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KODAI MATH. J.
36 (2013), 397–408

TAGGED PARTICLE DYNAMICS IN STOCHASTIC RANKING PROCESS

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Abstract

We consider a stochastic ranking process, which is a mathematical model of the ranking in the web page of online bookstores or posting web pages. We give a scaling limit of tagged particle dynamics. In this limit the scaled tagged particles jumps to the top of the list when its own Poisson clock rings and moves deterministically along a curve otherwise. This curve is characteristic curve of a system of quasi linear PDE, which is mentioned in [11, 14]. We also give a scaling limit of multi-tagged particle dynamics, in which the motion of the particles are independent.

1. Introduction

We consider a stochastic ranking process or Poisson embedding of the move-to-front rules, which is an algorithm for a self-organizing linear list of a finite number of items. The list is updated in the following way. Each item has an independent Poisson clock, whose rate depends on type of the item. If the Poisson clock of the i-th item rings, then we move it to the top of the list and accordingly each of the items located in front of the i-th item backwards simultaneously by one rank; those behind do not move at all. In this paper, we treat this process as an "interacting particle system". We fasten a tag to a "particle" (or tags to "particles") and observe the motion of "tagged particle" (or "tagged particles"). We give a scaling limit of tagged particle dynamics as the number of the items tends to infinity. In this limit the scaled tagged particle jumps to the top of the list when its own Poisson clock rings and moves deterministically along a curve otherwise. This curve is characteristic curve of a system of quasi linear PDE, which is mentioned in [11, 14]. We also give a scaling limit of multi-tagged particle dynamics, in which the motion of the particles are independent.

The move-to-front rule is introduced by Tsetlin [24] and studied [5, 16, 19, 20, 21]. It is also studied as least-recently-used cashing [1, 2, 3, 6, 7, 8, 9, 10, 17,

2010 *Mathematics Subject Classification*. 60K35.
Key words and phrases. Stochastic ranking process; tagged particle dynamics.
Received December 25, 2012; revised January 9, 2013.

18, 22, 23]. Recently it is reintroduced and studied as a mathematical model of the ranking in the web page of online bookstores or in the posting web pages [11, 12, 13, 14, 15].

The distribution of the scaling limit of tagged particle dynamics is discussed and obtained in [1, 9, 17] (as the scaling limit of search cost for the move-to-front rules). Precisely, the stationary distribution is obtained [1, 9, 10, 17]. In [1, 9, 17] the distribution of stationary search cost for the move-to-front rules is discussed and the scaling limit (a fluid limit) is obtained. Furthermore in [1] the distribution of the scaling limit of general search cost and the independence of the motion of the multi-tagged particle (propagation of chaos) is obtained.

Let $\{\nu_i; i \in \mathbf{N}\}$ be independent Poisson random measures on $[0, \infty)$ with intensity $w_i(s)\,ds$. We assume that the set of intensities is finite, i.e., there exists K such that $\{w_i; i \in \mathbf{N}\} = \{\tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_K\}$. Let $(x_1^N, x_2^N, \dots x_N^N)$ be a permutation of $1, 2, \dots N$. We define stochastic ranking process $X^N = (X_1^N, X_2^N, \dots, X_N^N)$ by

$$X_i^N(t) = x_i^N + \sum_{j=1}^{N} \int_0^t \mathbf{1}(X_j(s-) > X_i(s-))\nu_j(ds) + \int_0^t (1 - X_i(s-))\nu_i(ds) \tag{1}$$

where $\mathbf{1}(A)$ is the indicator function of A. We regard $X_i^N(t)$ and x_i^N as positions of the i-th particle at time t and at time 0 respectively. Each particle has an independent Poisson clock with intensity w_i. If i-th particle's Poisson clock rings, then i-th particle jumps to the top. If a Poisson clock of a particle located behind the i-th particle rings, then the i-th particle jumps backward by one step.

