\begin{aligned}
\frac{\partial^2}{\partial s\partial t}\Phi_F(u_{s,t})|_{s,t=0} &= -\frac{\partial}{\partial s}\int_M h\left(d\Psi\left(\frac{\partial}{\partial t}\right), \sum_{i=1}^m \left\{\tilde{\nabla}_{e_i}\left(F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\sigma_{u_{s,t}}(e_i)\right)\right.\right. \\
&\qquad \left.\left. - F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\sigma_{u_{s,t}}(\nabla_{e_i}e_i)\right\}\right)\bigg|_{s,t=0} dv_g \\
&= -\int_M h\left(d\Psi\left(\frac{\partial}{\partial t}\right), \sum_{i=1}^m \left\{\tilde{\nabla}_{\partial/\partial s}\tilde{\nabla}_{e_i}\left(F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\sigma_{u_{s,t}}(e_i)\right)\right.\right. \\
&\qquad \left.\left. - \tilde{\nabla}_{\partial/\partial s}\left(F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\sigma_{u_{s,t}}(\nabla_{e_i}e_i)\right)\right\}\right)\bigg|_{s,t=0} dv_g
\end{aligned}$$

where we use the F-stationarity for the last equality. At P, we compute

$$\begin{aligned}
&h\left(d\Psi\left(\frac{\partial}{\partial t}\right), \sum_{i=1}^m \left\{\tilde{\nabla}_{\partial/\partial s}\tilde{\nabla}_{e_i}\left(F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\sigma_{u_{s,t}}(e_i)\right)\right.\right. \\
&\qquad \left.\left. - \tilde{\nabla}_{\partial/\partial s}\left(F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\sigma_{u_{s,t}}(\nabla_{e_i}e_i)\right)\right\}\right) \\
&= h\left(d\Psi\left(\frac{\partial}{\partial t}\right), \sum_{i=1}^m \tilde{\nabla}_{e_i}\tilde{\nabla}_{\partial/\partial s}\left(F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\sigma_{u_{s,t}}(e_i)\right)\right) \\
&\quad + h\left(d\Psi\left(\frac{\partial}{\partial t}\right), \sum_{i=1}^m R^N\left(d\Psi\left(\frac{\partial}{\partial s}\right), d\Psi(e_i)\right)\left(F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\sigma_{u_{s,t}}(e_i)\right)\right)
\end{aligned} \tag{19}$$

where we use $\left[\frac{\partial}{\partial s}, e_i\right] = 0$.

The first term in the right-hand side of (19) is

$$(20)\quad h\left(d\Psi\left(\frac{\partial}{\partial t}\right), \sum_{i=1}^{m} \tilde{\nabla}_{e_i}\tilde{\nabla}_{\partial/\partial s}\left(F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\sigma_{u_{s,t}}(e_i)\right)\right)$$

$$= \sum_{i=1}^{m} e_i h\left(d\Psi\left(\frac{\partial}{\partial t}\right), \tilde{\nabla}_{\partial/\partial s}\left(F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\sigma_{u_{s,t}}(e_i)\right)\right)$$

$$- \sum_{i=1}^{m} h\left(\tilde{\nabla}_{e_i}\, d\Psi\left(\frac{\partial}{\partial t}\right), \tilde{\nabla}_{\partial/\partial s}\left(F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\sigma_{u_{s,t}}(e_i)\right)\right)$$

The second term in the right-hand side of (20) is

$$(21)\quad \sum_{i=1}^{m} h\left(\tilde{\nabla}_{e_i}\, d\Psi\left(\frac{\partial}{\partial t}\right), \tilde{\nabla}_{\partial/\partial s}\left(F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\sigma_{u_{s,t}}(e_i)\right)\right)$$

$$= \left[\sum_{i=1}^{m} h\left(\tilde{\nabla}_{e_i}\, d\Psi\left(\frac{\partial}{\partial t}\right), \sigma_{u_{s,t}}(e_i)\right)\right] F''\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right) \frac{\partial}{\partial s}\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)$$

$$+ F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right) \sum_{i=1}^{m} h\left(\tilde{\nabla}_{e_i}\, d\Psi\left(\frac{\partial}{\partial t}\right), \tilde{\nabla}_{\partial/\partial s}\sigma_{u_{s,t}}(e_i)\right)$$

$$= \left[\sum_{i=1}^{m} h\left(\tilde{\nabla}_{e_i}\, d\Psi\left(\frac{\partial}{\partial t}\right), \sigma_{u_{s,t}}(e_i)\right)\right] F''\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)$$

