B. LU AND Z. LIU
KODAI MATH. J.
36 (2013), 343–362

RELATIVE INJECTIVITY AND FLATNESS OF COMPLEXES

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Abstract

A complex C is said to be FR-injective (resp., FR-flat) if $\underline{\mathrm{Ext}}^1(D, C) = 0$ (resp., $\overline{\mathrm{Tor}}_1(C, D) = 0$) for any finitely represented complex D. We prove that a complex C is FR-injective (resp., FR-flat) if and only if C is exact and $\mathrm{Z}_m(C)$ is FR-injective (resp., FR-flat) in R-Mod for all $m \in \mathbf{Z}$. We show that the class of FR-injective complexes is closed under direct limits and the class of FR-flat complexes is closed under direct products over any ring R. We use this result to prove that every complex have FR-flat preenvelopes and FR-injective covers.

1. Introduction

The homological theory of complexes of modules has been studied by many authors such as Avramov, Enochs, Foxby, García Rozas, Goddard, Jenda, Oyonarte and Xu (see [2, 5–7, 10, 11, 13]). As we know, the concepts of direct products, direct sums and direct limits play important roles in the investigations of the category of complexes of modules. For example, if $\mathcal{C}$ is a finitely accessible category and $\mathcal{A}$ is a class of objects of $\mathcal{C}$ closed under direct limits and pure epimorphic images, then $\mathcal{A}$ is covering; if $\mathcal{C}$ is a finitely accessible additive category with products and $\mathcal{A}$ is a class of objects of $\mathcal{C}$ closed under products and pure subobjects, then $\mathcal{A}$ is a preenveloping class [3]. So it is an important question to investigate the closure of the class of some complexes (such as injective complexes, flat complexes) under direct products, direct sums and direct limits.

Injective and flat complexes play important roles in the studies of the category of complexes. It is well known that the direct sum of any family of injective complexes of left R-modules is injective if and only if R is left Noetherian; the direct product of any family of flat complexes of right R-

Mathematics Subject Classification. 16E05; 18G35; 18G15.

Key words and phrases. n-Presented complex; Finitely represented complex; FR-injective complex; FR-flat complex; Cotorsion theory; Preenvelope; (Pre)Cover.

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This research was supported by National Natural Science Foundation of China (No. 11001222, 10961021).

Received May 25, 2012; revised January 17, 2013.

modules is flat if and only if R is left coherent. Thus, the class of injective complexes (resp. flat complexes) is not closed under direct sums (resp. direct products) in general.

Our main purpose in this paper is to introduce and investigate a generalization of injective complex (resp. flat complex) of modules which is closed under direct limits (resp. direct products) over any ring. We call it FR-injective complex (resp. FR-flat complex). We show that a complex C is FR-injective (resp., FR-flat) if and only if C is exact and $Z_m(C)$ is FR-injective (resp., FR-flat) in R-Mod for all $m \in \mathbf{Z}$. We prove that (1) a complex C of right R-modules is FR-flat if and only if C^+ is FR-injective; (2) a complex C of left R-modules is FR-injective if and only if C^+ is FR-flat. We also show that a ring R is left Noetherian if and only if any FR-injective complex of left R-modules is injective and R is left coherent if and only if any FR-flat complex of right R-modules is flat.

The existence of (pre)envelopes and (pre)covers, not just in the setting of the categories of modules but for more general abelian categories, such as the category of complexes of modules which is one of the important categories where this problem could be studied. In this paper, we show that (1) every complex of R-modules has an FR-injective (resp. FR-flat) preenvelope; (2) every complex of R-modules has an FR-flat (resp. FR-injective) cover.

2. Preliminaries

Throughout this paper, R denotes a ring with unity, R-Mod denotes the category of R-modules and $\mathscr{C}(R)$ denotes the abelian category of complexes of R-modules. A complex

$$\cdots \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} C_{-1} \xrightarrow{\delta_{-1}} \cdots$$

of R-modules will be denoted by (C, δ) or C.

We will use subscripts to distinguish complexes. So if $\{C^i\}_{i \in I}$ is a family of complexes, C^i will be

$$\cdots \xrightarrow{\delta_2} C_1^i \xrightarrow{\delta_1} C_0^i \xrightarrow{\delta_0} C_{-1}^i \xrightarrow{\delta_{-1}} \cdots.$$

Given a left R-module M, we use the notation $D^m(M)$ to denote the complex

$$\cdots \to 0 \to M \xrightarrow{id} M \to 0 \to \cdots$$

with M in the mth and $(m-1)$th positions and set $\overline{M} = D^0(M)$. We also use the notation $S^m(M)$ to denote the complex with M in the mth place and 0 in the other places and set $\underline{M} = S^0(M)$.

Given a complex C and an integer m, $\sum^m C$ denotes the complex such that $(\sum^m C)_l = C_{l-m}$, and whose boundary operators are $(-1)^m \delta_{l-m}$. The lth homology module of C is the module $\mathrm{H}_l(C) = \mathrm{Z}_l(C)/\mathrm{B}_l(C)$ where $\mathrm{Z}_l(C) = \mathrm{Ker}(\delta_l^C)$ and $\mathrm{B}_l(C) = \mathrm{Im}(\delta_{l+1}^C)$. We set $\mathrm{H}^l(C) = \mathrm{H}_{-l}(C)$.

Let C and D be complexes of left R-modules. We will use $\mathrm{Hom}^{\cdot}(C,D)$ to denote the usual homomorphism complex of C and D, and let $\underline{\mathrm{Hom}}(C,D) = \mathrm{Z}(\mathrm{Hom}^{\cdot}(C,D))$. Then $\underline{\mathrm{Hom}}(C,D)$ can be made into a complex with $\mathrm{Hom}(C,D)_m$ the abelian group of morphisms from C to $\Sigma^{-m}D$ and with boundary operator given by $f \in \mathrm{Hom}(C,D)_m$, then $\delta_m(f) : C \to \Sigma^{-(m-1)}D$ with $\delta_m(f)_l = (-1)^m \delta^D f_l$ for any $l \in \mathbf{Z}$. And we put $C^+ = \underline{\mathrm{Hom}}(C, \overline{\mathbf{Q}/\mathbf{Z}})$. We note that the new functor $\underline{\mathrm{Hom}}(C,D)$ will have right derived functors whose values will be complexes. These values will be denoted $\underline{\mathrm{Ext}}^i(C,D)$. It is easy to see that $\underline{\mathrm{Ext}}^i(C,D)$ is the complex

$$\cdots \to \mathrm{Ext}^i(C, \Sigma^{-(m+1)}D) \to \mathrm{Ext}^i(C, \Sigma^{-m}D) \to \mathrm{Ext}^i(C, \Sigma^{-(m-1)}D) \to \cdots$$

with boundary operator induced by the boundary operator of D.

Let C be a complex of right R-modules and D be a complex of left R-modules, $C \otimes^{\cdot} D$ denotes the usual tensor product of C and D. We define $C \overline{\otimes} D$ to be $\dfrac{(C \otimes^{\cdot} D)}{\mathrm{B}(C \otimes^{\cdot} D)}$ with the maps

$$\frac{(C \otimes^{\cdot} D)_m}{\mathrm{B}_m(C \otimes^{\cdot} D)} \to \frac{(C \otimes^{\cdot} D)_{m-1}}{\mathrm{B}_{m-1}(C \otimes^{\cdot} D)}, \quad x \otimes y \mapsto \delta^C(x) \otimes y,$$

where $x \otimes y$ is used to denote the coset in $\dfrac{(C \otimes^{\cdot} D)_m}{\mathrm{B}_m(C \otimes^{\cdot} D)}$. Then we get a complex.