We define the normalized position of X^N by

$$Y_i^N(t) = \frac{1}{N}(X_i^N(t) - 1).$$

Let us define $\lceil x \rceil = \lceil x \rceil^N$ and $\lfloor x \rfloor = \lfloor x \rfloor^N$ for $x \in [0, 1]$ by

$$\lceil x \rceil^N = \frac{l}{N}, \quad \lfloor x \rfloor^N = \frac{l-1}{N} \quad \text{such that } l \in \mathbf{Z} \quad \text{and} \quad \frac{l-1}{N} < x \le \frac{l}{N}.$$

We define $U^N = (U_1^N(x; s), U_2^N(x; s), \dots U_K^N(x; s))$ by

$$U_l^N\left(\frac{k}{N}; s\right) := \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}(w_i = \tilde{w}_l)\mathbf{1}\left(Y_i^N(s) \ge \frac{k}{N}\right),$$

for $k = 0, 1, 2, \dots, N$ and

$$U_l^N(x; s) := U_l^N(\lfloor x \rfloor; s) + N(x - \lfloor x \rfloor)\{U_l^N(\lceil x \rceil; s) - U_l^N(\lfloor x \rfloor; s)\},$$

for $0 \le x \le 1$ and $x \notin \{0, 1/N, 2/N, \dots (N-1)/N, 1\}$. Namely if $x \in \{0, 1/N, 2/N, \dots (N-1)/N, 1\}$ then $U_l^N(x,s)$ denotes the normalized number of scaled particles in $[x,1]$ at time s whose intensity is $\tilde{w}_l$. If $x \notin \{0, 1/N, 2/N, \dots (N-1)/N, 1\}$ then it is given by linear interpolation of $\{U_l^N(0;s), U_l^N(1/N;s), U_l^N(2/N;s), \dots, U_l^N((N-1)/N;s), U_l^N(1;s)\}$.

Let us consider the Cauchy problem for a system of quasi linear PDE

$$
\begin{aligned}
(2) \qquad & \frac{\partial}{\partial t} u_l(x,t) = -u_l(x,s)\tilde{w}_l(s) - \sum_{m=1}^{K} u_m(x,s)\tilde{w}_m(s)\frac{\partial}{\partial x} u_l(x,s), \\
& u_l(0,t) = f_l(0), \\
& u_l(1,t) = 0, \\
& u_l(x,0) = f_l(x),
\end{aligned}
$$

for $l = 1, 2, \dots, K$ where the initial functions f_l, $1 \le l \le K$ are smooth and decreasing, and satisfies that $f_l \ge 0$ and $\sum_{l=1}^{K} f_l(0) = 1$. In [13] it is proved that this system of PDE has a unique global classical solution. From now on, we denote by $u(x,t) = (u_1(x,t), \dots, u_K(x,t))$ the unique global solution of (2).

The following result is already proved in [11].

Proposition 1.1. *Assume that $U^N(x;0) \to u(x,0)$ $(N \to \infty)$ uniformly in $x \in [0,1]$ almost surely. Then the process $U^N(x;t) \to u(x,t)$ $(N \to \infty)$ uniformly in $x \in [0,1]$ and $t \in [0,T]$ for all T with probability one.*

We give a scaling limit of tagged particle dynamics.

Theorem 1.2. *Assume that $U^N(x;0) \to u(x,0)$ $(N \to \infty)$ uniformly in $x \in [0,1]$ almost surely, and $\frac{1}{N} x_1^N \to y_1$ $(N \to \infty)$ almost surely. Then the scaled tagged particle motion $Y_1^N(t) \to Y_1(t)$ uniformly in $t \in [0,T]$ almost surely for all $T \ge 0$, where Y_1 is the solution of*

$$
Y_1(t) = y_1 + \sum_{l=1}^{K} \int_0^t u_l(Y_1(s-), s)\tilde{w}_l(s)\, ds - \int_0^t Y_1(s-)\nu_1(ds).
$$

Furthermore, assume that $U^N(x;0) \to u_0(x)$ $(N \to \infty)$ uniformly in $x \in [0,1)$ almost surely, and for some L, $\left(\frac{1}{N} x_1^N, \frac{1}{N} x_2^N, \dots, \frac{1}{N} x_L^N\right) \to (y_1, y_2, \dots, y_L)$ $(N \to \infty)$ almost surely. Then the scaled tagged particle system $(Y_1^N(t), Y_2^N(t), \dots, Y_L^N(t)) \to (Y_1(t), Y_2(t), \dots Y_L(t))$ uniformly in $t \in [0,T]$ almost surely for all $T \ge 0$, where Y_i, $i = 1, 2, \dots, L$ are the solutions of

$$
Y_i(t) = y_i + \sum_{l=1}^{K} \int_0^t u_l(Y_i(s-), s)\tilde{w}_l(s)\, ds - \int_0^t Y_i(s-)\nu_i(ds).
$$

The last equation expresses what is mentioned previously: a scaled particle moves deterministically obeying the same ODE as for the corresponding characteristic curve of the system of PDE (2) except for its successive Poisson epochs at each of which it jumps to the top independently of the motion of the other tagged particles.