$$\left[\frac{1}{2}\sum_{i,j}\frac{\partial}{\partial s}\left[h(d\Psi(e_i), d\Psi(e_j)) - \frac{1}{m}\|du_{u_{s,t}}\|^2 g(e_i, e_j)\right] T_{u_{s,t}}(e_i, e_j)\right]$$

$$+ F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right) \sum_{i=1}^{m} h\left(\tilde{\nabla}_{e_i}\, d\Psi\left(\frac{\partial}{\partial t}\right), \tilde{\nabla}_{\partial/\partial s}\sigma_{u_{s,t}}(e_i)\right)$$

$$= \left[\sum_{i=1}^{m} h\left(\tilde{\nabla}_{e_i}\, d\Psi\left(\frac{\partial}{\partial t}\right), \sigma_{u_{s,t}}(e_i)\right)\right] F''\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)$$

$$\times \left[\sum_{j=1}^{m} h\left(\tilde{\nabla}_{e_j}\, d\Psi\left(\frac{\partial}{\partial s}\right), \sigma_{u_{s,t}}(e_j)\right)\right]$$

$$+ F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right) \sum_{i=1}^{m} h\left(\tilde{\nabla}_{e_i}\, d\Psi\left(\frac{\partial}{\partial t}\right),\right.$$

$$\left.\tilde{\nabla}_{\partial/\partial s}\left[\sum_{j=1}^{m} h(du_{s,t}(e_i), du_{s,t}(e_j))\, du_{s,t}(e_j) - \frac{1}{m}\|du_{s,t}\|^2\, du_{s,t}(e_i)\right]\right)$$

$$= \left[\sum_{i=1}^{m} h\left(\tilde{\nabla}_{e_i}\, d\Psi\left(\frac{\partial}{\partial t}\right), \sigma_{u_{s,t}}(e_i)\right)\right] F''\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\left[\sum_{j=1}^{m} h\left(\tilde{\nabla}_{e_j}\, d\Psi\left(\frac{\partial}{\partial s}\right), \sigma_{u_{s,t}}(e_j)\right)\right]$$

$$+ F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\sum_{i,j=1}^{m} h\left(\tilde{\nabla}_{e_i}\, d\Psi\left(\frac{\partial}{\partial t}\right), \tilde{\nabla}_{e_j}\, d\Psi\left(\frac{\partial}{\partial s}\right)\right) T_{u_{s,t}}(e_i, e_j)$$

$$+ F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\sum_{i,j=1}^{m} h\left(\tilde{\nabla}_{e_i}\, d\Psi\left(\frac{\partial}{\partial t}\right), d\Psi(e_j)\right) h\left(\tilde{\nabla}_{e_i}\, d\Psi\left(\frac{\partial}{\partial s}\right), d\Psi(e_j)\right)$$

$$+ F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\sum_{i,j=1}^{m} h\left(\tilde{\nabla}_{e_i}\, d\Psi\left(\frac{\partial}{\partial t}\right), d\Psi(e_j)\right) h\left(d\Psi(e_i), \tilde{\nabla}_{e_j}\, d\Psi\left(\frac{\partial}{\partial s}\right)\right)$$

$$- F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\frac{2}{m}\sum_{i} h\left(\tilde{\nabla}_{e_i}\, d\Psi\left(\frac{\partial}{\partial t}\right), d\Psi(e_i)\right)\sum_{j} h\left(\tilde{\nabla}_{e_j}\, d\Psi\left(\frac{\partial}{\partial s}\right), d\Psi(e_j)\right),$$

where we use that

$$\sum_{i,j=1}^{m} g(e_i, e_j) T_{u_{s,t}}(e_i, e_j) = 0$$

for the third equality. Let X_1, X_2, X_3, X_4 and X_5 be compactly supported vector fields on M such that

$$g(X_1, Y) = F''\left(\frac{\|T_u\|^2}{4}\right)\langle \tilde{\nabla} W, \sigma_u\rangle h(\sigma_u(Y), V),$$

$$g(X_2, Y) = F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{j=1}^{m} h(V, du(e_j)) h(\tilde{\nabla}_Y W, du(e_j)),$$

$$g(X_3, Y) = F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{j=1}^{m} h(V, du(e_j)) h(du(Y), \tilde{\nabla}_{e_j} W),$$

$$g(X_4, Y) = F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{j=1}^{m} h(V, \tilde{\nabla}_{e_j} W) T_u(Y, e_j),$$

$$g(X_5, Y) = F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{j=1}^{m} h(V, du(Y)) h(du(e_j), \tilde{\nabla}_{e_j} W),$$

for any vector field Y on M, respectively. For the first term in the right-hand side of (20), we have