Given a complex C of left R-modules. Then we have two functors $- \overline{\otimes} C : \mathscr{C}_R \to \mathscr{C}_{\mathbf{Z}}$ and $\underline{\mathrm{Hom}}(C,-) : {}_R\mathscr{C} \to \mathscr{C}_{\mathbf{Z}}$, where $\mathscr{C}_R$ (resp., ${}_R\mathscr{C}$) denotes the category of complexes of right R-modules (resp., left R-modules). Since $- \overline{\otimes} C : \mathscr{C}_R \to \mathscr{C}_{\mathbf{Z}}$ is a right exact functor, we can construct left derived functors, which we denote by $\overline{\mathrm{Tor}}_1(-,C)$.

General background materials can be found in [5] or [11].

Let $\mathcal{C}$ be an abelian category with enough projectives and injectives. Given a class $\mathcal{F}$ of objects of $\mathcal{C}$, write $\mathcal{F}^{\perp} = \{C \in Ob(\mathcal{C}) \mid \mathrm{Ext}^1(F,C) = 0 \text{ for all } F \in \mathcal{F}\}$ and ${}^{\perp}\mathcal{F} = \{C \in Ob(\mathcal{C}) \mid \mathrm{Ext}^1(C,F) = 0 \text{ for all } F \in \mathcal{F}\}$. A pair $(\mathcal{A},\mathcal{B})$ of classes of objects of $\mathcal{C}$ is called a cotorsion pair (cotorsion theory) [7] if $\mathcal{A}^{\perp} = \mathcal{B}$ and ${}^{\perp}\mathcal{B} = \mathcal{A}$. Two simple examples of cotorsion pairs in the category of R-modules are ($\mathcal{P}roj$, R-Mod) and (R-Mod, $\mathcal{I}nj$), where $\mathcal{P}roj$ ($\mathcal{I}nj$) is the class of projective (injective) R-modules. A cotorsion pair $(\mathcal{A},\mathcal{B})$ is said to be hereditary [7] if the following equivalent conditions are hold:

(1) If $0 \to L' \to L \to L'' \to 0$ is exact with $L', L \in \mathcal{B}$, then L'' is also in $\mathcal{B}$;

(2) If $0 \to L' \to L \to L'' \to 0$ is exact with $L'', L \in \mathcal{A}$, then L' is also in $\mathcal{A}$;

(3) If $\mathrm{Ext}^i(A,B) = 0$ for all $i \geq 1$ and all $A \in \mathcal{A}$, $B \in \mathcal{B}$.

Given a class $\mathcal{F}$ of objects of $\mathcal{C}$. Following [9], a morphism $\phi : X \to F$ of $\mathcal{C}$ is called an $\mathcal{F}$-preenvelope of X if $F \in \mathcal{F}$ and $\mathrm{Hom}(F,F') \to \mathrm{Hom}(X,F') \to 0$ is exact for all $F' \in \mathcal{F}$. If, moreover, any $f : F \to F$ such that $f\phi = \phi$ is an automorphism of F then $\phi : X \to F$ is called an $\mathcal{F}$-envelope of X. An $\mathcal{F}$-precover and an $\mathcal{F}$-cover of X are defined dually. It is immediate that envelopes and covers, if they exist, are unique up to isomorphism, and that if $\mathcal{F}$ contains all injective (projective) objects, then $\mathcal{F}$-(pre)envelopes ($\mathcal{F}$-(pre)covers) are always injective (surjective). We say a class $\mathcal{F}$ of objects of $\mathcal{C}$ is (pre)enveloping if every object of $\mathcal{C}$ has an $\mathcal{F}$-(pre)envelope. Dually, we have the concept of a (pre)covering class.

3. *n*-Presented complexes and some isomorphisms

In this section, we first introduce and study the concept of *n*-presented complexes. Moreover, some isomorphisms are established which will be used to prove the main results of this paper.

Definition 3.1 ([11, Definition 4.1.1]). A complex C is called finitely generated if, in the case where we can write $C = \sum_{i \in I} D^i$ with $D^i \in \mathscr{C}(R)$ subcomplexes of C, there exists a finite subset $J \subseteq I$ such that $C = \sum_{i \in J} D^i$.

A complex C is called finitely presented if C is finitely generated and for every exact sequence of complexes $0 \to K \to L \to C \to 0$ with L finitely generated, K is also finitely generated.

Lemma 3.2 ([11, Lemma 4.1.1]). *A complex C is finitely generated if and only if C is bounded and C_m is finitely generated in R-*Mod *for all $m \in \mathbf{Z}$.*

*A complex C is finitely presented if and only if C is bounded and C_m is finitely presented in R-*Mod *for all $m \in \mathbf{Z}$.*

It is clear that we have the following results:

Lemma 3.3. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of complexes. Then the following statements hold*:

(1) *If A is finitely generated and B is finitely presented, then C is finitely presented*;

(2) *If A and C are finitely presented, then so is B*;

(3) *If R is left coherent ring, and B, C are finitely presented, then so is A.*

Recall that a complex P is said to be projective if P is exact and $\mathrm{Z}_i(P)$ is projective in R-Mod for all $i \in \mathbf{Z}$.

Lemma 3.4. *Let C be a complex. Then the following statements are equivalent*:

(1) *C is finitely presented*;

(2) *There exists an exact sequence $0 \to L \to P \to C \to 0$ of complexes, where P is finitely generated projective, and L is finitely generated*;

(3) *There exists an exact sequence $P^1 \to P^0 \to C \to 0$ of complexes, where P^0, P^1 are finitely generated projective, and P^0_m, P^1_m are free for all $m \in \mathbf{Z}$.*

An R-module M is called n-presented if it has a finite n-presentation, i.e., there is an exact sequence

$$F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

in which each F_i is finitely generated free.

Now, we extend the notion of n-presented modules to that of complexes and characterize such complexes.

Definition 3.5. Let $n \geq 0$ be an integer. A complex C is said to be n-presented if there is an exact sequence $P^n \to P^{n-1} \to \cdots \to P^1 \to P^0 \to C \to 0$ of complexes, where P^i is finitely generated projective, and P^i_m is free for $i = 0, 1, \ldots, n$ and all $m \in \mathbf{Z}$.

Remark 3.6. (1) A complex C is n-presented if and only if C is bounded and C_m is n-presented in R-Mod for all $m \in \mathbf{Z}$;

(2) A complex C is n-presented if and only if there is an exact sequence of complexes

$$0 \to K^n \to P^{n-1} \to \cdots \to P^1 \to P^0 \to C \to 0$$

where P^i is finitely generated projective, P^i_m is free for $i = 0, 1, \ldots, n-1$ and all $m \in \mathbf{Z}$, K^n is finitely generated;

(3) A complex C is n-presented ($n \geq 1$) if and only if there is an exact sequence of complexes

$$0 \to K \to P \to C \to 0,$$

where K is $(n-1)$-presented and P is finitely generated projective.

Theorem 3.7. *Let $n \geq 1$ be an integer and $0 \to K \to P \to C \to 0$ an exact sequence of complexes. Then*

(1) *If P is n-presented and K is $(n-1)$-presented, then C is n-presented;*

(2) *If K and C are n-presented, then so is P;*

(3) *If C is n-presented and P is $(n-1)$-presented, then K is $(n-1)$-presented.*

Proof. It is similar to the proof of [12, Theorem 2.1.2] by Remark 3.6 (1). □

Let I be a set. An R-module M is called I-graded if there exists a family $\{M_i\}_{i \in I}$ of submodules of M such that $M = \bigoplus_{i \in I} M_i$. A $\mathbf{Z}$-graded module is simply called a graded module. General background about graded modules can be found in [4].