2. Proof of the main results

2.1. Proof of the Proposition 1.1

As being mentioned in Introduction Proposition 1.1 is already proved in [11]. Here we give another proof, where we derive PDE (2) directly from the stochastic integral equation (1) by using Itô formula.

In order to define the characteristics of the quasi linear PDE

$$\frac{\partial}{\partial t}u_l(x,t) + a(x,t,u)\frac{\partial}{\partial x}u_l(x,t) = h_l(x,t,u)$$

for $l = 1, 2, \dots, K$, with some boundary conditions for $f = (f_1, f_2, \dots, f_K)$, we follow [4, Chap. 3]. In our case, $a(x,t,u) = \sum_{m=1}^{K} u_m(x,t)\tilde{w}(t)$ and $h_l(x,t,u) = -u_l(x,t)\tilde{w}_l(t)$. Let us consider the system of ODE

$$\begin{cases} \dfrac{d}{dt}z = a(z,t,v), \\ \dfrac{d}{dt}v_l = h_l(z,t,v), \end{cases}$$

for $l = 1, 2, \dots, K$ with initial condition

$$z(t_0) = y_0, \quad v(t_0) = \bar{f}(y_0)$$

where (t_0, y_0) and $\bar{f}(y_0)$ corresponds to the boundary conditions of PDE.

We assume that for each fixed (x,t), there is an unique initial condition (t_0, y_0) such that $z(t) = x$, entailing that our quasi linear PDE has an unique global solution $u_l(x,t) = v_l(z(t),t)$ (see [13]).

We extend $U_l^N(x;t)$ by

$$U_l^N(x;t) = \begin{cases} U_l^N(0;t) & \text{if } x \le 0, \\ U_l^N(1;t) & \text{if } x \ge 1, \end{cases}$$

for notational convenience. We define $V^{N,n}(x;t) = (V_1^{N,n}(x;t), V_2^{N,n}(x;t), \dots V_K^{N,n}(x;t))$ by

$$V_l^{N,n}(x;t) := \frac{1}{2\frac{n}{N}}\int_{x-n/N}^{x+n/N} U_l^N(y;t)\,dy$$

for $1 \ll n \ll N$. By the definition of U^N, we have $|U_l^N(x;t) - U_l^N(y;t)| \le |x-y|$ for all l and t. Therefore we have

$$|V_l^{N,n}(x;t) - U_l^N(x;t)| \le \frac{1}{2\frac{n}{N}} \int_{x-n/N}^{x+n/N} |U_l^N(y;t) - U_l^N(x;t)|\, dy \le \frac{n}{2N}, \tag{3}$$

for all $1 \le l \le K$, $x \in \mathbf{R}$, $t \ge 0$.

We note that

$$\frac{\partial}{\partial x} U_l^N(z(s);s) = N\{U_l^N(\lceil z(s)\rceil; s) - U_l^N(\lfloor z(s)\rfloor; s)\},$$

due to the definition of U^N, especially linear interpolation. We also note that

$$\begin{aligned} &-N\{U_l^N(\lceil z(s)\rceil; s) - U_l^N(\lfloor z(s)\rfloor; s)\} \\ &= \sum_{j=1}^N \mathbf{1}(w_j = \tilde{w}_l)\mathbf{1}(Y_j^N(s) = \lfloor z(s)\rfloor) \\ &= \mathbf{1}(l\text{-th particle is located between } z(t)-1 \text{ and } z(t)). \end{aligned} \tag{4}$$