$$
\begin{aligned}
(22)\quad & \sum_{i=1}^{m} e_i h\left(d\Psi\left(\frac{\partial}{\partial t}\right), \tilde{\nabla}_{\partial/\partial s}\left(F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\sigma_{u_{s,t}}(e_i)\right)\right) \\
&= \sum_{i=1}^{m} e_i h\left(F''\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\left(\frac{1}{4}\frac{\partial}{\partial s}\|T_{u_{s,t}}\|^2\right)\sigma_{u_{s,t}}(e_i), d\Psi\left(\frac{\partial}{\partial t}\right)\right) \\
&\quad + \sum_{i=1}^{m} e_i h\left(F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\tilde{\nabla}_{\partial/\partial s}\sigma_{u_{s,t}}(e_i), d\Psi\left(\frac{\partial}{\partial t}\right)\right) \\
&= \sum_{i=1}^{m} e_i h\left(F''\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\left(\frac{1}{4}\frac{\partial}{\partial s}\left[\|u_{s,t}^{*}h\|^2 - \frac{1}{m}\|du_{s,t}\|^4\right]\right)\sigma_{u_{s,t}}(e_i), d\Psi\left(\frac{\partial}{\partial t}\right)\right) \\
&\quad + \sum_{i=1}^{m} e_i h\left(F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\tilde{\nabla}_{\partial/\partial s}\sigma_{u_{s,t}}(e_i), d\Psi\left(\frac{\partial}{\partial t}\right)\right) \\
&= \sum_{i=1}^{m} e_i \left\{F''\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\sum_{j=1}^{m} h\left(\sigma_{u_{s,t}}(e_i), d\Psi\left(\frac{\partial}{\partial t}\right)\right)\right. \\
&\qquad\qquad \left.\times h\left(\tilde{\nabla}_{e_j}\, d\Psi\left(\frac{\partial}{\partial s}\right), \sigma_{u_{s,t}}(e_j)\right)\right\} \\
&\quad + \sum_{i=1}^{m} e_i \left\{F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\sum_{j=1}^{m} h\left(\tilde{\nabla}_{e_i}\, d\Psi\left(\frac{\partial}{\partial s}\right), d\Psi(e_j)\right)\right. \\
&\qquad\qquad \left.\times h\left(d\Psi\left(\frac{\partial}{\partial t}\right), d\Psi(e_j)\right)\right\} \\
&\quad + \sum_{i=1}^{m} e_i \left\{F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\sum_{j=1}^{m} T_{u_{s,t}}(e_i,e_j) h\left(d\Psi\left(\frac{\partial}{\partial t}\right), \tilde{\nabla}_{e_j}\, d\Psi\left(\frac{\partial}{\partial s}\right)\right)\right\} \\
&\quad + \sum_{i=1}^{m} e_i \left\{F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\sum_{j=1}^{m} h\left(d\Psi(e_i), \tilde{\nabla}_{e_j}\, d\Psi\left(\frac{\partial}{\partial s}\right)\right)\right. \\
&\qquad\qquad \left.\times h\left(d\Psi\left(\frac{\partial}{\partial t}\right), d\Psi(e_j)\right)\right\} \\
&\quad - \frac{2}{m}\sum_{i=1}^{m} e_i \left\{F'\left(\frac{\|T_{u_{s,t}}\|^2}{4}\right)\sum_{j=1}^{m} h\left(d\Psi(e_j), \tilde{\nabla}_{e_j}\, d\Psi\left(\frac{\partial}{\partial s}\right)\right)\right. \\
&\qquad\qquad \left.\times h\left(d\Psi\left(\frac{\partial}{\partial t}\right), d\Psi(e_i)\right)\right\},
\end{aligned}
$$

when $s = t = 0$, (22) becomes

$$\begin{aligned}(23)\qquad & \sum_{i=1}^m e_i g(X_1, e_i) + \sum_{i=1}^m e_i g(X_2, e_i) + \sum_{i=1}^m e_i g(X_3, e_i) \\ & + \sum_{i=1}^m e_i g(X_4, e_i) - \frac{2}{m}\sum_{i=1}^m e_i g(X_5, e_i) \\ & = \sum_{i=1}^m g(\nabla_{e_i} X_1, e_i) + \sum_{i=1}^m g(\nabla_{e_i} X_2, e_i) + \sum_{i=1}^m g(\nabla_{e_i} X_3, e_i) \\ & + \sum_{i=1}^m g(\nabla_{e_i} X_4, e_i) - \frac{2}{m}\sum_{i=1}^m g(\nabla_{e_i} X_5, e_i) \\ & = div(X_1) + div(X_2) + div(X_3) + div(X_4) - \frac{2}{m} div(X_5).\end{aligned}$$