Lemma 3.8. *Let $\{C^i\}_{i \in I}$ be a family of complexes, D a finitely generated complex. Then $\underline{\mathrm{Hom}}(D, \bigoplus_{i \in I} C^i) \cong \bigoplus_{i \in I} \underline{\mathrm{Hom}}(D, C^i)$ as complexes.*

Proof. Firstly,

$$\alpha : \bigoplus_{i\in I} \mathrm{Hom}^{\cdot}(D, C^i) \to \mathrm{Hom}^{\cdot}\left(D, \bigoplus_{i\in I} C^i\right)$$

is an isomorphism by $x = (x^i)_{i\in I} \mapsto \sum_{i\in I} \mathrm{Hom}^{\cdot}(D, \varepsilon^i)(x^i) = \sum_{i\in I} \varepsilon^i x^i$, where $x = (x^i)_{i\in I} \in (\bigoplus_{i\in I} \mathrm{Hom}^{\cdot}(D, C^i))_l = \bigoplus_{i\in I}(\mathrm{Hom}^{\cdot}(D, C^i))_l$ with $x^i \in \mathrm{Hom}^{\cdot}(D, C^i)_l$ and $\varepsilon^j : C^j \mapsto \bigoplus_{i\in I} C^i$ is the natural embedding (see [4, Proposition 2.5.16]).

Secondly, we will show that $\underline{\mathrm{Hom}}(D, \bigoplus_{i\in I} C^i) \cong \bigoplus_{i\in I} \underline{\mathrm{Hom}}(D, C^i)$. We define a morphism

$$\gamma = \alpha|_{\bigoplus_{i\in I} \underline{\mathrm{Hom}}(D, C^i)} : \bigoplus_{i\in I} \underline{\mathrm{Hom}}(D, C^i) \to \underline{\mathrm{Hom}}\left(D, \bigoplus_{i\in I} C^i\right).$$

We claim that γ is a graded isomorphism of graded modules with degree 0. We first show that $\underline{\mathrm{Hom}}(D, \bigoplus_{i\in I} C^i) \cong \bigoplus_{i\in I} \underline{\mathrm{Hom}}(D, C^i)$ as graded modules. Note that $\underline{\mathrm{Hom}}(D, \bigoplus_{i\in I} C^i) = \mathrm{Z}(\mathrm{Hom}^{\cdot}(D, \bigoplus_{i\in I} C^i)) \cong \mathrm{Z}(\bigoplus_{i\in I} \mathrm{Hom}^{\cdot}(D, C^i)) \cong \bigoplus_{i\in I} \mathrm{Z}(\mathrm{Hom}^{\cdot}(D, C^i)) = \bigoplus_{i\in I} \underline{\mathrm{Hom}}(D, C^i)$ since D is finitely generated. And $\gamma = \alpha|_{\bigoplus_{i\in I} \underline{\mathrm{Hom}}(D, C^i)}$ is a monomorphism since α is a isomorphism. So γ is a graded isomorphism of graded modules with degree 0.

On the other hand, for any $(x^i)_{i\in I} \in (\bigoplus_{i\in I} \underline{\mathrm{Hom}}(D, C^i))_l$,

$$\begin{aligned}
\gamma\delta^{\bigoplus_{i\in I} \underline{\mathrm{Hom}}(D, C^i)}(x^i)_{i\in I} &= \alpha\delta^{\bigoplus_{i\in I} \underline{\mathrm{Hom}}(D, C^i)}(x^i)_{i\in I} = \alpha(\delta^{\underline{\mathrm{Hom}}(D, C^i)}(x^i))_{i\in I} \\
&= \sum_{i\in I} \mathrm{Hom}(D, \varepsilon^i)\delta^{\underline{\mathrm{Hom}}(D, C^i)}(x^i) \\
&= \sum_{i\in I} \varepsilon^i(-1)^l\delta^{C^i}(x^i) = (-1)^l \sum_{i\in I} \varepsilon^i\delta^{C^i}(x^i),
\end{aligned}$$

and

$$\begin{aligned}
\delta^{\underline{\mathrm{Hom}}(D, \oplus_{i\in I} C^i)}\gamma(x^i)_{i\in I} &= \delta^{\underline{\mathrm{Hom}}(D, \oplus_{i\in I} C^i)}\alpha(x^i)_{i\in I} = \delta^{\underline{\mathrm{Hom}}(D, \oplus_{i\in I} C^i)}\left(\sum_{i\in I} \varepsilon^i x^i\right) \\
&= (-1)^l\delta^{\oplus_{i\in I} C^i}\left(\sum_{i\in I} \varepsilon^i x^i\right) = (-1)^l \sum_{i\in I} \delta^{\oplus_{i\in I} C^i}\varepsilon^i x^i \\
&= (-1)^l \sum_{i\in I} \varepsilon^i\delta^{C^i}(x^i).
\end{aligned}$$

Thus γ is an isomorphism of complexes, and hence $\underline{\mathrm{Hom}}(D, \bigoplus_{i\in I} C^i) \cong \bigoplus_{i\in I} \underline{\mathrm{Hom}}(D, C^i)$. □

LEMMA 3.9. *Let* $\{C^i\}_{i\in I}$ *be a direct system of complexes,* D *a finitely presented complex. Then* $\mathrm{Hom}(D, \varinjlim C^i) \cong \varinjlim \mathrm{Hom}(D, C^i)$.

Proof. It follows from Stenström [15, Chap. V, Proposition 3.4] since $\mathscr{C}(R)$ is locally finitely generated in the sense of [15]. □

LEMMA 3.10. *Let $\{C^i\}_{i\in I}$ be a family of complexes, D a finitely presented complex. Then $D\,\overline{\otimes}\,\prod_{i\in I} C^i \cong \prod_{i\in I}(D\,\overline{\otimes}\,C^i)$ as complexes.*

Proof. Firstly,

$$\alpha : D \otimes^{\cdot} \prod_{i\in I} C^i \to \prod_{i\in I}(D \otimes^{\cdot} C^i)$$

is an isomorphism by $x \mapsto ((D \otimes^{\cdot} \pi^i)(x))_{i\in I}$, where $x = d\otimes c \in (D\otimes^{\cdot} \prod_{i\in I} C^i)_l$ and $\pi^j : \prod_{i\in I} C^i \to C^j$ is the natural projection (see [4, Proposition 2.5.17).

Secondly, we will show that $D\,\overline{\otimes}\,\prod_{i\in I} C^i \cong \prod_{i\in I}(D\,\overline{\otimes}\,C^i)$. Since we have the following commutative diagram:

$$\begin{array}{ccccc}
\left(D\otimes^{\cdot}\prod_{i\in I} C^i\right)_l & \longrightarrow & \dfrac{\left(D\otimes^{\cdot}\prod_{i\in I} C^i\right)_l}{B_l\left(D\otimes^{\cdot}\prod_{i\in I} C^i\right)} & \longrightarrow & 0 \\
{\scriptstyle \alpha_l}\big\downarrow & & {\scriptstyle \beta_l}\big\downarrow & & \\
\left(\prod_{i\in I} D\otimes^{\cdot} C^i\right)_l & \longrightarrow & \dfrac{\left(\prod_{i\in I} D\otimes^{\cdot} C^i\right)_l}{B_l\left(\prod_{i\in I} D\otimes^{\cdot} C^i\right)} & \longrightarrow & 0,
\end{array}$$

where $\beta_l : \dfrac{\left(D\otimes^{\cdot}\prod_{i\in I} C^i\right)_l}{B_l\left(D\otimes^{\cdot}\prod_{i\in I} C^i\right)} \to \dfrac{\left(\prod_{i\in I} D\otimes^{\cdot} C^i\right)_l}{B_l\left(\prod_{i\in I} D\otimes^{\cdot} C^i\right)}$ is given by the assignment

$$d\otimes c + B\left(D\otimes^{\cdot}\prod_{i\in I} C^i\right) \to \alpha(d\otimes c) + B\left(\prod_{i\in I} D\otimes^{\cdot} C^i\right)$$

for any $d\otimes c \in (D\otimes^{\cdot} \prod_{i\in I} C^i)_l$, and let β be a graded homomorphism induced by β_l. Thus β is a graded isomorphism of graded modules with degree 0. Moreover,

$$\begin{aligned}
&\beta\delta^{D\overline{\otimes}\prod_{i\in I}C^i}\left(d\otimes c + B\left(D\otimes^{\cdot}\prod_{i\in I} C^i\right)\right)\\
&\quad = \beta(\delta^D(d)\otimes c) = \alpha(\delta^D(d)\otimes c) = (\delta^D(d)\otimes \pi^i(c))_{i\in I}
\end{aligned}$$

and

$$\delta^{\prod_{i\in I}(D\bar{\otimes}C^i)}\beta\left(d\otimes c+B\left(\prod_{i\in I}D\otimes^{\cdot}C^i\right)\right)$$
$$=\delta^{\prod_{i\in I}(D\bar{\otimes}C^i)}\left(\alpha(d\otimes c)+B\left(\prod_{i\in I}D\otimes^{\cdot}C^i\right)\right)$$
$$=\delta^{\prod_{i\in I}(D\bar{\otimes}C^i)}\alpha(d\otimes c)=(\delta^{D\bar{\otimes}C^i}\alpha(d\otimes c))_{i\in I}=(\delta^D(d)\otimes\pi^i(c))_{i\in I}.$$