Increment of $V_l^{N,n}(z(t);t)$ is divided into following five factors; (i) due to increment of $z(t)$ and increment of $V_l^{N,n}(z(t);t)$ is $\frac{dz}{dt}\frac{\partial}{\partial x}V_l^{N,n}(z(t);t)\,dt$, (ii) due to jump of a particle located behind $z(t)+\frac{x}{N}$ (in the sense of normalized position) when one of the l-th type particle is located between $z(t)+\frac{x-1}{N}$ and $z(t)+\frac{x}{N}$ for $-n \le x \le n$ and increment of $V_l^{N,n}(z(t);t)$ is $\frac{1}{2nN}$, (iii) the error of (ii) which is caused by the linear interpolation at the edge of mollifier and the order of the error is $O\left(\frac{1}{nN}\right)$, (iv) due to jump of l-th type particle located behind $z(t)+\frac{n}{N}$ and increment of $V_l^{N,n}(z(t);t)$ is $-\frac{1}{N}$, (v) due to jump of l-th type particle located behind $z(t)+\frac{x}{N}$ for $-n \le x \le n-1$ and increment of $V_l^{N,n}(z(t);t)$ is $-\frac{n+x}{2nN}$. Therefore by using Itô formula, we have

$$\begin{aligned} &V_l^{N,n}(z(t);t) \\ &\quad = V_l^{N,n}(y_0;t_0) + M_l^{N,n}(t) + O\left(\frac{1}{n}\right) + \int_{t_0}^t \frac{dz}{dt}(s-)\frac{\partial}{\partial x}V_l^{N,n}(z(s-);s-)\,ds \\ &\qquad - \int_{t_0}^t \frac{1}{2nN}\sum_{x=-n}^n N\left\{U_l^N\left(\lfloor z(s-)\rfloor + \frac{x}{N}; s-\right) - U_l^N\left(\lceil z(s-)\rceil + \frac{x}{N}; s-\right)\right\} \end{aligned}$$

$$\times \sum_{j=1}^{N} \mathbf{1}\left(Y_j^N(s-) > z(s-) + \frac{x}{N} \right) w_j(s-)\, ds$$

$$- \int_{t_0}^{t} \frac{1}{N} \sum_{j=1}^{N} \mathbf{1}(w_j = \tilde{w}_l) \mathbf{1}\left(Y_j^N(s-) > z(s-) + \frac{n}{N} \right) w_j(s-)\, ds$$

$$- \int_{t_0}^{t} \frac{1}{N} \sum_{j=1}^{N} \sum_{x=-n}^{n} \frac{n+x}{2n} \mathbf{1}(w_j = \tilde{w}_l) \mathbf{1}\left(Y_j^N(s-) = \lfloor z(s-) \rfloor + \frac{x}{N} \right) w_j(s-)\, ds,$$

where $M_l^{N,n}$ is martingale term defined by

$$M_l^{N,n}(t) := -\int_{t_0}^{t} \frac{1}{2nN} \sum_{x=-n}^{n} N \left\{ U_l^N\left(\lfloor z(s) \rfloor + \frac{x}{N}; s \right) - U_l^N\left(\lceil z(s) \rceil + \frac{x}{N}; s \right) \right\}$$

$$\times \sum_{j=1}^{N} \mathbf{1}\left(Y_j^N(s-) > z(s-) + \frac{x}{N} \right) \tilde{v}_j(ds)$$

$$- \int_{t_0}^{t} \frac{1}{N} \sum_{j=1}^{N} \mathbf{1}(w_j = \tilde{w}_l) \mathbf{1}\left(Y_j^N(s-) > z(s-) + \frac{n}{N} \right) \tilde{v}_j(ds)$$

$$- \int_{t_0}^{t} \frac{1}{N} \sum_{j=1}^{N} \sum_{x=-n}^{n} \frac{n+x}{2n} \mathbf{1}(w_j = \tilde{w}_l) \mathbf{1}\left(Y_j^N(s-) = \lfloor z(s-) \rfloor + \frac{x}{N} \right) \tilde{v}_j(ds)$$

and $\tilde{v}_j(ds) = v_j(ds) - w_j(s-)\, ds$.

Since $w_j = \sum_{l=1}^{K} \mathbf{1}(w_j = \tilde{w}_l) \tilde{w}_l$ for all j, by using the definition of U^N we have

$$\sum_{j=1}^{N} \frac{1}{N} \mathbf{1}(Y_j^N(s-) > x) w_j(s-) = \sum_{l=1}^{K} \sum_{j=1}^{N} \mathbf{1}(w_j = \tilde{w}_l) \frac{1}{N} \mathbf{1}(Y_j^N(s-) > x) \tilde{w}_l(s-)$$