By Green's theorem the integral of (23) vanishes. Theorem follows from (19)–(23). □

6. Stable maps into spheres

In this section we prove the following theorem

THEOREM 6.1. *Let $u : M^m \to S^n$ be an F-stationary map from a compact Riemannian manifold M into the n-dimensional standard sphere S^n. Assume that*

$$(24)\qquad \int_{M^m} \|T_u\|^2 \left\{ F''\left(\frac{\|T_u\|^2}{4}\right)\|T_u\|^2 + (4-n)F'\left(\frac{\|T_u\|^2}{4}\right)\right\} dv_g < 0.$$

Then u is unstable.

Proof. In order to prove the instability of $u : M^m \to S^n$, we need to consider some special variational vector fields along u. To do this, choosing a local orthonormal frame field $\{\epsilon_\alpha\}$, $\alpha = 1, \ldots, n$ around a point P on S^n with ${}^{S^n}\nabla_{\epsilon_\alpha}\epsilon_\beta|_P = 0$ and choosing ϵ_{n+1} such that $\{\epsilon_\alpha, \epsilon_{n+1}\}$ is an orthonormal frame field of R^{n+1}. Meanwhile, taking a fixed orthonormal basis E_A, $A = 1, \ldots, n+1$ of R^{n+1} and setting

$$(25)\qquad V_A = \sum_{\alpha=1}^n v_A^\alpha \epsilon_\alpha, \quad v_A^\alpha = \langle E_A, \epsilon_\alpha\rangle, \quad v_A^{n+1} = \langle E_A, \epsilon_{n+1}\rangle,$$

where $\langle\, ,\rangle$ denotes the canonical Euclidean inner product. We shall consider the second variation

$$I(V_A,V_A)=\int_M F''\left(\frac{\|T_u\|^2}{4}\right)\langle\tilde{\nabla}V_A,\sigma_u\rangle\langle\tilde{\nabla}V_A,\sigma_u\rangle\,dv_g \tag{26}$$
$$+\int_M F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i,j=1}^m h(\tilde{\nabla}_{e_i}V_A,\tilde{\nabla}_{e_j}V_A)T_u(e_i,e_j)\,dv_g$$
$$+\int_M F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i,j=1}^m h(\tilde{\nabla}_{e_i}V_A,du(e_j))h(\tilde{\nabla}_{e_i}V_A,du(e_j))\,dv_g$$
$$+\int_M F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i,j=1}^m h(\tilde{\nabla}_{e_i}V_A,du(e_j))h(du(e_i),\tilde{\nabla}_{e_j}V_A)\,dv_g$$
$$-\frac{2}{m}\int_M F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i,j=1}^m h(\tilde{\nabla}_{e_i}V_A,du(e_i))h(du(e_j),\tilde{\nabla}_{e_j}V_A)\,dv_g$$
$$+\int_M F'\left(\frac{\|T_u\|^2}{4}\right)\sum_i h(R^{S^n}(V_A,du(e_i))V_A,\sigma_u(e_i))\,dv_g.$$

At P, we compute

$$\tilde{\nabla}_{e_i}V_A={}^{S^n}\nabla_{du(e_i)}V_A=-v_A^{n+1}\,du(e_i). \tag{27}$$

From (25) and (27), we compute the following equations:

$$F''\left(\frac{\|T_u\|^2}{4}\right)\sum_A\langle\tilde{\nabla}V_A,\sigma_u\rangle\langle\tilde{\nabla}V_A,\sigma_u\rangle \tag{28}$$
$$=F''\left(\frac{\|T_u\|^2}{4}\right)\sum_{A,i,j}h(\tilde{\nabla}_{e_i}V_A,\sigma_u(e_i))h(\tilde{\nabla}_{e_j}V_A,\sigma_u(e_j))$$
$$=F''\left(\frac{\|T_u\|^2}{4}\right)\sum_{A,i,j}v_A^{n+1}v_A^{n+1}h(du(e_i),\sigma_u(e_i))h(du(e_j),\sigma_u(e_j))$$
$$=F''\left(\frac{\|T_u\|^2}{4}\right)\|T_u\|^4$$

and

$$F'\left(\frac{\|T_u\|^2}{4}\right)\sum_A\sum_{i,j=1}^m h(\tilde{\nabla}_{e_i}V_A,\tilde{\nabla}_{e_j}V_A)T_u(e_i,e_j) \tag{29}$$
$$=F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i,j=1}^m\sum_A v_A^{n+1}v_A^{n+1}h(du(e_i),du(e_j))T_u(e_i,e_j)$$
$$=F'\left(\frac{\|T_u\|^2}{4}\right)\|T_u\|^2$$