Therefore, β is an isomorphism of complexes. □

THEOREM 3.11. *Let $n\geq 1$ be an integer, D an n-presented complex and $(C^i)_{i\in I}$ a direct system of complexes. Then $\mathrm{Ext}^{n-1}(D,\varinjlim C^i)\cong\varinjlim\mathrm{Ext}^{n-1}(D,C^i)$.*

Proof. We do an induction on n. If $n=1$, then the result follows from Lemma 3.9.

Let $n=2$ and D be an 2-presented complex. Then there exists an exact sequence of complexes $0\to L\to P\to D\to 0$ with P finitely generated projective and L finitely presented. Thus there is a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}(P,\varinjlim C^i) & \longrightarrow & \mathrm{Hom}(L,\varinjlim C^i) & \longrightarrow & \mathrm{Ext}^1(D,\varinjlim C^i) & \longrightarrow & 0\\
\cong\downarrow & & \cong\downarrow & & \downarrow & & \\
\varinjlim\mathrm{Hom}(P,C^i) & \longrightarrow & \varinjlim\mathrm{Hom}(L,C^i) & \longrightarrow & \varinjlim\mathrm{Ext}^1(D,C^i) & \longrightarrow & 0.
\end{array}$$

Since $\mathrm{Hom}(P,\varinjlim C^i)\cong\varinjlim\mathrm{Hom}(P,C^i)$ and $\mathrm{Hom}(L,\varinjlim C^i)\cong\varinjlim\mathrm{Hom}(L,C^i)$ by Lemma 3.9, we have $\mathrm{Ext}^1(D,\varinjlim C^i)\cong\varinjlim\mathrm{Ext}^1(D,C^i)$.

If $n>2$, then it follows from the standard homological method. Therefore, $\mathrm{Ext}^{n-1}(D,\varinjlim C^i)\cong\varinjlim\mathrm{Ext}^{n-1}(D,C^i)$. □

THEOREM 3.12. *Let $n\geq 1$ be an integer, D an n-presented complex and $(N^\alpha)_{\alpha\in I}$ a family of complexes. Then $\overline{\mathrm{Tor}}_{n-1}(\prod_{\alpha\in I}N^\alpha,D)\cong\prod_{\alpha\in I}\overline{\mathrm{Tor}}_{n-1}(N^\alpha,D)$.*

Proof. We do an induction on n. If $n=1$, then the result follows from Lemma 3.10.

Let $n=2$ and D be an 2-presented complex. Then there exists an exact sequence of complexes $0\to L\to P\to D\to 0$ with P finitely generated projective and L finitely presented. Thus there is a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \longrightarrow & \overline{\mathrm{Tor}}_1\left(\prod_{\alpha\in I} N^\alpha, D\right) & \longrightarrow & \left(\prod_{\alpha\in I} N^\alpha\right) \overline{\otimes} L & \longrightarrow & \left(\prod_{\alpha\in I} N^\alpha\right) \overline{\otimes} P \\
& \downarrow & & \cong\downarrow & & \cong\downarrow \\
0 \longrightarrow & \prod_{\alpha\in I} \overline{\mathrm{Tor}}_1(N^\alpha, D) & \longrightarrow & \prod_{\alpha\in I}(N^\alpha \overline{\otimes} L) & \longrightarrow & \prod_{\alpha\in I}(N^\alpha \overline{\otimes} P).
\end{array}$$

Since $(\prod_{\alpha\in I} N^\alpha) \overline{\otimes} L \cong \prod_{\alpha\in I}(N^\alpha \overline{\otimes} L)$ and $(\prod_{\alpha\in I} N^\alpha) \overline{\otimes} P \cong \prod_{\alpha\in I}(N^\alpha \overline{\otimes} P)$ by Lemma 3.10, we have $\overline{\mathrm{Tor}}_1(\prod_{\alpha\in I} N^\alpha, D) \cong \prod_{\alpha\in I} \overline{\mathrm{Tor}}_1(N^\alpha, D)$.

If $n > 2$, then it follows from the standard homological method. Therefore, $\overline{\mathrm{Tor}}_{n-1}(\prod_{\alpha\in I} N^\alpha, D) \cong \prod_{\alpha\in I} \overline{\mathrm{Tor}}_{n-1}(N^\alpha, D)$. □

Lemma 3.13 ([11, Lemma 4.2.2]). *Let R and S be rings, L a complex of right S-modules, K a complex of (R, S)-bimodules and P a complex of left R-modules. Suppose that P is finitely presented and L is injective as complexes of right S-modules. Then $\underline{\mathrm{Hom}}(K, L) \overline{\otimes} P \cong \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(P, K), L)$ as complexes. This isomorphism is functorial in P, K and L.*

Theorem 3.14. (1) *Let R and S be rings, n a fixed positive integer, A an n-presented complex of left R-modules, B a complex of (R, S)-bimodules, C an injective complex of right S-modules. Then $\underline{\mathrm{Hom}}(\underline{\mathrm{Ext}}^{n-1}(A, B), C) \cong \overline{\mathrm{Tor}}_{n-1}(\underline{\mathrm{Hom}}(B, C), A)$.*

(2) *Let R and S be rings, n a fixed positive integer, A a complex of left R-modules, B a complex of right (R, S)-bimodules, C an injective complex of right S-modules. Then $\underline{\mathrm{Ext}}^n(A, \underline{\mathrm{Hom}}(B, C) \cong \underline{\mathrm{Hom}}(\overline{\mathrm{Tor}}_n(B, A), C)$.*

Proof. (1) We do an induction on n. If $n = 1$, then the result follows from Lemma 3.13.

Let $n = 2$ and A be an 2-presented complex. Then there exists an exact sequence of complexes $0 \to K \to P \to A \to 0$ with P finitely generated projective and K finitely presented in $\mathscr{C}(R)$. Thus we have the commutative diagram with exact rows by Lemma 3.13:

$$\begin{array}{ccccccc}
0 \longrightarrow & \underline{\mathrm{Hom}}(\underline{\mathrm{Ext}}^1(A, B), C) & \longrightarrow & \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(K, B), C) & \longrightarrow & \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(P, B), C) \\
& \downarrow & & \cong\downarrow & & \cong\downarrow \\
0 \longrightarrow & \overline{\mathrm{Tor}}_1(\underline{\mathrm{Hom}}(B, C), A) & \longrightarrow & \underline{\mathrm{Hom}}(B, C) \overline{\otimes} K & \longrightarrow & \underline{\mathrm{Hom}}(B, C) \overline{\otimes} P.
\end{array}$$

Hence, $\underline{\mathrm{Hom}}(\underline{\mathrm{Ext}}^1(A, B), C) \cong \overline{\mathrm{Tor}}_1(\underline{\mathrm{Hom}}(B, C), A)$.

If $n > 2$, then it follows from the standard homological method. Therefore, $\underline{\mathrm{Hom}}(\underline{\mathrm{Ext}}^{n-1}(A, B), C) \cong \overline{\mathrm{Tor}}_{n-1}(\underline{\mathrm{Hom}}(B, C), A)$.