$$= \sum_{l=1}^{K} U_l^N(\lceil x \rceil; s) \tilde{w}_l(s-).$$

By using this identity and the definition of $V_l^{N,n}$, we have

$$V_l^{N,n}(z(t); t)$$

$$= V_l^{N,n}(y_0; t_0) + M_l^{N,n}(t) + O\left(\frac{1}{n} \right) + \int_{t_0}^{t} \frac{dz}{dt}(s-) \frac{\partial}{\partial x} V_l^{N,n}(z(s-); s-)\, ds$$

$$- \int_{t_0}^{t} \frac{1}{2n} \sum_{x=-n}^{n} N \left\{ U_l^N\left(\lfloor z(s-) \rfloor + \frac{x}{N}; s- \right) - U_l^N\left(\lceil z(s-) \rceil + \frac{x}{N}; s- \right) \right\}$$

$$\times \sum_{m=1}^{K} U_m^N\left(\lceil z(s-)\rceil + \frac{x}{N}; s-\right)\tilde{w}_m(s-)\, ds$$
$$- \int_{t_0}^{t} V_l^{N,n}(z(s-); s-)\tilde{w}_l(s-)\, ds$$

We recall that

$$\frac{\partial}{\partial x} U_l^N(z(s); s) = N\{U_l^N(\lceil z(s)\rceil; s) - U_l^N(\lfloor z(s)\rfloor; s)\}.$$

Hence we have

$$\frac{\partial}{\partial x} V_l^{N,n}(z(s); s) = \frac{1}{2n}\sum_{x=-n}^{n} N\left\{U_l^N\left(\lceil z(s)\rceil + \frac{x}{N}; s\right) - U_l^N\left(\lfloor z(s)\rfloor + \frac{x}{N}; s\right)\right\}.$$

By the definition of z, we have

$$\frac{d}{dt} z(t) = a_l(z, t, v) = \sum_{m=1}^{K} v_m(z(t), t)\tilde{w}_m(t).$$

We also recall that

$$|V_l^{N,n}(x; t) - U_l^N(x; t)| \le \frac{n}{2N}.$$

By using these two identities and inequality, we have

$$\begin{aligned}
&V_l^{N,n}(z(t); t)\\
&\quad = V_l^{N,n}(y_0; t_0) + M_l^{N,n}(t) + O\left(\frac{1}{n}\right)\\
&\qquad + \int_{t_0}^{t} \frac{1}{2n}\sum_{x=-n}^{n} N\left\{U_l^N\left(\lceil z(s-)\rceil + \frac{x}{N}; s-\right) - U_l^N\left(\lfloor z(s-)\rfloor + \frac{x}{N}; s-\right)\right\}\\
&\qquad \times \left\{\sum_{m=1}^{K} v_m(z(s-), s-)\tilde{w}_m(s-) - \sum_{m=1}^{K} V_m^{N,n}(z(s-); s-)\tilde{w}_m(s-) + O\left(\frac{n}{N}\right)\right\} ds\\
&\qquad - \int_{t_0}^{t} V_l^{N,n}(z(s-); s-)\tilde{w}_l(s-)\, ds.
\end{aligned}$$

We set $W_l^N(x, t) := \sup_{0\le s\le t}|U_l^N(x, s) - v_l(x, s)|$. By using (3), (4), we have

$$W_l^N(z(t),t) \le W_l^N(y_0;t_0) + |M_l^{N,n}(t)| + O\left(\frac{1}{n}\right) + O\left(\frac{n}{N}\right)$$
$$+ \int_{t_0}^t \sum_{m=1}^K W_m^N(z(s-);s-)\tilde{w}_m(s-)\,ds$$
$$+ \int_{t_0}^t W_l^N(z(s-);s-)\tilde{w}_l(s-)\,ds,$$

for $1 \ll n \ll N$.