and

$$F'\left(\frac{\|T_u\|^2}{4}\right)\sum_A\sum_{i,j=1}^m h(\tilde{\nabla}_{e_i}V_A, du(e_j))h(\tilde{\nabla}_{e_i}V_A, du(e_j)) \tag{30}$$
$$= F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i,j=1}^m\sum_A v_A^{n+1}v_A^{n+1}h(du(e_i), du(e_j))h(du(e_i), du(e_j))$$
$$= F'\left(\frac{\|T_u\|^2}{4}\right)\|u^*h\|^2$$

and

$$F'\left(\frac{\|T_u\|^2}{4}\right)\sum_A\sum_{i,j=1}^m h(\tilde{\nabla}_{e_i}V_A, du(e_j))h(du(e_i), \tilde{\nabla}_{e_j}V_A) \tag{31}$$
$$= F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i,j=1}^m\sum_A v_A^{n+1}v_A^{n+1}h(du(e_i), du(e_j))h(du(e_i), du(e_j))$$
$$= F'\left(\frac{\|T_u\|^2}{4}\right)\|u^*h\|^2$$

and

$$F'\left(\frac{\|T_u\|^2}{4}\right)\sum_A\sum_{i,j=1}^m h(\tilde{\nabla}_{e_i}V_A, du(e_i))h(\tilde{\nabla}_{e_j}V_A, du(e_j)) \tag{32}$$
$$= F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i,j=1}^m\sum_A v_A^{n+1}v_A^{n+1}h(du(e_i), du(e_i))h(du(e_j), du(e_j))$$
$$= F'\left(\frac{\|T_u\|^2}{4}\right)\|du\|^4$$

and

$$F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i=1}^m\sum_A h(R^{S^n}(V_A, du(e_i))V_A, \sigma_u(e_i)) \tag{33}$$
$$= F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i=1}^m\sum_{A,\alpha,\beta} v_A^{\alpha}v_A^{\beta}h(R^{S^n}(\epsilon_\alpha, du(e_i))\epsilon_\beta, \sigma_u(e_i))$$
$$= F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i=1}^m\sum_{\alpha} h(R^{S^n}(\epsilon_\alpha, du(e_i))\epsilon_\alpha, \sigma_u(e_i))$$

$$= F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i=1}^{m}\sum_{\alpha}[h(\epsilon_\alpha, \sigma_u(e_i))h(\epsilon_\alpha, du(e_i)) - h(du(e_i), \sigma_u(e_i))h(\epsilon_\alpha, \epsilon_\alpha)]$$

$$= (1-n)F'\left(\frac{\|T_u\|^2}{4}\right)\|T_u\|^2.$$

From (26)–(33), we get

$$\sum_{A=1}^{n+1} I(V_A, V_A) = \int_{M^m} \|T_u\|^2\left\{F''\left(\frac{\|T_u\|^2}{4}\right)\|T_u\|^2 + (4-n)F'\left(\frac{\|T_u\|^2}{4}\right)\right\} dv_g. \tag{34}$$

By (34) and the assumption, we have

$$\sum_{A=1}^{n+1} I(V_A, V_A) < 0 \tag{35}$$

and u is unstable. □

Corollary 6.2. *Assume that* (i) $F'' \le 0$ *and* $n \ge 5$, *or* (ii) $F'' < 0$ *and* $n = 4$. *Then any stable F-stationary map from a compact Riemannian manifold M to* S^n *is a weakly conformal map.*

7. Stable maps from spheres

In this section we prove the following theorem

Theorem 7.1. *Let* $u : S^m \to N$ *be an F-stationary map. Assume that*

$$\int_{S^m} \|T_u\|^2\left\{F''\left(\frac{\|T_u\|^2}{4}\right)\|T_u\|^2 + (4-m)F'\left(\frac{\|T_u\|^2}{4}\right)\right\} dv_g < 0. \tag{36}$$

Then u is unstable.