(2) It follows by similar arguments since $\underline{\mathrm{Hom}}(A \overline{\otimes} B, C) \cong \underline{\mathrm{Hom}}(A, \underline{\mathrm{Hom}}(B, C))$ for any complex A, B and C. □

Remark 3.15. It is not hard to see that

$$\underline{\mathrm{Hom}}\left(D, \prod_{i\in I} C^i\right) \cong \prod_{i\in I} \underline{\mathrm{Hom}}(D, C^i),$$

$$D \mathbin{\overline{\otimes}} \bigoplus_{i\in I} C^i \cong \bigoplus_{i\in I} (D \mathbin{\overline{\otimes}} C^i),$$

$$\underline{\mathrm{Ext}}^n\left(D, \prod_{i\in I} C^i\right) \cong \prod_{i\in I} \underline{\mathrm{Ext}}^n(D, C^i),$$

and

$$\overline{\mathrm{Tor}}_n\left(\bigoplus_{\alpha\in I} C^i, D\right) \cong \bigoplus_{\alpha\in I} \overline{\mathrm{Tor}}_n(C^i, D)$$

$$\overline{\mathrm{Tor}}_n(\varinjlim N^\alpha, D) \cong \varinjlim \overline{\mathrm{Tor}}_n(N^\alpha, D)$$

for a fixed positive integer n, any complex D, any family $\{C^i\}_{i\in I}$ of complexes and direct system $\{N^\alpha\}_{\alpha\in\Lambda}$ by analogy with the proof of the above results.

4. *FR*-injective complexes and *FR*-flat complexes

In the following, we use the word "finitely represented" instead of "2-presented" in R-Mod.

Definition 4.1. We will say that a complex C is finitely represented (i.e. 2-presented) if there is an exact sequence of complexes $0 \to K \to P \to C \to 0$ with P finitely generated projective, K is finitely presented.

Let R be a ring. A left R-module M is said to be *FR*-injective if $\mathrm{Ext}^1(F, M) = 0$ for any finitely represented module F; a right R-module N is called *FR*-flat if $\mathrm{Tor}_1(N, F) = 0$ for any finitely represented module F. These modules have been studied by Ding, Mao and Zhou (see [14, 16]). Note that in [16], *FR*-injective (resp., *FR*-flat) is called $(2,0)$-injective (resp., $(2,0)$-flat).

We now want to define an *FR*-injective (*FR*-flat) complex. For such a definition, we have two options. We could define such a complex by analogy with the definition of *FR*-injective (resp., *FR*-flat) modules (i.e., Definition 4.2), or we could use a modification of the definition of injective complexes (resp., flat complexes). Recall that a complex C is called injective (resp., flat) if C is exact and C_m is injective (resp., flat) for all $m \in \mathbf{Z}$. The following results (Theorem 4.9 and Theorem 4.10) will show that the two definitions are equivalent.

Definition 4.2. (1) A complex C is called *FR*-injective if $\underline{\mathrm{Ext}}^1(D, C) = 0$ for any finitely represented complex D;

(2) A complex C is called FR-flat if $\overline{\mathrm{Tor}}_1(C, D) = 0$ for any finitely represented complex D.

Remark 4.3. (1) It is obvious that injective complexes are FR-injective and flat complexes are FR-flat. However, the converse is not true in general. See Example 4.12 or Theorem 4.11;

(2) It is clear that the class FR-injective complexes is closed under direct products and the class of FR-flat complexes is closed under direct limits by Remak 3.15;

(3) A complex C is FR-injective if and only if $\mathrm{Ext}^1(F, C) = 0$ for any finitely represented complex F.

Proof. Here we only prove (3) of the Remark 4.3. Note that $\underline{\mathrm{Ext}}^1(F, C)$ is the complex

$$\cdots \to \mathrm{Ext}^1(F, \Sigma^{-(m+1)}C) \to \mathrm{Ext}^1(F, \Sigma^{-m}C) \to \mathrm{Ext}^1(F, \Sigma^{-(m-1)}C) \to \cdots$$

and also is the complex

$$\cdots \to \mathrm{Ext}^1(\Sigma^{(m+1)}F, C) \to \mathrm{Ext}^1(\Sigma^{m}F, C) \to \mathrm{Ext}^1(\Sigma^{(m-1)}F, C) \to \cdots.$$

So if $\underline{\mathrm{Ext}}^1(F, C) = 0$, then $\mathrm{Ext}^1(F, C) = 0$. Now we prove the converse. Let $\mathrm{Ext}^1(F, C) = 0$ for any finitely represented complex F. Then $\mathrm{Ext}^1(\Sigma^m F, C) = 0$ since $\Sigma^m F$ is finitely represented for all $m \in \mathbf{Z}$. Therefore $\underline{\mathrm{Ext}}^1(F, C) = 0$. □

Proposition 4.4. *Let R be a ring and $0 \to A \to B \to C \to 0$ an exact sequence of complexes. Then*:

(1) *If A and C are FR-injective, then so is B*;

(2) *If A and C are FR-flat, then so is B.*

Proof. It is obvious by the definitions. □

Proposition 4.5. *Let $\{B^i\}_{i \in I}$ and $\{C^i\}_{i \in I}$ be families of complexes of R-modules. Then*

(1) *$\prod_{i \in I} C^i$ is FR-injective if and only if each C^i is FR-injective*;

(2) *$\bigoplus_{i \in I} B^i$ is FR-flat if and only if each B^i is FR-flat.*

Proof. (1) It follows from the isomorphism $\underline{\mathrm{Ext}}^1(N, \prod_{i \in I} C^i) \cong \prod_{i \in I} \underline{\mathrm{Ext}}^1(N, C^i)$, where N is a complex of R-modules.

(2) It follows from the isomorphism $\overline{\mathrm{Tor}}_1(\bigoplus_{i \in I} B^i, N) \cong \bigoplus_{i \in I} \overline{\mathrm{Tor}}_1(B^i, N)$, where N is a complex of R-modules. □

It is well known that a complex C is flat if and only if C^+ is injective over any ring R, and a ring R is left Noetherian if and only if a complex C of left R-modules being injective is equivalent to that C^+ being flat.

Here we have the similar result over any ring R.

THEOREM 4.6. *Let R be a ring. Then*

(1) *A complex C of right R-modules is FR-flat if and only if C^+ is FR-injective*;

(2) *A complex C of left R-modules is FR-injective if and only if C^+ is FR-flat.*

Proof. (1) It follows from the isomorphism $\underline{\mathrm{Ext}}^1(D, C^+) \cong \overline{\mathrm{Tor}}_1(C, D)^+$ for any complex D.

(2) Note that $\overline{\mathrm{Tor}}_1(C^+, D) \cong \underline{\mathrm{Ext}}^1(D, C^+)$ for any finitely represented complex C by Theorem 3.14. Hence C is FR-injective if and only if C^+ is FR-flat. □

PROPOSITION 4.7. *Let C be a complex of left R-modules. Then C is FR-injective if and only if C_m is FR-injective in R-*Mod *for all $m \in \mathbf{Z}$ and* $\mathrm{Hom}^{\cdot}(F, C)$ *is exact for any finitely represented complex F.*

Proof. ($\Rightarrow$) Suppose that (C, δ) is FR-injective and let

$$0 \to C_m \xrightarrow{\alpha} X \to G \to 0 \tag{1}$$

be an extension in R-Mod, where G is a finitely represented module. We will show that the sequence (1) splits.