By using (4), it is standard to estimate the martingale term $M_l^{N,n}(t)$ by

$$E\left[\sup_{t\in[t_0,T]} M_l^{N,n}(t)^2\right]$$
$$\le 4E[\langle M_l^{N,n}\rangle_T] = \frac{4}{N}\int_{t_0}^T E\left[\frac{1}{N}\sum_{i=1}^N\left\{\frac{1}{2n}\sum_{x=-n}^n \mathbf{1}\left(Y_i^N(s-) > z(s-) + \frac{x}{N}\right)\right.\right.$$
$$\times \sum_{j=1}^N \mathbf{1}(w_j = \tilde{w}_l)\mathbf{1}\left(Y_k^N = \lfloor z(s-)\rfloor + \frac{x}{N}\right)$$
$$- \mathbf{1}\left(Y_i^N(s-) \ge z(s-) + \frac{n}{N}\right)\mathbf{1}(w_i = \tilde{w}_l)$$
$$\left.\left. - \sum_{x=-n}^n \frac{n+x}{2n}\mathbf{1}\left(Y_i^N(s-) \ge z(s-) + \frac{x}{N}\right)\mathbf{1}(w_i = \tilde{w}_l)\right\}^2 w_i(s)\right]ds.$$

It is easy to see that the absolute value of the expression in the large braces is at most 1. Hence we have

$$E\left[\sup_{t\in[t_0,T]} M_l^{N,n}(t)^2\right] \le \frac{4}{N}\frac{1}{N}\sum_{i=1}^N\sqrt{\int_0^T w_i(s)^2\,ds} \le \frac{C}{N},$$

for some constant C.

We conclude that

$$\sum_{l=1}^K E[W_l^N(z(T),T)^2] \le 6\sum_{l=1}^K E[W_l^N(y_0,t_0)^2] + \frac{6KC}{N} + O\left(\frac{1}{n}\right) + O\left(\frac{n}{N}\right)$$
$$+ \int_{t_0}^T 6(K+1)\sum_{l=1}^K E[W_l^N(z(s),s)^2]\tilde{w}_l(s)\,ds,$$

for $1 \ll n \ll N$. By using Gronwall's inequality, we have

$$\sum_{l=1}^{K} E[W_l^N(z(t),t)^2]$$
$$\leq 6\left\{\sum_{l=1}^{K} E[W_l^N(y_0,t_0)^2] + \frac{KC}{N} + O\left(\frac{1}{n}\right) + O\left(\frac{n}{N}\right)\right\}(1 + C'Te^{C'T})$$

for some constant C' and $1 \ll n \ll N$. Note that C' is independent of (y_0, t_0). We also note that z depends on the boundary condition (y_0, t_0). Since $U_l^N(y_0; t_0) \to v_l(y_0, t_0)$ uniformly in (y_0, t_0), we have $U_l^N(z(t); t) \to u_l(z(t), t)$ in L^2 uniformly in $t \in [t_0, T]$ and uniformly in (y_0, t_0). If we take $A = ([0,1] \cap \mathbf{Q} \times \{0\}) \cup (\{0\} \times [0,T] \cap \mathbf{Q})$, which is a countable subset of the boundary of $[0,1] \times [0,T]$, then it is easy to see that $B = \{(z(t), t); t \in [t_0, T], (y_0, t_0) \in A\}$ is a dense subset of $[0,1] \times [0,T]$. By taking subsequence, we have $U_l^N(x; t) \to u_l(x, t)$ uniformly in $(x, t) \in B$ a.s.. Since U_l^N and u_l are non increasing and continuous functions of x, we have $U_l^N(x; t) \to u_l(x, t)$ uniformly in $t \in [0, T]$ and $x \in [0, 1]$ a.s.. $\square$

2.2. Proof of the Theorem 1.2

It is easy to see that Y_i^N has an expression

$$Y_i^N(t) = y_i^N + \sum_{j=1}^{N} \int_0^t \frac{1}{N} \mathbf{1}(Y_j^N(s-) > Y_i^N(s-)) w_j(s-)\, ds$$
$$- \int_0^t Y_i^N(s-) v_i(ds) + M_i^N(t)$$

where $y_i^N = \dfrac{x_i^N - 1}{N}$, $M_i^N(t) = \sum_{j=1}^N \int_0^t \dfrac{1}{N} \mathbf{1}(Y_j^N(s-) > Y_i^N(s-)) \tilde{v}_j(ds)$ and $\tilde{v}_j(ds) = v_j(ds) - w_j(s-)\, ds$. Since $w_j = \sum_{l=1}^K \mathbf{1}(w_j = \tilde{w}_l)\tilde{w}_l$ for all j, by the definition of U^N, we have

$$\sum_{j=1}^{N} \frac{1}{N} \mathbf{1}(Y_j^N(s-) > Y_i^N(s-)) w_j(s-)$$
$$= \sum_{l=1}^{K} \sum_{j=1}^{N} \mathbf{1}(w_j = \tilde{w}_l) \frac{1}{N} \mathbf{1}(Y_j^N(s-) > Y_i^N(s-)) \tilde{w}_l(s-)$$
$$= \sum_{l=1}^{K} U_l^N(\lceil Y_i^N(s-) \rceil; s) \tilde{w}_l(s-).$$