Proof. In order to prove the instability of $u : S^m \to N$, we need to consider some special variational vector fields along u. To do this, choosing a local orthonormal frame field $\{e_i\}$, $i = 1, \dots, m$ around a point P on S^m with ${}^{S^m}\nabla_{e_i} e_j|_P = 0$ and choosing e_{m+1} such that $\{e_i, e_{m+1}\}$ is an orthonormal frame field of R^{m+1}. Meanwhile, taking a fixed orthonormal basis E_A, $A = 1, \dots, m+1$ of R^{m+1} and setting

$$V_A = \sum_{i=1}^{m} v_A^i e_i, \quad v_A^i = \langle E_A, e_i\rangle, \quad v_A^{m+1} = \langle E_A, e_{m+1}\rangle, \tag{37}$$

where $\langle\, ,\rangle$ denotes the canonical Euclidean inner product. Then $du(V_A) \in \Gamma(u^{-1}TN)$ and

$$\sum_A v_A^i v_A^j = \sum_A \langle E_A, e_i\rangle\langle E_A, e_j\rangle = \delta_{ij}, \tag{38}$$

$$\nabla_{e_i} V_A = -v_A^{m+1} e_i, \tag{39}$$

$$\tilde{\nabla}_{e_i}\, du(V_A) = -v_A^{m+1}\, du(e_i) + v_A^l \tilde{\nabla}_{e_i}\, du(e_l). \tag{40}$$

By using the condition $\tau_F(u) = -\delta\left(F'\left(\frac{\|T_u\|^2}{4}\right)\sigma_u\right) = 0$ and (38), we have

$$\begin{aligned}
&\int_{S^m} \sum_{A=1}^{m+1} F'\left(\frac{\|T_u\|^2}{4}\right)\langle(\triangle\, du)(V_A), \sigma_u(V_A)\rangle\, dv_g \\
&= \int_{S^m} \sum_A v_A^i v_A^j F'\left(\frac{\|T_u\|^2}{4}\right)\langle(\triangle\, du)(e_i), \sigma_u(e_j)\rangle\, dv_g \\
&= \sum_i \int_{S^m} F'\left(\frac{\|T_u\|^2}{4}\right)\langle(\triangle\, du)(e_i), \sigma_u(e_i)\rangle \\
&= \int_{S^m} F'\left(\frac{\|T_u\|^2}{4}\right)\langle(\triangle\, du), \sigma_u\rangle \\
&= \int_{S^m} \left\langle \delta\, du, \delta\left(F'\left(\frac{\|T_u\|^2}{4}\right)\sigma_u\right)\right\rangle \\
&= 0.
\end{aligned} \tag{41}$$

It follows from Weitzenböck formula that

$$-\sum_{k=1}^m R^N(du(X), du(e_k))\, du(e_k) + du(Ric^{S^m}(X)) = (\triangle\, du)(X) + (\nabla^2\, du)(X). \tag{42}$$

where X is any smooth vector field on S^m and $(\nabla^2\, du)(X) = \sum_{i=1}^m [\nabla_{e_i}\nabla_{e_i}\, du - \nabla_{\nabla_{e_i} e_i}\, du](X)$. With respect to the variational vector field $du(V_A)$ along u, it follows form (41) and (42) that

$$\begin{aligned}
&\sum_A I(du(V_A), du(V_A)) \\
&= \int_M F''\left(\frac{\|T_u\|^2}{4}\right)\sum_A \langle\tilde{\nabla}\, du(V_A), \sigma_u\rangle^2\, dv_g \\
&+ \int_M F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i,j,A} h(\tilde{\nabla}_{e_i}\, du(V_A), \tilde{\nabla}_{e_j}\, du(V_A))T_u(e_i, e_j)\, dv_g \\
&+ \int_M F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i,j,A} h(\tilde{\nabla}_{e_i}\, du(V_A), du(e_j))h(\tilde{\nabla}_{e_i}\, du(V_A), du(e_j))\, dv_g
\end{aligned} \tag{43}$$

$$+\int_M F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i,j,A} h(\tilde{\nabla}_{e_i}\, du(V_A), du(e_j))h(du(e_i), \tilde{\nabla}_{e_j}\, du(V_A))\, dv_g$$

$$-\frac{2}{m}\int_M F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i,j,A} h(\tilde{\nabla}_{e_i}\, du(V_A), du(e_i))h(du(e_j), \tilde{\nabla}_{e_j}\, du(V_A))\, dv_g$$

$$-\int_M F'\left(\frac{\|T_u\|^2}{4}\right)\sum_i h(du(Ric^{S^m}(e_i)), \sigma_u(e_i))\, dv_g$$

$$+\int_M F'\left(\frac{\|T_u\|^2}{4}\right)\sum_i h((\nabla^2\, du)(e_i),)\sigma_u(e_i))\, dv_g.$$