By the factor theorem (see [1, Theorem 3.6]), we have the following commutative diagram:

$$\begin{array}{ccccc}
C_{m-1} & \xrightarrow{\eta} & \mathrm{Coker}(\delta_m) & \longrightarrow & 0 \\
{\scriptstyle\delta_{m-1}}\downarrow & \swarrow{\scriptstyle\theta} & & & \\
C_{m-2} & & & &
\end{array}$$

where $\eta : C_{m-1} \to \mathrm{Coker}(\delta_m)$ is the natural epimorphism. We form the pushout of $C_m \xrightarrow{\alpha} X$ and $C_m \xrightarrow{\delta_m} C_{m-1}$ to obtain a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_m & \xrightarrow{\alpha} & X & \longrightarrow & G & \longrightarrow & 0 \\
 & & {\scriptstyle\delta_m}\downarrow & & {\scriptstyle\mu}\downarrow & & \| & & \\
0 & \longrightarrow & C_{m-1} & \xrightarrow{\nu} & P & \longrightarrow & G & \longrightarrow & 0 \\
 & & {\scriptstyle\eta}\downarrow & & {\scriptstyle g}\downarrow & & & & \\
 & & \mathrm{Coker}(\delta_m) & = & \mathrm{Coker}(\delta_m) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

So we have the commutative diagram:

$$
\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{m+2} & \xrightarrow{id} & C_{m+2} & \longrightarrow & 0 & & \\
 & & {\scriptstyle \delta_{m+2}}\downarrow & & {\scriptstyle \delta_{m+2}}\downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{m+1} & \xrightarrow{id} & C_{m+1} & \longrightarrow & 0 & & \\
 & & {\scriptstyle \delta_{m+1}}\downarrow & & {\scriptstyle \alpha\delta_{m+1}}\downarrow & & \downarrow & & \\
0 & \longrightarrow & C_m & \xrightarrow{\alpha} & X & \longrightarrow & G & \longrightarrow & 0 \\
 & & {\scriptstyle \delta_m}\downarrow & & {\scriptstyle \mu}\downarrow & & \| & & \\
0 & \longrightarrow & C_{m-1} & \xrightarrow{\nu} & P & \longrightarrow & G & \longrightarrow & 0 \\
 & & {\scriptstyle \delta_{m-1}}\downarrow & & {\scriptstyle \theta g}\downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{m-2} & \xrightarrow{id} & C_{m-2} & \longrightarrow & 0 & & \\
 & & {\scriptstyle \delta_{m-2}}\downarrow & & {\scriptstyle \delta_{m-2}}\downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{m-3} & \xrightarrow{id} & C_{m-3} & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}
$$

and can form the complex

$$W = \cdots \to C_{m+2} \to C_{m+1} \to X \to P \to C_{m-2} \to \cdots.$$

Thus we have an exact sequence of complexes

$$0 \to C \to W \to D^m(G) \to 0. \tag{2}$$

Since G is a finitely represented module, we may deduce from Remark 3.6 that $D^m(G)$ is a finitely represented complex. By our hypothesis the sequence (2) splits in the category of complexes and so the sequence (1) splits in the category of modules. Therefore C_m is an FR-injective module.

For a finitely represented complex F we have that $\mathrm{Hom}^{\cdot}(F, C)$ is exact if and only if for each m, each morphism of complexes

$$f : F \to \Sigma^{-m} C$$

is homotopic to 0. This is equivalent to the requirement that for each m and each morphism of complexes

$$f : F \to \Sigma^{-m} C,$$

the sequence

$$0 \to \Sigma^{-m} C \to M(f) \to \Sigma^{-1} F \to 0$$

splits, or, equivalently, for each m and each morphism of complexes

$$f : F \to \Sigma^{-m}C$$

the sequence

$$0 \to C \to \Sigma^m M(f) \to \Sigma^{m-1}F \to 0$$

splits where $M(f)$ denotes the mapping cone of f. Since F is finitely represented, so is $\Sigma^{m-1}F$. By hypothesis, $\mathrm{Ext}^1(\Sigma^{m-1}F, C) = 0$. Hence the sequence $0 \to C \to \Sigma^m M(f) \to \Sigma^{m-1}F \to 0$ splits and $\mathrm{Hom}^{\cdot}(F, C)$ is an exact complex.

($\Leftarrow$). Suppose C_m is an FR-injective module for all $m \in \mathbf{Z}$ and $\mathrm{Hom}^{\cdot}(F, C)$ is exact for every finitely represented complex F. An exact sequence

$$0 \to C \to W \to F \to 0$$

of complexes with F finitely represented splits at the module level. So this sequence is isomorphic to

$$0 \to C \to M(f) \to F \to 0,$$

where $f : \Sigma^1 F \to C$ is a morphism of complexes. Since $\mathrm{Hom}^{\cdot}(\Sigma^1 F, C)$ is exact, the sequence

$$0 \to C \to M(f) \to F \to 0$$

splits by [11, Lemma 2.3.2], so

$$0 \to C \to W \to F \to 0$$

also splits. □

PROPOSITION 4.8. *Let C be an exact complex of left R-modules such that $Z_t(C)$ is FR-injective in R-*Mod *for all $t \in \mathbf{Z}$. Then* $\mathrm{Hom}^{\cdot}(F, C)$ *is exact for any finitely represented complex F.*

Proof. If F is a finitely represented complex, then F is bounded by Remark 3.6. Hence we can assume that F has the following form:

$$\cdots \to 0 \to F_m \to F_{m-1} \to \cdots \to F_2 \to F_1 \to F_0 \to 0 \to \cdots.$$

Now consider the complex

$$\mathrm{Hom}^{\cdot}(F, C) = \cdots \xrightarrow{\delta_{m+1}} \prod_{i \in \mathbf{Z}} \mathrm{Hom}(F_i, C_{m+i}) \xrightarrow{\delta_m} \prod_{i \in \mathbf{Z}} \mathrm{Hom}(F_i, C_{m-1+i}) \xrightarrow{\delta_{m-1}} \cdots.$$

It is enough to show that $\mathrm{Ker}(\delta_{m-1}) \subseteq \mathrm{Im}(\delta_m)$ for each $m \in \mathbf{Z}$. Let $g \in \mathrm{Ker}(\delta_{m-1})$. Then $\delta_{m-1}(g) = (\delta^C_{m-1+t} g_t - (-1)^{m-1} g_{t-1}\delta^F_t)_{t \in \mathbf{Z}} = 0$. In the following procedure we are going to construct a morphism f satisfying $f \in \mathrm{Hom}^{\cdot}(F, C)_m = \prod_{i \in \mathbf{Z}} \mathrm{Hom}(F_i, C_{m+i})$ and $\delta_m(f) = (\delta^C_{m+t} f_t - (-1)^m f_{t-1}\delta^F_t)_{t \in \mathbf{Z}} =$

$(g_t)_{t\in\mathbf{Z}}$. Since $g_t = 0$ for $t \le -1$, we take $f_t = 0$ if $t \le -1$. If $t = 0$, then $\delta^C_{m-1} g_0 = 0$, and so $\operatorname{Im} g_0 \subseteq \operatorname{Ker}(\delta^C_{m-1}) = \operatorname{Im}(\delta^C_m)$. Since $Z_m(C)$ is *FR*-injective and F_0 is a finitely represented module, there exists a homomorphism $f_0 : F_0 \to C_m$ such that $\delta^C_m f_0 = g_0$. That is, the diagram

$$\begin{array}{ccccccccc} & & & & & & F_0 & & \\ & & & & & \overset{f_0}{\swarrow} & \downarrow g_0 & & \\ 0 & \longrightarrow & Z_m(C) & \longrightarrow & C_m & \underset{\delta^C_m}{\longrightarrow} & Z_{m-1}(C) & \longrightarrow & 0 \end{array}$$

commutes.