Therefore we have

$$Y_i^N(t) = y_i^N + \sum_{l=1}^{K} \int_0^t U_l^N(\lceil Y_i^N(s-) \rceil; s-)\tilde{w}_l(s-)\, ds$$
$$+ \int_0^t Y_i^N(s-)v_i(ds) + M_i^N(t).$$

We define $Z_i^N(t) := Y_i^N(t) - Y_i(t)$, then we have

$$Z_i^N(t) = (y_i^N - y_i) + \sum_{l=1}^{K} \int_0^t \{U_l^N(\lceil Y_i^N(s-) \rceil; s-) - u_l(Y_i(s-), s-)\}\tilde{w}_l(s-)\, ds$$
$$- \int_0^t Z_i^N(s-)w_i(s-)\, ds + M_i^N(t) - \int_0^t Z_i^N(s-)\tilde{v}_i(ds)$$

Since $U_l^N(x; s) \to u_l(x, s)$ uniformly in $x \in [0, 1]$, $s \in [0, T]$ a.s. and $u_l \in C^1$,

$$\begin{aligned} &|U_l^N(\lceil Y_i^N(s-) \rceil; s-) - u_l(Y_i(s-), s-)| \\ &\quad = |u_l(Y_i^N(s-), s-) - u_l(Y_i(s-), s-) + o(1)| \\ &\quad \le C|Z_i^N(s-)| + o(1), \end{aligned}$$

where constant C is given by

$$C = \sup_{x \in [0,1], t \in [0,T], l \in \{1,2,\dots,K\}} \left| \frac{\partial}{\partial x} u_l(x, t) \right|,$$

and $o(1) \to 0$ as $N \to \infty$. It is standard to see that $\int_0^t Z_i^N(s)\tilde{v}_i(ds)$ and $M_i^N(t)$ are martingales and

$$E\left[\left(\int_0^t Z_i^N(s)\tilde{v}_i(ds)\right)^2\right] = \int_0^t E[Z_i^N(s)^2] w_i(s)\, ds,$$
$$E[M^N(t)^2] = E\left[\frac{1}{N^2} \sum_{j=1}^{N} \int_0^t \mathbf{1}(Y_j^N(s-) > Y_i^N(s-)) w_j(s-)\, ds\right]$$
$$\le \frac{1}{N} \int_0^t \frac{1}{N} \sum_{j=1}^{N} w_j(s-)\, ds.$$

Since $y_i^N \to y_i$, by using Cauchy-Schwarz inequality we have

$$E\left[\sup_{t \in [0,T]} Z_i^N(t)^2\right] \le o(1) + (K+4)C^2 \int_0^T E[Z_i^N(s-)^2]\, ds \int_0^T \left(\sum_{l=1}^{K} \tilde{w}_l(s)\, ds\right)$$
$$+ (K+4) \int_0^T E[Z_i^N(s-)^2]\, ds \int_0^T w_i(s-)^2\, ds$$

$$
\begin{aligned}
&+4(K+4)\frac{1}{N}\int_0^T \frac{1}{N}\sum_{j=1}^N w_j(s-)\,ds \\
&+4(K+4)\int_0^T E[Z_i^N(s-)^2]w_i(s-)\,ds \\
\le\ & o(1)+C'\int_0^T E\left[\sup_{u\in[0,s]} Z_i^N(u)^2\right] ds.
\end{aligned}
$$

for some constant C'. By using Gronwall's inequality, we have

$$
E\left[\sup_{t\in[0,T]} Z_i^N(t)^2\right] \le o(1)(1+C'Te^{C'T}),
$$

i.e., $Y_i^N \to Y_i$ in L^2 and uniformly in $t\in[0,T]$. By taking subsequence, $Y_i^N \to Y_i$ uniformly in $t\in[0,T]$ a.s.. □

Acknowledgment. The author would like to thank Professor K. Uchiyama for helping him with valuable suggestions. The author also would like to thank the anonymous referee for his/her careful reading of the paper.

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