At P, we compute

$$\begin{aligned}
(44)\qquad & F''\left(\frac{\|T_u\|^2}{4}\right)\sum_A \langle \tilde{\nabla}\, du(V_A), \sigma_u\rangle^2 \\
&= \sum_A F''\left(\frac{\|T_u\|^2}{4}\right)\left[\sum_i \langle \tilde{\nabla}_{e_i}\, du(V_A), \sigma_u(e_i)\rangle\right]^2 \\
&= F''\left(\frac{\|T_u\|^2}{4}\right)\sum_A \\
&\quad \times \left[\sum_i (-v_A^{m+1} h(du(e_i), \sigma_u(e_i)) + v_A^l h(\tilde{\nabla}_{e_i}\, du(e_l), \sigma_u(e_i)))\right]^2 \\
&= F''\left(\frac{\|T_u\|^2}{4}\right)\sum_A v_A^{m+1} v_A^{m+1}\left[\sum_i h(du(e_i), \sigma_u(e_i))\right]^2 \\
&\quad - 2F''\left(\frac{\|T_u\|^2}{4}\right)\sum_A v_A^{m+1}\left[\sum_i h(du(e_i), \sigma_u(e_i))\right] \\
&\quad \times \left\{\sum_l v_A^l \left[\sum_i h(\tilde{\nabla}_{e_i}\, du(e_l), \sigma_u(e_i)))\right]\right\} \\
&\quad + F''\left(\frac{\|T_u\|^2}{4}\right)\sum_A \left[\sum_l v_A^l\left[\sum_i h((\nabla_{e_i}\, du)(e_l), \sigma_u(e_i))\right]\right]^2 \\
&= F''\left(\frac{\|T_u\|^2}{4}\right)\left[\|T_u\|^4 + \sum_l\left[\sum_i h((\nabla_{e_l}\, du)(e_i), \sigma_u(e_i))\right]^2\right],
\end{aligned}$$

where we use the symmetry of $\nabla\, du$ in the last equality.

$$(45)\quad F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i,j,A} h(\tilde{\nabla}_{e_i}\, du(V_A), \tilde{\nabla}_{e_j}\, du(V_A))T_u(e_i, e_j)$$

$$= F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{A}[h(-v_A^{m+1}\, du(e_i) + v_A^k \tilde{\nabla}_{e_i}\, du(e_k),$$

$$-v_A^{m+1}\, du(e_j) + v_A^l \tilde{\nabla}_{e_j}\, du(e_l))T_u(e_i, e_j)]$$

$$= F'\left(\frac{\|T_u\|^2}{4}\right)\left[\|T_u\|^2 - 2\sum_{A} v_A^{m+1} v_A^k h(du(e_i), \tilde{\nabla}_{e_j}\, du(e_k))T_u(e_i, e_j)\right.$$

$$\left. + \sum_{A} v_A^k v_A^l h(\tilde{\nabla}_{e_i}\, du(e_k), \tilde{\nabla}_{e_j}\, du(e_l))T_u(e_i, e_j)\right]$$

$$= F'\left(\frac{\|T_u\|^2}{4}\right)\left[\|T_u\|^2 + \sum_{i,j,k} h((\nabla_{e_k}\, du)(e_i), (\nabla_{e_k}\, du)(e_j))T_u(e_i, e_j)\right]$$

and

$$(46)\quad F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i,j,A} h(\tilde{\nabla}_{e_i}\, du(V_A), du(e_j))h(\tilde{\nabla}_{e_i}\, du(V_A), du(e_j))$$

$$= F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{A}[h(-v_A^{m+1}\, du(e_i) + v_A^k \tilde{\nabla}_{e_i}\, du(e_k), du(e_j))$$

$$\times h(-v_A^{m+1}\, du(e_i) + v_A^k \tilde{\nabla}_{e_i}\, du(e_k), du(e_j))]$$

$$= F'\left(\frac{\|T_u\|^2}{4}\right)[\|u^*h\|^2 + v_A^k v_A^l h(\tilde{\nabla}_{e_i}\, du(e_k), du(e_j))h(\tilde{\nabla}_{e_i}\, du(e_l), du(e_j)$$

$$- 2v_A^{m+1} v_A^k h(du(e_i), du(e_j))h(\tilde{\nabla}_{e_i}\, du(e_k), du(e_j))]$$

$$= F'\left(\frac{\|T_u\|^2}{4}\right)\left[\|u^*h\|^2 + \sum_{i,j,k} h((\nabla_{e_k}\, du)(e_i), du(e_j))h((\nabla_{e_k}\, du)(e_i), du(e_j)\right]$$

and

$$(47)\quad F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i,j,A} h(\tilde{\nabla}_{e_i}\, du(V_A), du(e_j))h(du(e_i), \tilde{\nabla}_{e_j}\, du(V_A))$$

$$= F'\left(\frac{\|T_u\|^2}{4}\right)\left[\|u^*h\|^2 + \sum_{i,j,k} h((\nabla_{e_k}\, du)(e_i), du(e_j))h(du(e_i), (\nabla_{e_k}\, du)(e_j)\right]$$