If $t = 1$, then $\delta^C_m(g_1 - (-1)^{m-1} f_0 \delta^F_1) = \delta^C_m g_1 - (-1)^{m-1} \delta^C_m f_0 \delta^F_1 = 0$, and so $\operatorname{Im}(g_1 - (-1)^{m-1} f_0 \delta^F_1) \subseteq \operatorname{Ker}(\delta^C_m)$. Set $h_1 = g_1 - (-1)^{m+1} f_0 \delta^F_1$. Since $Z_{m+1}(C)$ is *FR*-injective and F_1 is a finitely represented complex, we have the following commutative diagram.

$$\begin{array}{ccccccccc} & & & & & & F_1 & & \\ & & & & & \overset{f_1}{\swarrow} & \downarrow h_1 & & \\ 0 & \longrightarrow & Z_{m+1}(C) & \longrightarrow & C_{m+1} & \underset{\delta^C_{m-1}}{\longrightarrow} & Z_m(C) & \longrightarrow & 0. \end{array}$$

That is, $g_1 = \delta^C_{m-1} f_1 - (-1)^m f_0 \delta^F_1$. Repeating this procedure, we deduce that $f \in \operatorname{Im} \delta_m$ and $\delta_m(f) = g$. □

THEOREM 4.9. *Let C be a complex of left R-modules. Then C is FR-injective if and only if C is exact and $Z_m(C)$ is FR-injective in R-*Mod *for all $m \in \mathbf{Z}$.*

Proof. (⇒). Suppose that C is an *FR*-injective complex. Since $\mathrm{H}^i(C) = \operatorname{Ext}^1(S^{1-i}(R), C)$ for all $i \in \mathbf{Z}$ and $S^{1-i}(R)$ is finitely represented, then C is exact. We only need to prove that $\operatorname{Ext}^1(G, Z_m(C)) = 0$ for every finitely represented module G. Consider the exact sequence

$$0 \to K \to P \to G \to 0$$

with P a finitely generated projective module and K a finitely presented module. It yields an exact sequence of complexes

$$0 \to S^m(K) \to S^m(P) \to S^m(G) \to 0.$$

By the hypothesis $\operatorname{Ext}^1(S^m(G), C) = 0$. So

$$\operatorname{Hom}(S^m(P), C) \to \operatorname{Hom}(S^m(K), C) \to 0$$

is exact.

Let $f : K \to Z_m(C)$ be an R-homomorphism. We define $\alpha_m : K \to C_m$ by $\alpha_m = \lambda f$ where λ is the inclusion map and $\alpha_j = 0$ for $j \ne m$. In this way we obtain a morphism of complexes $\alpha : S^m(K) \to C$. Then there exists a morphism $\beta : S^m(P) \to C$ such that the diagram

$$\begin{array}{ccc} S^m(K) & \longrightarrow & S^m(P) \\ {\scriptstyle\alpha}\downarrow & \swarrow{\scriptstyle\beta} & \\ C & & \end{array}$$

commutes. Hence we have the commutative diagram

$$\begin{array}{ccc} K & \longrightarrow & P \\ {\scriptstyle\lambda f}\downarrow & \swarrow{\scriptstyle\beta_m} & \\ C_m & & \end{array}$$

and $\delta_m \beta_m = 0$, which implies that $\operatorname{Im} \beta_m \subseteq \operatorname{Ker}(\delta_m)$. So we define $g : P \to \operatorname{Ker}(\delta_m)$ by $g = \beta_m$. Thus the sequence

$$\operatorname{Hom}(P, \mathrm{Z}_m(C)) \to \operatorname{Hom}(K, \mathrm{Z}_m(C)) \to 0$$

is exact.

On the other hand, we have an exact sequence

$$\operatorname{Hom}(P, \mathrm{Z}_m(C)) \to \operatorname{Hom}(K, \mathrm{Z}_m(C)) \to \operatorname{Ext}^1(G, \mathrm{Z}_m(C)) \to 0.$$

Therefore, $\operatorname{Ext}^1(G, \mathrm{Z}_m(C)) = 0$ and we have established our result.

($\Leftarrow$). Since C is exact we have an exact sequence

$$0 \to \mathrm{Z}_m(C) \to C_m \to \mathrm{Z}_{m-1}(C) \to 0$$

for each $m \in \mathbf{Z}$. Now $\mathrm{Z}_m(C)$ and $\mathrm{Z}_{m-1}(C)$ are FR-injective which implies that C_m is FR-injective. The result now follows from Propositions 4.7 and 4.8. □

Theorem 4.10. *Let C be a complex of left R-modules. Then C is FR-flat if and only if C is exact and $\mathrm{Z}_m(C)$ is FR-flat in* R-Mod *for all $m \in \mathbf{Z}$.*

Proof. ($\Rightarrow$). Since C is FR-flat, C^+ is FR-injective. Then C^+ is exact and $\mathrm{Z}_m(C^+) \cong \mathrm{Z}_{-m-1}(C)^+$ is FR-injective in R-Mod by Theorem 4.9. And so C is exact and $\mathrm{Z}_m(C)$ is FR-flat in R-Mod for all $m \in \mathbf{Z}$.

($\Leftarrow$). Let C be exact with $\mathrm{Z}_m(C)$ FR-flat in R-Mod for all $m \in \mathbf{Z}$. Then C^+ is exact and $\mathrm{Z}_m(C)^+ = \mathrm{B}_{-m}(C^+) \cong \mathrm{Z}_{-m-1}(C^+)$ is FR-injective in R-Mod for all $m \in \mathbf{Z}$. Thus C^+ is FR-injective by Theorem 4.9. Hence C is FR-flat by Theorem 4.6. □

As we know, the direct limit of injective complexes need not be injective and the direct product of flat complexes is not necessarily flat in general. Here we give an interesting result which establishes the transfer of the FR-injectivity to the direct limit and the FR-flatness to the direct product.

THEOREM 4.11. *The following are true for any ring R.*

(1) *Every direct limit of FR-injective complexes of left R-modules is FR-injective*;

(2) *Every direct product of FR-flat complexes of right R-modules is FR-flat*;

Proof. (1) Since $\varinjlim \mathrm{Ext}^1(F, C^i) \cong \mathrm{Ext}^1(F, \varinjlim C^i)$ for any finitely represented complex F and any direct system $\{C^i\}_{i \in J}$ of complexes of left R-modules by Theorem 3.11, then every direct limit of FR-injective complexes of left R-modules is FR-injective.

(2) Since $\overline{\mathrm{Tor}}_1(\prod N^\alpha, F) \cong \prod \overline{\mathrm{Tor}}_1(N^\alpha, F)$ for any finitely represented complex F and any family $\{N^\alpha\}_{\alpha \in \Lambda}$ of complexes of right R-modules by Theorem 3.12, then every direct product of FR-flat complexes of right R-modules is FR-flat. □

Example 4.12. If R is not Noetherian, then we can form a direct limit $\varinjlim M_i$ of injective R-modules $\{M_i\}_{i \in I}$ which is not injective, but is necessarily FR-injective. Hence $D^0(\varinjlim M_i)$ is an FR-injective complex but is not an injective complex by Theorem 4.9.

It is clear that if R is left Noetherian then the class of FR-injective left R-modules coincides with the class of injective left R-modules and if R is left coherent then the class of FR-flat left R-modules coincides with the class of flat left R-modules.

COROLLARY 4.13. *Let R be a ring. Then the following conditions are equivalent*:

(1) *R is left Noetherian*;

(2) *A complex C of left R-modules is FR-injective if and only if C is injective.*

Proof. (1) ⇒ (2). It follows from Theorem 4.9 since a complex C is injective if and only if C is exact and $\mathrm{Z}_m(C)$ is injective in R-Mod for all $m \in \mathbf{Z}$.

(2) ⇒ (1). It follows from Theorem 4.11 and the fact that a ring R is left noetherian if and only if every direct limit of injective left R-modules is injective. □

COROLLARY 4.14. *Let R be a ring. Then the following conditions are equivalent*:

(1) *R is left coherent*;

(2) *A complex C of right R-modules is FR-flat if and only if C is flat.*

Proof. Note that a ring R is left coherent if and only if every direct product of flat right R-modules is flat. Then the result is clear by Theorem 4.10 and Theorem 4.11 since a complex C is flat if and only if C is exact and $\mathrm{Z}_m(C)$ is flat in R-Mod for all $m \in \mathbf{Z}$. □

Recall from [11] that a exact sequence $0 \to S \to C \to C/S \to 0$ in $\mathscr{C}(R)$ is pure if $0 \to \underline{\mathrm{Hom}}(P, S) \to \underline{\mathrm{Hom}}(P, C) \to \underline{\mathrm{Hom}}(P, C/S) \to 0$ is exact for any finitely presented complex P. In this case, S is said to be a pure subcomplex of C and C/S is said to be a pure quotient complex of C.