and

$$F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i,j,A} h(\tilde{\nabla}_{e_i}\ du(V_A), du(e_i))h(du(e_j), \tilde{\nabla}_{e_j}\ du(V_A))$$
$$= F'\left(\frac{\|T_u\|^2}{4}\right)\left[\|du\|^4 + \sum_{i,j,k} h((\nabla_{e_k}\ du)(e_i), du(e_i))h(du(e_j), (\nabla_{e_k}\ du)(e_j)\right] \tag{48}$$

and

$$F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i} h(du(Ric^{S^m}(e_i)), \sigma_u(e_i))$$
$$= (m-1)F'\left(\frac{\|T_u\|^2}{4}\right)h(du(e_i), \sigma_u(e_i)))$$
$$= (m-1)F'\left(\frac{\|T_u\|^2}{4}\right)\|T_u\|^2 \tag{49}$$

and

$$F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i} h((\nabla^2\ du)(e_i), \sigma_u(e_i))$$
$$= F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i,k} h(\tilde{\nabla}_{e_k}\tilde{\nabla}_{e_k}\ du(e_i), \sigma_u(e_i))$$
$$= e_k\left\{F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i} h((\nabla_{e_k}\ du)(e_i), \sigma_u(e_i))\right\}$$
$$- F''\left(\frac{\|T_u\|^2}{4}\right)\sum_{k}\left[\sum_{i} h((\nabla_{e_k}\ du)(e_i), \sigma_u(e_i))\right]^2$$
$$- F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i,j,k} h((\nabla_{e_k}\ du)(e_i), (\nabla_{e_k}\ du)(e_j))T_u(e_i, e_j)$$
$$- F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i,j,k} h((\nabla_{e_k}\ du)(e_i), du(e_j))h((\nabla_{e_k}\ du)(e_i), du(e_j)$$
$$- F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i,j,k} h((\nabla_{e_k}\ du)(e_i), du(e_j))h(du(e_i), (\nabla_{e_k}\ du)(e_j))$$
$$+ \frac{2}{m}F'\left(\frac{\|T_u\|^2}{4}\right)\sum_{i,j,k} h((\nabla_{e_k}\ du)(e_i), du(e_i))h(du(e_j), (\nabla_{e_k}\ du)(e_j)). \tag{50}$$

By (43)–(50), we get

$$(51)\qquad \sum_A I(du(V_A), du(V_A)) = \int_{S^m} \|T_u\|^2 \left\{ F''\left(\frac{\|T_u\|^2}{4}\right)\|T_u\|^2 + (4-m)F'\left(\frac{\|T_u\|^2}{4}\right)\right\} dv_g.$$

By (51) and the assumption, we have

$$(52)\qquad \sum_A I(du(V_A), du(V_A)) < 0$$

and u is unstable. □

COROLLARY 7.2. *Assume that* (i) $F'' \le 0$ *and* $m \ge 5$, *or* (ii) $F'' < 0$ *and* $m = 4$. *Then any stable* F*-stationary map from* S^m *is a weakly conformal map.*

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REFERENCES

[1] M. ARA, Geometry of F-harmonic maps, Kodai Math. J. **22** (1999), 243–263.
[2] P. BAIRD AND J. EELLS, A conservation law for harmonic maps, Geometry Symposium Utrecht, 1980, Lecture notes in mathematics **894**, Springer, Berlin, 1981, 1–25.
[3] P. BAIRD, Stess-energy tensors and the Linchnerowicz Laplacian, J. Geo. and Phys. **58** (2008), 1329–1342.
[4] Y. X. DONG AND S. S. WEI, On vanishing theorems for vector bundle valued p-forms and their applications, Comm. Math. Phys. **304** (2011), 329–368.
[5] Y. X. DONG, H. Z. LIN AND G. L. YANG, Liouville theorems for F-harmonic maps and their applications, 2011, arXiv:1111.1882v1[math.DG].
[6] S. KAWAI AND N. NAKAUCHI, Some result for stationary maps of a functional related to pullback metrics, Nonlinear analysis **74** (2011), 2284–2295.
[7] J. C. LIU, Liouville theorems of stable F-harmonic maps from compact convex hypersurfaces, Hiroshima Math. J. **36** (2006), 221–234.
[8] N. NAKAUCHI, A variational problem related to conformal maps, Osaka J. Math. **48** (2011), 719–741.
[9] N. NAKAUCHI AND Y. TAKENAKA, A variational problem for pullback metrics, Ricerche Math. **60** (2011), 219–235.

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