Proposition 4.15. *The class of all FR-injective complexes and the class of all FR-flat complexes are closed under pure subcomplexes and pure quotient complexes.*

Proof. Let B be a pure subcomplex of an FR-injective complex A and D a finitely represented complex. Then we have the exact sequence

$$0 \to \underline{\mathrm{Hom}}(D, B) \to \underline{\mathrm{Hom}}(D, A) \to \underline{\mathrm{Hom}}(D, A/B) \to 0.$$

Thus $\underline{\mathrm{Ext}}^1(D, B) = 0$ since $\underline{\mathrm{Ext}}^1(D, A) = 0$. Therefore, B is FR-injective.

Let S be a pure subcomplex of an FR-flat complex C. Then the pure exact sequence $0 \to S \to C \to C/S \to 0$ induces the split exact sequence $0 \to (C/S)^+ \to C^+ \to S^+ \to 0$. Thus S^+ is FR-injective since C^+ is FR-injective by Theorem 4.6. So S is FR-flat by Theorem 4.6 again.

Let C be a pure quotient complex of an FR-injective complex A. Then we have an exact sequence $0 \to K \to A \to C \to 0$. Then C^+ is FR-flat by Theorem 4.6. So C is FR-injective by Theorem 4.6 again. It is easy to see that the class of all FR-flat complexes is also closed under pure quotient complexes. □

Lemma 4.16 ([3]). (1) *Let $\mathcal{C}$ be a finitely accessible category and $\mathcal{A}$ a class of objects of $\mathcal{C}$ closed under direct limits and pure epimorphic images. Then $\mathcal{A}$ is covering.*

(2) *Let $\mathcal{C}$ be a finitely accessible additive category with products and $\mathcal{A}$ a class of objects of $\mathcal{C}$ closed under products and pure subobjects. Then $\mathcal{A}$ is preenveloping.*

Theorem 4.17. *The following are true for any ring R.*
(1) *Every complex of left R-modules has an FR-injective preenvelope*;
(2) *Every complex of right R-modules has an FR-flat cover*;
(3) *Every complex of left R-modules has an FR-injective cover*;
(4) *Every complex of right R-modules has an FR-flat preenvelope.*

Proof. It follows from Theorem 4.11, Proposition 4.15, Lemma 4.16 and Remark 4.3 since $\mathscr{C}(R)$ is finitely accessible in the sense of [3]. □

In the following, $\mathcal{F}\mathscr{F}$ (resp. $\mathcal{F}\mathscr{I}$) denotes the class of all FR-flat complexes (resp. FR-injective complexes). Then $({}^{\perp}\mathcal{F}\mathscr{I}, \mathcal{F}\mathscr{I})$ and $(\mathcal{F}\mathscr{F}, \mathcal{F}\mathscr{F}^{\perp})$ are cotorsion theories for the category of complexes without any conditions.

Theorem 4.18. *The following are equivalent for a ring R*:
(1) $({}^{\perp}\mathcal{F}\mathscr{I}, \mathcal{F}\mathscr{I})$ *is a hereditary cotorsion theory*;

(2) $(\mathcal{F}\mathscr{F}, \mathcal{F}\mathscr{F}^{\perp})$ *is a hereditary cotorsion theory*;

(3) $\underline{\mathrm{Ext}}^2(D, C) = 0$ *for any finitely represented complex* D *and any FR-injective complex* C *of left* R*-modules*;

(4) $\overline{\mathrm{Tor}}_2(B, D) = 0$ *for any finitely represented complex* D *and any FR-flat complex* B *of right* R*-modules.*

Proof. (1) $\Rightarrow$ (2) Let $0 \to A \to B \to C \to 0$ be an exact sequence of complexes with B and C FR-flat. Then we have an induced exact sequence $0 \to C^+ \to B^+ \to A^+ \to 0$. Since B^+ and C^+ are FR-injective by Theorem 4.6, so is A^+ by (1). Hence A is FR-flat. Therefore, $(\mathcal{F}\mathscr{F}, \mathcal{F}\mathscr{F}^{\perp})$ is a hereditary cotorsion theory.

(2) $\Rightarrow$ (1) Let $0 \to A \to B \to C \to 0$ be an exact sequence of complexes with A and B FR-injective. Then we get an exact sequence $0 \to C^+ \to B^+ \to A^+ \to 0$. By Theorem 4.6, A^+ and B^+ are FR-flat, and so C^+ is FR-flat by (2). Hence C is FR-injective by Theorem 4.6 again. Therefore, $({}^{\perp}\mathcal{F}\mathscr{I}, \mathcal{F}\mathscr{I})$ is a hereditary cotorsion theory.

(1) $\Rightarrow$ (3) It follows from [11, Theorem 1.2.10].

(3) $\Rightarrow$ (1) Let $0 \to A \to B \to C \to 0$ be an exact sequence of complexes with A and B FR-injective. For any finitely represented complex D, we have the exact sequence

$$0 = \underline{\mathrm{Ext}}^1(D, B) \to \underline{\mathrm{Ext}}^1(D, C) \to \underline{\mathrm{Ext}}^2(D, A) = 0$$

Then $\underline{\mathrm{Ext}}^1(D, C) = 0$, and so C is FR-injective.

(2) $\Rightarrow$ (4) For any FR-flat complex of right R-modules N, there is an exact sequence $0 \to K \to P \to N \to 0$ with P projective. Then K is FR-flat by (2), and so $\overline{\mathrm{Tor}}_2(N, D) \cong \overline{\mathrm{Tor}}_1(K, D) = 0$ for any finitely represented complex D.

(4) $\Rightarrow$ (2) Similar to (3) $\Rightarrow$ (1). □

References

[1] F. W. Anderson and K. R. Fuller, Rings and categories of modules, Springer, Berlin, 1992.

[2] L. Avramov and H. B. Foxby, Homological dimensions of unbounded complexes, J. Pure Appl. Algebra **71** (1991), 129–155.

[3] S. Crivei, M. Prest and B. Torrecillas, Covers in finitely accessible categories, Proc. Amer. Math. Soc. **138** (2010), 1213–1221.

[4] L. W. Christensen, H. B. Foxby and H. Holm, Derived category methods in commutative algebra, preparing.

[5] E. E. Enochs and J. R. García Rozas, Tensor products of complexes, Math. J. Okayama Univ. **39** (1997), 17–39.

[6] E. E. Enochs and J. R. García Rozas, Flat covers of complexes, J. Algebra **210** (1998), 86–102.

[7] E. E. Enochs and L. Oyonarte, Covers, envelopes and cotorsion theories, Nova Science Publishere, Inc. New York, 2002.

[8] E. E. Enochs, O. M. G. Jenda and J. Xu, Orthogonality in the category of complexes, Math. J. Okayama Univ. **38** (1996), 25–46.

[9] E. E. Enochs, Injective and flat covers, envelopes and resolvents, Israel J. Math. **39** (1981), 189–209.
[10] H. B. Foxby, A homological theory of complexes of modules, Preprint series **19**, Department of Mathematics, Copenhagen University, 1981.
[11] J. R. García Rozas, Covers and envelopes in the category of complexes, Chapman & Hall/CRC, Boca Raton, FL, 1999.
[12] S. Glaz, Commutative coherent rings, Lect. notes math. **1371**, Springer-Verlag, Berlin, 1989.
[13] M. A. Goddard, Projective covers of complexes, Ring theory, World Science Publishing, 1993, 172–181.
[14] L. X. Mao and N. Q. Ding, Relative projective modules and relative injective modules, Comm. Algebra **34** (2006), 2403–2418.
[15] S. Stenström, Rings of quotients, Springer-Verlag, New York, 1975.
[16] D. X. Zhou, On n-coherent rings and (n,d)-rings, Comm. Algebra **32** (2004), 2425–2441.

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